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FIXED POINT THEORY IN MULTIPLICATIVE METRIC SPACE AND ITS APPLICATION IN BOUNDARY VALUE PROBLEMS AND CYCLIC MAPPINGS

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Abstract. In this paper, building on the recent work of Stojan Radenović and Bessem Samet, we establish several fixed point theorems in multiplicative metric spaces for Jaggi-type and weak contractions. These results are derived using a function $\varphi : [1,\infty) \to [0,\infty)$ satisfying $\varphi(s) \geq b(\ln s)^c$ for $s > 1$, with b and c positive constants. The theorems presented here are not equivalent to their classical analogues in standard metric spaces. To demonstrate the applicability of our results, we provide an example involving a boundary value problem for a second-order differential equation and the determination of a unique fixed point for a cyclic mapping.

Keywords: multiplicative metric space; fixed point; weak contraction; Jaggi contraction; rational contraction.

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1. Introduction and Preliminaries

Multiplicative calculus, introduced by Grossman and Katz [12] in 1972, laid the foundation for multiplicative metric spaces, later developed by Bashirov *et al.* [7]. In this framework, for a non-empty set $\mathbf{W}$, the classical additive distance is replaced by a multiplicative distance $\rho_m : \mathbf{W} \times \mathbf{W} \to [1,\infty)$, where distance between two identical elements is 1 instead of 0 and the

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triangle inequality takes a multiplicative form. Multiplicative metric spaces generalize classical metrics and have been extensively studied, including Banach, Chatterjea, Kannan, Weak and various rational contractions. These spaces are particularly effective in fixed point theory, where classical results, such as the Banach contraction principle, have been reformulated within the multiplicative framework, offering robust techniques for nonlinear analysis, iterative processes, and the study of integral equations.

The formal definition of a multiplicative metric space is given below.

Definition 1.1 ([7])**.** Let $\mathbf{W}$ be a non-empty set. A function $\rho_m : \mathbf{W} \times \mathbf{W} \to [1, \infty)$ is called a multiplicative metric if, for all $\omega, \varpi, \sigma \in \mathbf{W}$, the following conditions hold:

(i) $\rho_m(\omega, \varpi) = 1$ if and only if $\omega = \varpi$;

(ii) $\rho_m(\omega, \varpi) = \rho_m(\varpi, \omega)$ (symmetry);

(iii) $\rho_m(\omega, \varpi) \leq \rho_m(\omega, \sigma)\, \rho_m(\sigma, \varpi)$ (multiplicative triangle inequality).

The pair $(\mathbf{W}, \rho_m)$ is called a multiplicative metric space.

Since its inception, the Banach Contraction Principle [5], introduced by Stefan Banach, has served as a cornerstone of fixed point theory.

Theorem 1.2 ([5])**.** *Let* $(\mathbf{W}, \rho)$ *be a complete metric space and let* $\mathscr{F} : \mathbf{W} \to \mathbf{W}$ *be a mapping. Suppose there exists a constant* $\lambda \in [0, 1)$ *such that*

$$\rho(\mathscr{F}\omega, \mathscr{F}\varpi) \leq \lambda \rho(\omega, \varpi), \text{ for all } \omega, \varpi \in \mathbf{W}. \tag{1.1}$$

Then $\mathscr{F}$ *admits a unique fixed point in* $\mathbf{W}$.

The Kannan contraction [14], introduced by R. Kannan in 1968, depends on the sum of distances from points to their images rather than on the distance between the points themselves.

Theorem 1.3 ([14])**.** *Let* $(\mathbf{W}, \rho)$ *be a complete metric space, and let* $T : \mathbf{W} \to \mathbf{W}$ *be a mapping such that*

$$\rho(T\omega, T\varpi) \leq \lambda \{\rho(\omega, T\omega) + \rho(\varpi, T\varpi)\} \quad \text{for all } \omega, \varpi \in \mathbf{W}, \tag{1.2}$$

where $0 \leq \lambda < \frac{1}{2}$.

Then T *has a unique fixed point in* $\mathbf{W}$.

In 1972, Chatterjea introduced a new class of contractions [9], defined in terms of cross distances rather than direct distances.

Theorem 1.4 ([9])**.** *Let $(\mathbf{W},\rho)$ be a complete metric space and let $\mathscr{F}:\mathbf{W}\to\mathbf{W}$ be a mapping. Suppose there exists a constant $\lambda\in\left[0,\frac{1}{2}\right)$ such that*

$$\rho(\mathscr{F}\omega,\mathscr{F}\varpi)\le\lambda\left\{\rho(\omega,\mathscr{F}\varpi)+\rho(\varpi,\mathscr{F}\omega)\right\},\ \textit{for all } \omega,\varpi\in\mathbf{W}. \tag{1.3}$$

Then $\mathscr{F}$ admits a unique fixed point in $\mathbf{W}$.

The rational contraction introduced by D. S. Jaggi [13] in 1977 generalizes the Banach contraction by incorporating a rational expression of distances between points and their images, ensuring the existence and uniqueness of fixed points in complete metric spaces.

Theorem 1.5 ([13])**.** *Let $(\mathbf{W},\rho)$ be a complete metric space and let $\mathscr{F}:\mathbf{W}\to\mathbf{W}$ be a continuous mapping satisfying*

$$\rho(\mathscr{F}\omega,\mathscr{F}\varpi)\le\alpha\,\rho(\omega,\varpi)+\beta\frac{\rho(\omega,\mathscr{F}\omega)\,\rho(\varpi,\mathscr{F}\varpi)}{\rho(\omega,\varpi)},\quad \textit{for all } \omega,\varpi\in\mathbf{W},\ \omega\ne\varpi, \tag{1.4}$$

where $\alpha,\beta\ge0$ with $\alpha+\beta<1$.

Then $\mathscr{F}$ has a unique fixed point in $\mathbf{W}$. Moreover, for any $\omega_0\in\mathbf{W}$, the sequence $\{\omega_n\}$ defined by $\omega_{n+1}=\mathscr{F}\omega_n$ converges to the fixed point.

The concept of altering distance functions was introduced by Khan et al. [17] in 1984, providing a useful generalization in the study of fixed point theory.

Definition 1.6 ([17])**.** A function $\psi:[0,\infty)\to[0,\infty)$ is called an *altering distance function* if it is continuous, non-decreasing and satisfies $\psi(t)=0$ if and only if $t=0$. Let the family of all such functions be denoted by Ψ.

Subsequently, in 1997, Alber and Guerre-Delabriere [4] introduced the notion of weak contractions for single-valued mappings in Hilbert spaces. This concept was later extended by B.E. Rhoades [23] in 2001 to the setting of complete metric spaces, significantly broadening its applicability.

Theorem 1.7 ([23])**.** *Let $(\mathbf{W},\rho)$ be a complete metric space and let $\mathscr{F}:\mathbf{W}\to\mathbf{W}$ be a mapping. Suppose that*

$$\rho(\mathscr{F}\omega,\mathscr{F}\varpi)\leq\rho(\omega,\varpi)-\psi(\rho(\omega,\varpi)),\ \textit{for all}\ \omega,\varpi\in\mathbf{W},\ \textit{where}\ \psi\in\Psi. \tag{1.5}$$

Then $\mathscr{F}$ admits a unique fixed point in $\mathbf{W}$. Moreover, for any initial point $\omega_0\in\mathbf{W}$, the iterative sequence $\{\mathscr{F}^n\omega_0\}$ converges to this fixed point.

In 1971, Ćirić [10] introduced the notions of *orbital continuity* and *orbital completeness*, which extend classical continuity and completeness by focusing on the behavior of a mapping along its iterates. These concepts have been instrumental in analyzing convergence of iterative sequences in fixed point theory, even for mappings that are not globally continuous.

Definition 1.8 ([10])**.** Let $(\mathbf{W},\rho)$ be a metric space and $\mathscr{F}:\mathbf{W}\to\mathbf{W}$ a mapping. For any $\omega_0\in\mathbf{W}$, the set

$$\mathscr{O}(\omega_0,\mathscr{F})=\{\omega_0,\mathscr{F}\omega_0,\mathscr{F}^2\omega_0,\mathscr{F}^3\omega_0,\dots\}$$

is called the *orbit of $\mathscr{F}$ at ω_0*.

A mapping $\mathscr{F}$ is *orbitally continuous at $\omega\in\mathbf{W}$* if, whenever $x=\lim_{k\to\infty}\mathscr{F}^{n_k}\omega_0$, it holds that $\mathscr{F}\omega=\lim_{k\to\infty}\mathscr{F}^{n_k+1}\omega_0$. $\mathscr{F}$ is *orbitally continuous on* $\mathbf{W}$ if it is orbitally continuous at every point of $\mathbf{W}$.

The space $(\mathbf{W},\rho)$ is said to be *$\mathscr{F}$-orbitally complete* if every Cauchy sequence of the form $\{\mathscr{F}^{n_k}\omega_0\}$, for any $\omega_0\in\mathbf{W}$, converges in $\mathbf{W}$. While completeness of $(\mathbf{W},\rho)$ implies $\mathscr{F}$-orbital completeness for any $\mathscr{F}$, the converse may not hold.

The following are fundamental definitions and results used in the study of multiplicative metric spaces.

Definition 1.9. Let $(\mathbf{W},\rho_m)$ be a multiplicative metric space, and let $\{\omega_n\}$ be a sequence in $\mathbf{W}$. The sequence $\{\omega_n\}$ is said to be *multiplicatively convergent* to $\omega^*\in\mathbf{W}$ if, for every $\varepsilon>1$, there exists a natural number p such that $\rho_m(\omega_n,\omega^*)<\varepsilon,$ for all $n\geq p$, i.e., $\lim_{n\to\infty}\rho_m(\omega_n,\omega^*)=1$.

Definition 1.10. A sequence $\{\omega_n\}$ in $(\mathbf{W},\rho_m)$ is called a *multiplicative Cauchy sequence* if, for every $\varepsilon>1$, there exists a natural number p such that $\rho_m(\omega_{n_1},\omega_{n_2})<\varepsilon,$ for all $n_1,n_2\geq p$, i.e., $\lim_{n_1,n_2\to\infty}\rho_m(\omega_{n_1},\omega_{n_2})=1$.

Definition 1.11. A multiplicative metric space $(\mathbf{W}, \rho_m)$ is said to be *multiplicatively complete* if every multiplicative Cauchy sequence in $\mathbf{W}$ converges to a point in $\mathbf{W}$.

Theorem 1.12 (Multiplicative Banach Contraction [18])**.** *Let $\mathscr{F}$ be a mapping on a complete multiplicative metric space $(\mathbf{W}, \rho_m)$ such that*

$$\rho_m(\mathscr{F}\omega, \mathscr{F}\varpi) \leq \rho_m(\omega, \varpi)^{\lambda}, \textit{ for all } \omega, \varpi \in \mathbf{W}, \quad \lambda \in [0,1). \tag{1.6}$$

Then $\mathscr{F}$ has a unique fixed point in $\mathbf{W}$.

Theorem 1.13 (Multiplicative Kannan Contraction [18])**.** *Let $\mathscr{F}$ be a mapping on a complete multiplicative metric space $(\mathbf{W}, \rho_m)$ such that*

$$\rho_m(\mathscr{F}\omega, \mathscr{F}\varpi) \leq \left\{\rho_m(\omega, \mathscr{F}\omega)\, \rho_m(\varpi, \mathscr{F}\varpi)\right\}^{\lambda}, \textit{ for all } \omega, \varpi \in \mathbf{W}, \quad \lambda \in \left[0, \tfrac{1}{2}\right). \tag{1.7}$$

Then $\mathscr{F}$ has a unique fixed point in $\mathbf{W}$.

Theorem 1.14 (Multiplicative Chatterjea Contraction [18])**.** *Let $\mathscr{F}$ be a mapping on a complete multiplicative metric space $(\mathbf{W}, \rho_m)$ such that*

$$\rho_m(\mathscr{F}\omega, \mathscr{F}\varpi) \leq \left\{\rho_m(\omega, \mathscr{F}\varpi)\, \rho_m(\varpi, \mathscr{F}\omega)\right\}^{\lambda}, \textit{ for all } \omega, \varpi \in \mathbf{W}, \quad \lambda \in \left[0, \tfrac{1}{2}\right). \tag{1.8}$$

Then $\mathscr{F}$ has a unique fixed point in $\mathbf{W}$.

Theorem 1.15 (Multiplicative Weak Contraction [18])**.** *Let $\mathscr{F}$ be a mapping on a complete multiplicative metric space $(\mathbf{W}, \rho_m)$ such that*

$$\rho_m(\mathscr{F}\omega, \mathscr{F}\varpi) \leq \frac{\rho_m(\omega, \varpi)}{\psi_2(\rho_m(\omega, \varpi))}, \textit{ for all } \omega, \varpi \in \mathbf{W}, \tag{1.9}$$

where $\psi_2 : [1, \infty) \to [1, \infty)$ is lower semi-continuous and $\psi_2(t) = 1$ if and only if $t = 1$. Then $\mathscr{F}$ has a unique fixed point in $\mathbf{W}$.

Definition 1.16. Let $(\mathbf{W}, \rho_m)$ be a multiplicative metric space and $\mathscr{F} : \mathbf{W} \to \mathbf{W}$ be a mapping. The space $(\mathbf{W}, \rho_m)$ is said to be *$\mathscr{F}$-orbitally complete* (with respect to $\mathscr{F}$) if for some $\omega_0 \in \mathbf{W}$, every multiplicative Cauchy sequence contained in the orbit

$$\mathscr{O}_{\mathscr{F}}(\omega_0) = \{\omega_0, \mathscr{F}\omega_0, \mathscr{F}^2\omega_0, \dots\}$$

converges in $\mathbf{W}$; that is, if $\{\mathscr{F}^n\omega_0\}$ is multiplicative Cauchy, then there exists $\omega^* \in \mathbf{W}$ such that

$$\lim_{n\to\infty} \rho_m(\mathscr{F}^n\omega_0, \omega^*) = 1.$$

Definition 1.17. Let $(\mathbf{W}, \rho_m)$ be a multiplicative metric space and $\mathscr{F} : \mathbf{W} \to \mathbf{W}$ be a mapping. Then $\mathscr{F}$ is said to be *orbitally continuous* if for any $\omega_0 \in \mathbf{W}$, whenever

$$\lim_{n\to\infty} \rho_m(\mathscr{F}^n\omega_0, \omega^*) = 1$$

for some $\omega^* \in \mathbf{W}$, it follows that

$$\lim_{n\to\infty} \rho_m(\mathscr{F}^{n+1}\omega_0, \mathscr{F}\omega^*) = 1.$$

Agarwal et al. [3] and S. Shukla [25] in 2016 observed that any multiplicative metric space can be associated with a standard metric space. Specifically, by taking the natural logarithm of a multiplicative metric, one obtains a classical metric. That is, if $(\mathbf{W}, \rho_m)$ is a multiplicative metric space, then defining $\rho(\omega, \varpi) = \ln \rho_m(\omega, \varpi)$ yields a metric ρ on $\mathbf{W}$.

Conversely, Došenović et al. [11] established that any metric space can be transformed into a multiplicative metric space by exponentiation. That is, if $(\mathbf{W}, \rho)$ is a metric space, then $\rho_m(\omega, \varpi) = e^{\rho(\omega,\varpi)}$ is a multiplicative metric on $\mathbf{W}$.

Theorem 1.18 ([11])**.** *Let $(\mathbf{W}, \rho_m)$ be a multiplicative metric space. Then the pair $(\mathbf{W}, \rho)$, with $\rho(\omega, \varpi) = \ln \rho_m(\omega, \varpi)$, forms a metric space. Conversely, if $(\mathbf{W}, \rho)$ is a metric space, then $(\mathbf{W}, \rho_m)$, with $\rho_m(\omega, \varpi) = e^{\rho(\omega,\varpi)}$, is a multiplicative metric space.*

From these observations, the connections between multiplicative metrics and classical metrics can be summarized as follows:

(i) If ρ_m is a multiplicative metric on $\mathbf{W}$, then $\rho = \ln \rho_m$ defines a standard metric on $\mathbf{W}$.
(ii) If $(\mathbf{W}, \rho)$ is a complete multiplicative metric space, then $(\mathbf{W}, \rho_m)$ is a complete metric space.
(iii) Conversely, if ρ is a metric on $\mathbf{W}$, then $\rho_m = e^{\rho}$ defines a multiplicative metric on $\mathbf{W}$.
(iv) If $(\mathbf{W}, \rho)$ is a complete metric space, then $(\mathbf{W}, \rho_m)$ is a complete multiplicative metric space.

These observations emphasize the strong correspondence between multiplicative metric spaces and classical metric spaces, demonstrating that both the fundamental metric properties and completeness are preserved under logarithmic and exponential transformations.

Consequently, it has been established that the multiplicative analogues of most classical fixed point theorems are, in fact, equivalent to their standard metric space counterparts. Some of these equivalences are summarized below:

Theorem 1.19 ([11])**.** *Theorem 1.2 (Banach contraction in a classical metric space) and Theorem 1.12 (Banach contraction in a multiplicative metric space) are equivalent.*

Theorem 1.20 ([11])**.** *Theorem 1.3 (Kannan contraction in a classical metric space) and Theorem 1.13 (Kannan contraction in a multiplicative metric space) are equivalent.*

Theorem 1.21 ([11])**.** *Theorem 1.4 (Chatterjea contraction in a standard metric space) and Theorem 1.14 (Chatterjea contraction in a multiplicative metric space) are equivalent.*

Theorem 1.22 ([11])**.** *Theorem 1.7 (Weak contraction in a standard metric space) and Theorem 1.15 (Weak contraction in a multiplicative metric space) are equivalent.*

Motivated by the survey in [11], a fundamental question arose in the study of multiplicative metric spaces: whether there exist formulations of classical fixed point theorems—such as those of Banach, or Kannan—within multiplicative metric spaces that are genuinely non-equivalent to their counterparts in standard metric spaces.

This problem remained open until recently, when Radenović and Samet [20] addressed it through the introduction of the following function class Φ.

Definition 1.23. Let Φ denote the collection of functions $\phi : [1,+\infty) \to [0,+\infty)$ satisfying

$$\phi(s) \geq b(\ln s)^c, \quad s > 1, \tag{1.10}$$

where b and c are positive constants.

Remark 1.24 ([20]). From the definition above, it immediately follows that

$$\phi(s) > 0, \text{ for all } s > 1.$$

Radenović and Samet further introduced the following subsets of Φ:

$$\Phi^{\uparrow} = \{\phi \in \Phi : \phi \text{ is non-decreasing}\},$$

$$\Phi_0 = \{\phi \in \Phi : \phi(1) = 0\}.$$

Using these function classes, they established new multiplicative fixed point theorems (Theorems (1.2) and (1.12) in [20]) that are genuinely non-equivalent to their classical counterparts in standard metric spaces. This open problem was resolved by Stojan Radenović and Bessem Samet [20] through the introduction of a function $\varphi : [1,+\infty) \to [0,\infty)$ satisfying

$$\varphi(s) \geq b(\ln s)^c, \quad s > 1,$$

where b and c are positive constants.

Theorem 1.25 ([20])**.** *Let $(\mathbf{W}, \rho_m)$ be a complete multiplicative metric space and let $\mathscr{F} : \mathbf{W} \to \mathbf{W}$ be a mapping satisfying the following conditions:*

(i) There exist a function $\varphi \in \Phi$ and a constant $\lambda \in (0,1)$ such that

$$\varphi\big(\rho_m(\mathscr{F}\omega, \mathscr{F}\varpi)\big) \leq \lambda\, \varphi\big(\rho_m(\omega,\varpi)\big), \quad \textit{for all } \omega, \varpi \in \mathbf{W} \textit{ with } \rho_m(\omega,\varpi) > 1.$$

(ii) For any $\omega, \varpi \in \mathbf{W}$, if $\lim_{n\to\infty} \rho_m(\mathscr{F}^n\omega, \varpi) = 1$, then there exists a subsequence $\{\mathscr{F}^{n_k}\omega\}$ of $\{\mathscr{F}^n\omega\}$ such that $\lim_{k\to\infty} \rho_m(\mathscr{F}^{n_{k+1}}\omega, \mathscr{F}\varpi) = 1$.

Then, the mapping $\mathscr{F}$ admits a unique fixed point in $\mathbf{W}$.

Theorem 1.26 ([20])**.** *Let $(\mathbf{W}, \rho_m)$ be a complete multiplicative metric space and let $\mathscr{F} : \mathbf{W} \to \mathbf{W}$ be a mapping satisfying the following conditions:*

(i) There exist a function $\varphi \in \Phi$ and a constant $\lambda \in (0,1)$ such that

$$\varphi\big(\rho_m(\mathscr{F}\omega, \mathscr{F}\varpi)\big) \leq \lambda \left\{\varphi(\rho_m(\omega, \mathscr{F}\omega)) + \varphi(\rho_m(\varpi, \mathscr{F}\varpi)\right\},$$

for all $\omega, \varpi \in \mathbf{W}$ with $\rho_m(\omega,\varpi) > 1$.

(ii) For any $\omega, \varpi \in \mathbf{W}$, if $\lim_{n\to\infty} \rho_m(\mathscr{F}^n\omega, \varpi) = 1$, then there exists a subsequence $\{\mathscr{F}^{n_k}\omega\}$ of $\{\mathscr{F}^n\omega\}$ such that $\lim_{k\to\infty} \rho_m(\mathscr{F}^{n_{k+1}}\omega, \mathscr{F}\varpi) = 1$.

Then, the mapping $\mathscr{F}$ admits a unique fixed point in $\mathbf{W}$.

The function φ ensures that the multiplicative fixed point theorems obtained through this approach are not necessarily equivalent to their classical counterparts in standard metric spaces, thus providing genuinely new results within the multiplicative framework.

We introduce the following notations and lemmas, which will be employed directly or indirectly in our main results.

$$\begin{aligned}
\Phi &= \Big\{\varphi \mid \varphi : [1,+\infty) \to [0,\infty),\ \varphi(s) \geq b(\ln s)^c \text{ for } s > 1,\ \text{with } b,c > 0\Big\},\\
\Phi_0 &= \Big\{\varphi \in \Phi \mid \varphi(t) = 0 \iff t = 1\Big\},\\
\Psi_m &= \Big\{\psi \mid \psi : [1,\infty) \to [1,\infty),\ \psi \text{ is continuous, non-decreasing, and}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \psi(t) = 1 \iff t = 1\Big\}.
\end{aligned}$$

Lemma 1.27. *Let $(\mathbf{W},\rho)$ and $(\mathbf{W},\rho_m)$ be corresponding metric and multiplicative metric spaces linked via logarithmic and exponential transformations. Then $\mathscr{F}$-orbital completeness and orbital continuity are invariant under these transformations.*

In particular, $(\mathbf{W},\rho)$ is $\mathscr{F}$-orbitally complete and $\mathscr{F}$ is orbitally continuous if and only if $(\mathbf{W},\rho_m)$ has the same property.

Proof. **Orbital completeness:** Let $(\mathbf{W},\rho)$ be a metric space and $(\mathbf{W},\rho_m)$ its associated multiplicative metric space, where

$$\rho_m(\omega,\varpi) = e^{\rho(\omega,\varpi)},\ \text{for all } \omega,\varpi \in \mathbf{W}.$$

Consider a sequence $\{\omega_n\}$ defined by $\omega_n = \mathscr{F}^n\omega_0$ for some $\omega_0 \in \mathbf{W}$ and a mapping $\mathscr{F} : \mathbf{W} \to \mathbf{W}$.

Suppose $\{\omega_n\}$ is a convergent Cauchy sequence in the standard metric space $(\mathbf{W},\rho)$. Then, for any $p \in \mathbb{N}$,

$$\rho_m(\omega_n,\omega_{n+p}) = e^{\rho(\omega_n,\omega_{n+p})}.$$

Since $\{\omega_n\}$ is Cauchy and convergent in $(\mathbf{W},\rho)$, we have $\rho(\omega_n,\omega_{n+p}) \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \rho_m(\omega_n,\omega_{n+p}) = e^0 = 1,$$

showing that $\{\omega_n\}$ is a multiplicatively Cauchy and convergent sequence in $(\mathbf{W},\rho_m)$. Therefore, if $(\mathbf{W},\rho)$ is $\mathscr{F}$-orbitally complete, then $(\mathbf{W},\rho_m)$ is also $\mathscr{F}$-orbitally complete.

Conversely, suppose $\{\omega_n\}$ is a convergent Cauchy sequence in $(\mathbf{W},\rho_m)$. Then, for any $k\in\mathbb{N}$,

$$\rho(\omega_n,\omega_{n+k}) = \ln\rho_m(\omega_n,\omega_{n+k}).$$

Since $\{\omega_n\}$ is multiplicatively Cauchy and convergent, $\rho_m(\omega_n,\omega_{n+k})\to 1$ as $n\to\infty$. Hence,

$$\lim_{n\to\infty}\rho(\omega_n,\omega_{n+k}) = \ln 1 = 0,$$

showing that $\{\omega_n\}$ is a Cauchy and convergent sequence in $(\mathbf{W},\rho)$. Therefore, if $(\mathbf{W},\rho_m)$ is $\mathscr{F}$-orbitally complete, then $(\mathbf{W},\rho)$ is also $\mathscr{F}$-orbitally complete.

Orbital continuity: Let $\mathscr{F}:\mathbf{W}\to\mathbf{W}$ be a mapping, where ρ is the standard metric on $\mathbf{W}$ and ρ_m is the corresponding multiplicative metric defined by

$$\rho_m(\omega,\varpi) = e^{\rho(\omega,\varpi)},\quad \text{equivalently,}\quad \rho(\omega,\varpi) = \ln\rho_m(\omega,\varpi),\quad \forall\omega,\varpi\in\mathbf{W}.$$

Suppose $\mathscr{F}$ is orbitally continuous on $(\mathbf{W},\rho)$. Let $\omega_0\in\mathbf{W}$ and assume

$$\lim_{n\to\infty}\rho_m(\mathscr{F}^n\omega_0,\omega^*) = 1.$$

Then using $\rho=\ln\rho_m$, we have

$$\lim_{n\to\infty}\rho(\mathscr{F}^n\omega_0,\omega^*) = \lim_{n\to\infty}\ln\rho_m(\mathscr{F}^n\omega_0,\omega^*) = \ln 1 = 0.$$

By orbital continuity of $\mathscr{F}$ in $(\mathbf{W},\rho)$,

$$\lim_{n\to\infty}\rho(\mathscr{F}^{n+1}\omega_0,\mathscr{F}\omega^*) = 0,$$

which gives

$$\lim_{n\to\infty}\rho_m(\mathscr{F}^{n+1}\omega_0,\mathscr{F}\omega^*) = e^0 = 1.$$

Thus, $\mathscr{F}$ is orbitally continuous on $(\mathbf{W},\rho_m)$.

Conversely, suppose $\mathscr{F}$ is orbitally continuous on $(\mathbf{W},\rho_m)$. Let $\omega_0\in\mathbf{W}$ and assume

$$\lim_{n\to\infty}\rho(\mathscr{F}^n\omega_0,\omega^*) = 0.$$

Then

$$\lim_{n\to\infty}\rho_m(\mathscr{F}^n\omega_0,\omega^*) = \lim_{n\to\infty}e^{\rho(\mathscr{F}^n\omega_0,\omega^*)} = e^0 = 1.$$

By orbital continuity in $(\mathbf{W}, \rho_m)$,

$$\lim_{n\to\infty} \rho_m(\mathscr{F}^{n+1}\omega_0, \mathscr{F}\omega^*) = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} \rho(\mathscr{F}^{n+1}\omega_0, \mathscr{F}\omega^*) = \ln 1 = 0.$$

Hence, $\mathscr{F}$ is orbitally continuous on $(\mathbf{W}, \rho)$.

Therefore, $\mathscr{F}$ is orbitally continuous on $(\mathbf{W}, \rho)$ if and only if it is orbitally continuous on $(\mathbf{W}, \rho_m)$.

Conclusion: Both $\mathscr{F}$-orbital completeness and orbital continuity are preserved under the logarithmic and exponential transformations linking $(\mathbf{W}, \rho)$ and $(\mathbf{W}, \rho_m)$. □

Using the approach of Došenović et al. [11], and to establish the equivalence between rational-type contractions in standard metric spaces and their multiplicative counterparts, we formulate the following multiplicative analogue of the rational contraction.

Lemma 1.28 (Multiplicative Rational Contraction)**.** *Let $\mathscr{F}$ be a continuous self-mapping on a complete multiplicative metric space $(\mathbf{W}, \rho_m)$. Suppose that*

$$\rho_m(\mathscr{F}\omega, \mathscr{F}\varpi) \leq (\rho_m(\omega, \varpi))^{\alpha} \exp\left(\beta \frac{\ln\left(\rho_m(\omega, \mathscr{F}\omega)\right) \ln\left(\rho_m(\varpi, \mathscr{F}\varpi)\right)}{\ln\left(\rho_m(\omega, \varpi)\right)}\right), \tag{1.11}$$

for all $\omega, \varpi \in \mathbf{W}$ with $\rho_m(\omega, \varpi) > 1$, where $\alpha, \beta \in [0, 1)$ satisfy $\alpha + \beta < 1$.

Then $\mathscr{F}$ admits a unique fixed point in $\mathbf{W}$.

Proof. Let $\omega_0 \in \mathbf{W}$ be arbitrary, and define the sequence $\{\omega_n\}$ in $\mathbf{W}$ by $\omega_n = \mathscr{F}^n\omega_0, \quad n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $\omega_{n_0} = \omega_{n_0+1}$, then ω_{n_0} is a fixed point of $\mathscr{F}$. Otherwise, assume that $\omega_n \neq \omega_{n+1}$ for all $n \geq 0$, that is, $\rho_m(\omega_n, \omega_{n+1}) > 1$.

Substituting $\omega = \omega_{n-1}$ and $\varpi = \omega_n$ into (1.11), we obtain

$$\begin{aligned}\rho_m(\omega_{n+1}, \omega_n) &= \rho_m(\mathscr{F}\omega_n, \mathscr{F}\omega_{n-1}) \\ &\leq (\rho_m(\omega_n, \omega_{n-1}))^{\alpha} \exp\left(\beta \frac{\ln \rho_m(\omega_n, \omega_{n+1}) \ln \rho_m(\omega_{n-1}, \omega_n)}{\ln \rho_m(\omega_{n-1}, \omega_n)}\right).\end{aligned}$$

Then, we can conclude

$$\rho_m(\omega_{n+1}, \omega_n) \leq (\rho_m(\omega_n, \omega_{n-1}))^{\alpha} \exp\left(\beta \ln \rho_m(\omega_n, \omega_{n+1})\right). \tag{1.12}$$

Taking natural logarithm on both sides of (1.12), we get

$$s_n \leq \alpha t_n + \beta s_n. \tag{1.13}$$

Where, $s_n = \ln \rho_m(\omega_n, \omega_{n+1})$ and $t_n = \ln \rho_m(\omega_n, \omega_{n-1})$.

Rearranging (1.13) yields

$$s_n \leq \frac{\alpha}{1-\beta} t_n.$$

Now, taking $\frac{\alpha}{1-\beta} = \lambda \in [0,1)$, we get

$$\ln \rho_m(\omega_n, \omega_{n+1}) \leq \lambda \ln \rho_m(\omega_n, \omega_{n-1}).$$

That is,

$$\rho_m(\omega_n, \omega_{n+1}) \leq (\rho_m(\omega_n, \omega_{n-1}))^{\lambda}$$

By induction, we obtain

$$\rho_m(\omega_{n+1}, \omega_n) \leq (\rho_m(\omega_1, \omega_0))^{\lambda^n}.$$

For $m > n$, using the multiplicative triangle inequality,

$$\begin{aligned}\rho_m(\omega_n, \omega_m) \leq \prod_{k=n}^{m-1} \rho_m(\omega_k, \omega_{k+1}) \leq \prod_{k=n}^{m-1} \rho_m(\omega_1, \omega_0)^{\lambda^k} \\ = \rho_m(\omega_1, \omega_0)^{\sum_{k=n}^{m-1} \lambda^k}.\end{aligned}$$

Since $0 \leq \lambda < 1$,

$$\sum_{k=n}^{m-1} \lambda^k \leq \sum_{k=n}^{\infty} \lambda^k = \frac{\lambda^n}{1-\lambda} \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\rho_m(\omega_n, \omega_m) \to 1 \quad \text{as } n \to \infty,$$

so $\{\omega_n\}$ is multiplicative Cauchy. Since $(\mathbf{W}, \rho_m)$ is complete, there exists $\omega^* \in \mathbf{W}$ such that

$$\rho_m(\omega_n, \omega^*) \to 1 \quad \text{as } n \to \infty.$$

Again, by continuity of $\mathscr{F}$,

$$\lim_{n\to\infty} \rho_m(\mathscr{F}\omega_n, \mathscr{F}\omega^*) = 1;$$

$$\implies \rho_m(\mathscr{F}\omega^*, \omega^*) = 1.$$

Thus, $\mathscr{F}\omega^* = \omega^*$. Suppose ϖ^* is another fixed point. Then

$$\rho_m(\omega^*, \varpi^*) = \rho_m(\mathscr{F}\omega^*, \mathscr{F}\varpi^*)$$

$$\le (\rho_m(\omega^*,\varpi^*))^{\alpha} \exp\Big(\beta \frac{\ln \rho_m(\omega^*,\mathscr{F}\omega^*)\ln \rho_m(\varpi^*,\mathscr{F}\varpi^*)}{\ln \rho_m(\omega^*,\varpi^*)}\Big)$$
$$= (\rho_m(\omega^*,\varpi^*))^{\alpha}.$$

Since $\alpha \in [0,1)$ and $\rho_m(\omega^*,\varpi^*) \geq 1$, this implies $\rho_m(\omega^*,\varpi^*) = 1$, i.e., $\omega^* = \varpi^*$. Hence the fixed point is unique. □

Then, Theorem 1.5 (Rational contraction in a standard metric space) and Lemma 1.28 (Rational contraction in a multiplicative metric space) are equivalent.

Lemma 1.29 ([26]). *Let $(\mathbf{W},\rho_m)$ be a multiplicative metric space. Then the multiplicative metric $\rho_m : \mathbf{W}\times\mathbf{W} \to [1,\infty)$ is continuous.*

In this paper, we establish fixed point theorems for rational, and weak-type contractions within multiplicative metric spaces, employing a function $\varphi : [1,+\infty) \to [0,\infty)$ satisfying $\varphi(s) \geq b(\ln s)^c$, $s > 1$, $b,c > 0$, which prevents the equivalence of these results with the corresponding theorems in standard metric spaces.

2. Main Results

Theorem 2.1. *Let $(\mathbf{W},\rho_m)$ be an $\mathscr{F}$-orbitally complete multiplicative metric space, and let $\mathscr{F} : \mathbf{W} \to \mathbf{W}$ be a mapping satisfying the following conditions:*

(i) There exists a function $\varphi \in \Phi$ such that

$$\varphi(\rho_m(\omega,\varpi)) \leq \lambda_1 \varphi(\rho_m(\omega,\varpi)) + \lambda_2 \frac{\varphi(\rho_m(\omega,\mathscr{F}\omega))\,\varphi(\rho_m(\varpi,\mathscr{F}\varpi))}{\varphi(\rho_m(\omega,\varpi))}, \tag{2.1}$$

for all $\omega,\varpi \in \mathbf{W}$ with $\rho_m(\omega,\varpi) > 1$, where $0 \leq \lambda_1+\lambda_2 < 1$ and $\lambda_1,\lambda_2 \geq 0$.

(ii) The mapping $\mathscr{F}$ is orbitally continuous on $\mathbf{W}$.

Under these conditions, $\mathscr{F}$ has at least one fixed point in $\mathbf{W}$. Furthermore, if $\varphi \in \Phi_0$, then this fixed point is unique.

Proof. Consider an arbitrary point $\omega_0 \in \mathbf{W}$, and define the iterative sequence $\{\omega_n\}$ in $\mathbf{W}$ by $\omega_n = \mathscr{F}^n\omega_0, \quad n \geq 0$. If there exists some $n_0 \in \mathbb{N}$ such that $\omega_{n_0} = \omega_{n_0+1}$, then ω_{n_0} is a fixed point of $\mathscr{F}$. Assume, instead, that $\omega_n \neq \omega_{n+1}$ for all $n \geq 0$, i.e., $\rho_m(\omega_n,\omega_{n+1}) > 1$.

Substituting $\omega=\omega_0$ and $\varpi=\omega_1$ into (2.1), we obtain

$$\begin{aligned}\varphi\big(\rho_m(\mathscr{F}\omega_0,\mathscr{F}\omega_1)\big) &\le \lambda_1\varphi\big(\rho_m(\omega_0,\omega_1)\big)+\lambda_2\frac{\varphi\big(\rho_m(\omega_1,\mathscr{F}\omega_1)\big)\,\varphi\big(\rho_m(\omega_0,\mathscr{F}\omega_0)\big)}{\varphi\big(\rho_m(\omega_0,\omega_1)\big)}\\ &=\lambda_1\varphi\big(\rho_m(\omega_0,\omega_1)\big)+\lambda_2\frac{\varphi\big(\rho_m(\omega_1,\omega_2)\big)\,\varphi\big(\rho_m(\omega_0,\omega_1)\big)}{\varphi\big(\rho_m(\omega_0,\omega_1)\big)}.\end{aligned}$$

Hence, we deduce

$$\varphi\big(\rho_m(\omega_1,\omega_2)\big)\le k\,\varphi\big(\rho_m(\omega_0,\omega_1)\big),\quad \text{where } k=\frac{\lambda_1}{1-\lambda_2}<1. \tag{2.2}$$

Next, taking $\omega=\omega_1$ and $\varpi=\omega_2$ in (2.2), we get

$$\varphi\big(\rho_m(\omega_2,\omega_3)\big)\le k\,\varphi\big(\rho_m(\omega_1,\omega_2)\big)\le k^2\,\varphi\big(\rho_m(\omega_0,\omega_1)\big).$$

Proceeding inductively, we obtain

$$\varphi\big(\rho_m(\omega_n,\omega_{n+1})\big)\le k^n\,\varphi\big(\rho_m(\omega_0,\omega_1)\big),\quad n\ge 0.$$

As $\varphi(s)\ge b(\ln s)^c$ for $s>1$, we have

$$\begin{aligned}&b\left(\ln\rho_m(\omega_n,\omega_{n+1})\right)^c\le k^n\,\varphi\big(\rho_m(\omega_0,\omega_1)\big)\\ &\implies \ln\rho_m(\omega_n,\omega_{n+1})\le k^{n/c}\left(\frac{\varphi\big(\rho_m(\omega_0,\omega_1)\big)}{b}\right)^{1/c}=\lambda^n\left(\frac{\varphi\big(\rho_m(\omega_0,\omega_1)\big)}{b}\right)^{1/c},\end{aligned}$$

where $\lambda=k^{1/c}\in(0,1)$.

For all $n\ge 0$ and $p\ge 1$,

$$\begin{aligned}\rho_m(\omega_n,\omega_{n+p})&\le\rho_m(\omega_n,\omega_{n+1})...\rho_m(\omega_{n+p-1},\omega_{n+p})\\ \implies ln(\rho_m(\omega_n,\omega_{n+p}))\ &\le ln(\rho_m(\omega_n,\omega_{n+1}))+...+ln(\rho_m(\omega_{n+p-1},\omega_{n+p}))\\ &\le[\lambda^n+...+\lambda^{n+p-1}]\Big\{\frac{\varphi(\rho_m(\omega_0,\omega_1))}{b}\Big\}^{\frac{1}{c}}\\ &\le\frac{\lambda^n(1-\lambda^p)}{1-\lambda}\Big\{\frac{\varphi(\rho_m(\omega_0,\omega_1))}{b}\Big\}^{\frac{1}{c}}\\ &\le\frac{\lambda^n}{1-\lambda}\Big\{\frac{\varphi(\rho_m(\omega_0,\omega_1))}{b}\Big\}^{\frac{1}{c}}\\ \implies \rho_m(\omega_n,\omega_{n+p})&\le e^{\frac{\lambda^n}{1-\lambda}\left\{\frac{\varphi(\rho_m(\omega_0,\omega_1))}{b}\right\}^{\frac{1}{c}}}\to 1\ when\ n\to\infty.\end{aligned}$$

Consequently, the sequence $\{\omega_n\}$ forms a multiplicative Cauchy sequence in $\mathbf{W}$. Since $\mathbf{W}$ is $\mathscr{F}$-orbitally complete, there exists $\omega^* \in \mathbf{W}$ such that $\lim_{n\to\infty} \rho_m(\omega_n, \omega^*) = 1$.

Again, as $\mathscr{F}$ is orbitally continuous, we obtain

$$\lim_{n\to\infty} \rho_m(\mathscr{F}\omega_n, \mathscr{F}\omega^*) = 1.$$

Then, by the continuity of the multiplicative metric, Lemma 1.29, we have

$$\rho_m\Big(\lim_{n\to\infty} \omega_{n+1}, \mathscr{F}\omega^*\Big) = 1,$$

which implies

$$\lim_{n\to\infty} \omega_{n+1} = \mathscr{F}\omega^*.$$

Since $\lim_{n\to\infty} \omega_n = \omega^*$, it follows that

$$\mathscr{F}\omega^* = \omega^*.$$

Hence, $\mathscr{F}$ has a fixed point in $\mathbf{W}$.

Now, suppose that $\varphi \in \Phi_0$ and assume that there exists $\varpi^* \in \mathbf{W}$ such that $\mathscr{F}\varpi^* = \varpi^*$. Then, using (2.1), we get

$$\varphi\big(\rho_m(\mathscr{F}\omega^*, \mathscr{F}\varpi^*)\big) \le \lambda_1 \varphi\big(\rho_m(\omega^*, \varpi^*)\big) + \lambda_2 \frac{\varphi\big(\rho_m(\omega^*, \mathscr{F}\omega^*)\big)\, \varphi\big(\rho_m(\varpi^*, \mathscr{F}\varpi^*)\big)}{\varphi\big(\rho_m(\omega^*, \varpi^*)\big)}.$$

Since $\mathscr{F}\omega^* = \omega^*$ and $\mathscr{F}\varpi^* = \varpi^*$, it follows that

$$\varphi\big(\rho_m(\omega^*, \varpi^*)\big) \le \lambda_1 \varphi\big(\rho_m(\omega^*, \varpi^*)\big).$$

Hence,

$$(1-\lambda_1)\, \varphi\big(\rho_m(\omega^*, \varpi^*)\big) \le 0.$$

Since $0 \le \lambda_1 < 1$ and $\varphi \in \Phi_0$, we conclude that

$$\varphi\big(\rho_m(\omega^*, \varpi^*)\big) = 0 \quad \Longrightarrow \quad \rho_m(\omega^*, \varpi^*) = 1,$$

which implies $\omega^* = \varpi^*$.

Therefore, $\mathscr{F}$ admits a unique fixed point in $\mathbf{W}$. $\square$

We show that Theorem 2.1 is not equivalent to Lemma 1.28. Define the function $\rho_{\varphi(\rho_m)}$: $\mathbf{W}\times\mathbf{W}\to[0,+\infty)$ by

$$\rho_{\varphi(\rho_m)}(\omega,\varpi)=\begin{cases}0, & \text{if } \omega=\varpi,\\ \ln\left[\varphi(\rho_m(\omega,\varpi))\right], & \text{if } \omega\neq\varpi.\end{cases}$$

However, this function does not necessarily define a metric on $\mathbf{W}$. Therefore, in general, the contractive condition (2.1) cannot be transformed into (1.4) in a metric space. Consequently, the approach used in [11] is not applicable in this case. The following example illustrates this fact.

Example 2.2. Let $\mathbf{W}=\{a_1,a_2,a_3\}$. Define a mapping $\mathscr{F}:\mathbf{W}\to\mathbf{W}$ by

$$\mathscr{F}(a_1)=a_1,\quad \mathscr{F}(a_2)=a_3,\quad \mathscr{F}(a_3)=a_1.$$

Consider the multiplicative metric $\rho_m:\mathbf{W}\times\mathbf{W}\to[1,+\infty)$ defined as

$$\rho_m(a_1,a_1)=\rho_m(a_2,a_2)=\rho_m(a_3,a_3)=1,$$

$$\rho_m(a_1,a_2)=\rho_m(a_2,a_1)=4,$$

$$\rho_m(a_1,a_3)=\rho_m(a_3,a_1)=5,$$

$$\rho_m(a_2,a_3)=\rho_m(a_3,a_2)=7.$$

Notice that

$$\rho_m(a_1,a_2)\,\rho_m(a_2,a_3)=4\cdot7=28>5=\rho_m(a_1,a_3),$$

$$\rho_m(a_1,a_3)\,\rho_m(a_3,a_2)=5\cdot7=35>4=\rho_m(a_1,a_2),$$

$$\rho_m(a_2,a_1)\,\rho_m(a_1,a_3)=4\cdot5=20>7=\rho_m(a_2,a_3),$$

which confirms that ρ_m is indeed a multiplicative metric on $\mathbf{W}$.

Define $\varphi:[1,+\infty)\to[0,+\infty)$ by

$$\varphi(s)=\begin{cases}0, & s=1,\\ \frac{5}{4}s, & 1<s\leq 4,\\ \frac{7}{4}s, & 4<s\leq 5,\\ \frac{1}{7}s, & s>5.\end{cases}$$

Then, for all $s > 1$, we have $\varphi(s) \geq \frac{1}{7}s$.

Next, define

$$\rho_{\varphi(\rho_m)} : \mathbf{W} \times \mathbf{W} \to [0, \infty)$$

by

$$\rho_{\varphi(\rho_m)}(\omega, \varpi) = \begin{cases} 0, & \text{if } \omega = \varpi, \\ \ln\big(\varphi(\rho_m(\omega, \varpi))\big), & \text{if } \omega \neq \varpi. \end{cases}$$

Then,

$$\rho_{\varphi(\rho_m)}(a_1, a_2) = \ln\big(\varphi(\rho_m(a_1, a_2))\big) = \ln 5,$$

$$\rho_{\varphi(\rho_m)}(a_1, a_3) = \ln\big(\varphi(\rho_m(a_1, a_3))\big) = \ln \frac{35}{4},$$

$$\rho_{\varphi(\rho_m)}(a_2, a_3) = \ln\big(\varphi(\rho_m(a_2, a_3))\big) = 0.$$

Hence,

$$\rho_{\varphi(\rho_m)}(a_1, a_3) > \rho_{\varphi(\rho_m)}(a_1, a_2) + \rho_{\varphi(\rho_m)}(a_2, a_3),$$

which shows that $\rho_{\varphi(\rho_m)}$ does not satisfy the triangle inequality. Therefore, $\rho_{\varphi(\rho_m)}$ is not a metric.

Example 2.3. Now, let us take ρ_m same as in example 2.2.

Consider the function $\varphi : [1, +\infty) \to [0, +\infty)$ given by:

$$\varphi(s) = \begin{cases} 0, s = 1 \\ 4s, 1 < s \leq 4 \\ s, 4 < s \leq 5 \\ 4s, s > 5. \end{cases}$$

For all $s > 1$, we observe that $\varphi(s) \geq lns$, where $b = c = 1$.

Now, using condition(i) and by taking $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{4}$,we compute,

Case 1:(i,j)=(1,2)

$$\varphi(\rho_m(\mathscr{F}a_1, \mathscr{F}a_2)) = \varphi(\rho_m(a_1, a_3)) = 5$$

$$Also, \frac{1}{2}\varphi(\rho_m(a_1, a_2)) + \frac{1}{4}\frac{\varphi(\rho_m(a_1, \mathscr{F}a_1))\varphi(\rho_m(a_2, \mathscr{F}a_2))}{\varphi(\rho_m(a_1, a_2))} = 8$$

Case 2:(i,j)=(1,3)

$$\varphi(\rho_m(\mathscr{F}a_1,\mathscr{F}a_3)) = \varphi(\rho_m(a_1,a_1)) = 0$$
$$\frac{1}{2}\varphi(\rho_m(a_1,a_3)) + \frac{1}{4}\frac{\varphi(\rho_m(a_1,\mathscr{F}a_1))\varphi(\rho_m(a_3,\mathscr{F}a_3))}{\varphi(\rho_m(a_1,a_3))} = \frac{5}{2}$$

Case 3:(i,j)=(2,3)

$$\varphi(\rho_m(\mathscr{F}a_2,\mathscr{F}a_3)) = \varphi(\rho_m(a_3,a_1)) = 5$$
$$Also, \frac{1}{2}\varphi(\rho_m(a_2,a_3)) + \frac{1}{4}\frac{\varphi(\rho_m(a_2,\mathscr{F}a_2))\varphi(\rho_m(a_3,\mathscr{F}a_3))}{\varphi(\rho_m(a_2,a_3))} = 14$$

Hence, the mapping $\mathscr{F}$ satisfies condition (2.1) for all $a_i, a_j \in \mathbf{W}$. Moreover, for any $a_i \in \mathbf{W}$, we have $\lim_{n\to\infty} \mathscr{F}^n a_i = a_1$, which implies that every Cauchy sequence in $\mathbf{W}$ converges to a_1. Therefore, $\mathbf{W}$ is $\mathscr{F}$-orbitally complete.

Additionally, since $\lim_{n\to\infty} \rho_m(\mathscr{F}^n a_i, a_1) = 1$ for all i, it follows that all the hypotheses of Theorem 2.1 are satisfied. Consequently, a_1 is the unique fixed point of $\mathscr{F}$ in $\mathbf{W}$.

Corollary 2.4. Let $(\mathbf{W}, \rho_m)$ be a complete multiplicative metric space, and let $\mathscr{F} : \mathbf{W} \to \mathbf{W}$ be a mapping satisfying conditions (i) and (ii) of Theorem (2.1). Then, $\mathscr{F}$ admits at least one fixed point in $\mathbf{W}$. Furthermore, if $\varphi \in \Phi_0$, the fixed point is unique.

Proof. The result follows directly from the fact that the completeness of $\mathbf{W}$ implies its $\mathscr{F}$-orbital completeness. □

Theorem 2.5. *Let $(\mathbf{W}, \rho_m)$ be an $\mathscr{F}$-orbitally complete multiplicative metric space, and let $\mathscr{F} : \mathbf{W} \to \mathbf{W}$ be a mapping satisfying the following conditions:*

(i) There exist functions $\varphi \in \Phi$ and $\psi \in \Psi_m$ such that

$$\varphi\big(\rho_m(\mathscr{F}\omega, \mathscr{F}\varpi)\big) \leq \varphi\big(\rho_m(\omega,\varpi)\big) - \varphi\big(\psi(\rho_m(\omega,\varpi))\big), \tag{2.3}$$

for all $\omega, \varpi \in \mathbf{W}$ with $\rho_m(\omega,\varpi) > 1$.

(ii) The mapping $\mathscr{F}$ is orbitally continuous on $\mathbf{W}$.

Then $\mathscr{F}$ admits a fixed point in $\mathbf{W}$. Further if $\varphi \in \Phi_0$ then the fixed point is unique.

Proof. Let $\omega_0 \in \mathbf{W}$ be arbitrary, and define the sequence $\{\omega_n\}$ in $\mathbf{W}$ by $\omega_n = \mathscr{F}^n \omega_0, \quad n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $\omega_{n_0} = \omega_{n_0+1}$, then ω_{n_0} is a fixed point of $\mathscr{F}$. Otherwise, assume that $\omega_n \neq \omega_{n+1}$ for all $n \geq 0$, that is, $\rho_m(\omega_n, \omega_{n+1}) > 1$.

Applying the contraction (2.3) with $\omega = \omega_n$ and $\varpi = \omega_{n+1}$, we have

$$\begin{aligned} \varphi(\rho_m(\omega_{n+1}, \omega_{n+2})) =& \varphi(\rho_m(\mathscr{F}\omega_n, \mathscr{F}\omega_{n+1})) \\ \leq& \varphi(\rho_m(\omega_n, \omega_{n+1})) - \varphi(\psi(\rho_m(\omega_n, \omega_{n+1}))). \end{aligned} \tag{2.4}$$

Since $\varphi(\psi(\rho_m(\omega_n, \omega_{n+1}))) > 0$, it follows that

$$\varphi(\rho_m(\omega_{n+1}, \omega_{n+2})) < \varphi(\rho_m(\omega_n, \omega_{n+1})).$$

Hence, the sequence $\{\varphi(\rho_m(\omega_n, \omega_{n+1}))\}$ is strictly decreasing and bounded below by 0. Therefore, it converges:

$$\lim_{n\to\infty} \varphi(\rho_m(\omega_n, \omega_{n+1})) = \ell \geq 0.$$

Taking limits in the (2.4) gives

$$\ell = \lim_{n\to\infty} \varphi(\rho_m(\omega_{n+1}, \omega_{n+2})) \leq \ell - \lim_{n\to\infty} \varphi(\psi(\rho_m(\omega_n, \omega_{n+1}))).$$

Thus,

$$\lim_{n\to\infty} \varphi(\psi(\rho_m(\omega_n, \omega_{n+1}))) = 0.$$

Using the property $\varphi(s) \geq b(\ln s)^c$ for $s > 1$, we obtain

$$b\big(\ln \psi(\rho_m(\omega_n, \omega_{n+1}))\big)^c \leq \varphi(\psi(\big(\rho_m(\omega_n, \omega_{n+1})\big))) \to 0,$$

which implies

$$\ln \psi(\rho_m(\omega_n, \omega_{n+1})) \to 0 \implies \psi(\rho_m(\omega_n, \omega_{n+1})) \to 1.$$

And hence

$$\rho_m(\omega_n, \omega_{n+1}) \to 1.$$

Hence, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and $\delta > 0$ such that for all $k \geq N$,

$$0 < \ln \rho_m(\omega_k, \omega_{k+1}) < \delta,$$

where $\delta > 0$ can be chosen small enough depending on $m - n$.

Then, for all $m > n \geq N$, we have

$$\rho_m(\omega_n, \omega_m) \leq \prod_{k=n}^{m-1} \rho_m(\omega_k, \omega_{k+1})$$

Then,

$$\ln \rho_m(\omega_n, \omega_m) \leq \sum_{k=n}^{m-1} \ln \rho_m(\omega_k, \omega_{k+1}) < (m-n)\delta.$$

By choosing δ sufficiently small so that $(m-n)\delta < \varepsilon$, it follows that

$$\ln \rho_m(\omega_n, \omega_m) \leq \sum_{k=n}^{m-1} \ln \rho_m(\omega_k, \omega_{k+1}) < \varepsilon.$$

Exponentiating both sides yields

$$\rho_m(\omega_n, \omega_m) < e^{\varepsilon}.$$

Hence,

$$\rho_m(\omega_n, \omega_m) \to 1.$$

Therefore, $\{\omega_n\}$ is a multiplicative Cauchy sequence in $\mathbf{W}$. Since $(\mathbf{W}, \rho_m)$ is $\mathscr{F}$-orbitally complete, there exists $\omega^* \in \mathbf{W}$ such that

$$\lim_{n\to\infty} \rho_m(\omega_n, \omega^*) = 1.$$

By the orbital continuity of $\mathscr{F}$, it follows that

$$\lim_{n\to\infty} \rho_m(\mathscr{F}\omega_n, \mathscr{F}\omega^*) = 1.$$

That is,

$$\lim_{n\to\infty} \omega_{n+1} = \lim_{n\to\infty} \mathscr{F}\omega_n = \mathscr{F}\omega^*.$$

$$\implies \mathscr{F}\omega^* = \omega^*.$$

Thus, ω^* is a fixed point of $\mathscr{F}$.

Next, to prove uniqueness, assume that there exists another point $\varpi^* \in \mathbf{W}$ such that $\mathscr{F}\varpi^* = \varpi^*$ and $\varphi \in \Phi_0$. Then, from the contractive condition, we obtain

$$\varphi\big(\rho_m(\mathscr{F}\omega^*, \mathscr{F}\varpi^*)\big) \leq \varphi\big(\rho_m(\omega^*, \varpi^*)\big) - \psi\big(\varphi(\rho_m(\omega^*, \varpi^*))\big).$$

Since $\mathscr{F}\omega^* = \omega^*$ and $\mathscr{F}\varpi^* = \varpi^*$, this reduces to

$$\varphi\big(\rho_m(\omega^*, \varpi^*)\big) \leq \varphi\big(\rho_m(\omega^*, \varpi^*)\big) - \varphi\big(\psi(\rho_m(\omega^*, \varpi^*))\big),$$

As $\psi \in \Psi_m$ and $\varphi \in \Phi_0$, this yields

$$\varphi(\rho_m(\omega^*, \varpi^*)) = 0 \quad \Longrightarrow \quad \rho_m(\omega^*, \varpi^*) = 1,$$

and hence $\omega^* = \varpi^*$.

Therefore, $\mathscr{F}$ admits a unique fixed point in $\mathbf{W}$. $\square$

Example 2.6. Let $\mathbf{W} = \{a_1, a_2, a_3\}$. Define a mapping $\mathscr{F} : \mathbf{W} \to \mathbf{W}$ as

$$\mathscr{F}a_1 = a_1, \quad \mathscr{F}a_2 = a_1, \quad \mathscr{F}a_3 = a_2.$$

Consider the function $\rho_m : \mathbf{W} \times \mathbf{W} \to [1, +\infty)$ defined by

$$\rho_m(a_1, a_1) = \rho_m(a_2, a_2) = \rho_m(a_3, a_3) = 1,$$

$$\rho_m(a_1, a_2) = \rho_m(a_2, a_1) = 2,$$

$$\rho_m(a_1, a_3) = \rho_m(a_3, a_1) = 3,$$

$$\rho_m(a_2, a_3) = \rho_m(a_3, a_2) = 4.$$

It can be easily verified that ρ_m satisfies the multiplicative triangle inequality:

$$\rho_m(a_1, a_2)\rho_m(a_2, a_3) = 8 > 3 = \rho_m(a_1, a_3),$$

$$\rho_m(a_1, a_3)\rho_m(a_3, a_2) = 12 > 2 = \rho_m(a_1, a_2),$$

$$\rho_m(a_2, a_1)\rho_m(a_1, a_3) = 6 > 4 = \rho_m(a_2, a_3),$$

confirming that ρ_m is a multiplicative metric on $\mathbf{W}$.

Now, define the functions $\varphi : [1, +\infty) \to [0, +\infty)$ and $\psi : [0, \infty) \to [1, \infty)$ as

$$\varphi(s) = \begin{cases} 0, & s = 1, \\ s, & 1 < s \leq 2, \\ 2s, & 2 < s \leq 3, \\ s^2, & s > 3, \end{cases}$$

$$\psi(x) = \frac{x+1}{2}.$$

Observe that $\varphi(s) \geq \ln s$ for all $s > 1$, with $b = c = 1$.

We verify condition (i) of Theorem 2.5:

Case 1: $(i,j) = (1,2)$

$$\varphi(\rho_m(\mathscr{F}a_1,\mathscr{F}a_2)) = 0 < \varphi(\rho_m(a_1,a_2)) - \varphi(\psi(\rho_m(a_1,a_2))) = \frac{1}{2}.$$

Case 2: $(i,j) = (1,3)$

$$\varphi(\rho_m(\mathscr{F}a_1,\mathscr{F}a_3)) = 2 < \varphi(\rho_m(a_1,a_3)) - \varphi(\psi(\rho_m(a_1,a_3))) = 4.$$

Case 3: $(i,j) = (2,3)$

$$\varphi(\rho_m(\mathscr{F}a_2,\mathscr{F}a_3)) = 2 < \varphi(\rho_m(a_2,a_3)) - \varphi(\psi(\rho_m(a_2,a_3))) = 11.$$

Hence, $\mathscr{F}$ satisfies condition (2.3) for all $a_i, a_j \in \mathbf{W}$.

Furthermore, for any $a_i \in \mathbf{W}$, we have $\lim_{n\to\infty} \mathscr{F}^n a_i = a_1$, showing that every Cauchy sequence in $\mathbf{W}$ converges to a_1. Therefore, $\mathbf{W}$ is $\mathscr{F}$-orbitally complete.

Finally, since

$$\lim_{k\to\infty} \rho_m(\mathscr{F}^{n_k}a_i, a_1) = 1 \implies \lim_{k\to\infty} \rho_m(\mathscr{F}^{n_k+1}a_i, \mathscr{F}a_1) = 1,$$

all the criteria of Theorem 2.5 are satisfied. Consequently, a_1 is the unique fixed point of $\mathscr{F}$.

Corollary 2.7. Let $(\mathbf{W}, \rho_m)$ be complete metric space and let $F : \mathbf{W} \to \mathbf{W}$ be a mapping which satisfy conditions (i) and (ii) of Theorem 2.5 then $\mathscr{F}$ has a fixed point. Further if $\varphi \in \Phi_0$ then the fixed point is unique.

Proof. The proof is similar to the proof of corollary 2.4. □

3. Cyclic Mappings

Introduced by Kirk et al. in 2003 [15], cyclic contractions guarantee the existence and uniqueness of a fixed point in the intersection of the underlying cyclic sets within a complete metric space.

In this section, making use of Theorem 2.1, we establish the existence and uniqueness of fixed points for the following class of cyclic mappings.

Let $\{S_i\}_{i=1}^p$ be a collection of nonempty closed subsets of a complete metric space $(\mathbf{W},\rho)$. Consider a mapping

$$\mathscr{F}:\bigcup_{i=1}^{p} S_i \to \bigcup_{i=1}^{p} S_i$$

such that $\mathscr{F}(S_i)\subseteq S_{i+1},$ for each $i=1,2,\ldots,p$, where $S_{p+1}=S_1$.

Now, from (1.27) if $(\mathbf{W},\rho)$ is complete we can conclude that $(\mathbf{W},\rho_m)$ is complete.

Theorem 3.1. *Let $\{S_i\}_{i=1}^p$ be a family of non-empty and closed subsets of a complete metric space $(\mathbf{W},\rho_m)$. Assume the following conditions are true:*

(i) For all $\omega\in S_i$ and $\varpi\in S_{i+1}$, where $1\le i\le p$, with $\rho_m(\omega,\varpi)>1$; there exist functions $\varphi\in\Phi$ and $\lambda_1,\lambda_2\in[0,1)$ such that $\lambda_1+\lambda_2<1$, we have

$$\varphi(\rho_m(\omega,\varpi))\le\lambda_1\varphi(\rho_m(\omega,\varpi))+\lambda_2\frac{\varphi(\rho_m(\omega,\mathscr{F}\omega))\,\varphi(\rho_m(\varpi,\mathscr{F}\varpi))}{\varphi(\rho_m(\omega,\varpi))},\tag{3.1}$$

then $\bigcap_{i=1}^p S_i$ is non-empty. Furthermore, if

(ii) the mapping $\mathscr{F}$ is orbitally continuous on $\mathbf{W}$,

then $\mathscr{F}$ admits a fixed point $\omega^*\in\bigcap_{i=1}^p S_i$. Furthermore, if $\varphi\in\Phi_0$, the fixed point is unique.

Proof. Let $\omega_0\in\mathbf{W}$ be arbitrary, and define the sequence $\{\omega_n\}$ in $\mathbf{W}$ by $\omega_n=\mathscr{F}^n\omega_0,\quad n\ge 0$. If there exists $n_0\in\mathbb{N}$ such that $\omega_{n_0}=\omega_{n_0+1}$, then ω_{n_0} is a fixed point of $\mathscr{F}$. Otherwise, assume that $\omega_n\ne\omega_{n+1}$ for all $n\ge 0$, that is, $\rho_m(\omega_n,\omega_{n+1})>1$.

Substituting $\omega=\omega_0$ and $\varpi=\omega_1$ into 3.1, we obtain

$$\begin{aligned}\varphi\big(\rho_m(\mathscr{F}\omega_0,\mathscr{F}\omega_1)\big)&\le\lambda_1\varphi\big(\rho_m(\omega_0,\omega_1)\big)+\lambda_2\frac{\varphi\big(\rho_m(\omega_1,\mathscr{F}\omega_1)\big)\,\varphi\big(\rho_m(\omega_0,\mathscr{F}\omega_0)\big)}{\varphi\big(\rho_m(\omega_0,\omega_1)\big)}\\&=\lambda_1\varphi\big(\rho_m(\omega_0,\omega_1)\big)+\lambda_2\frac{\varphi\big(\rho_m(\omega_1,\omega_2)\big)\,\varphi\big(\rho_m(\omega_0,\omega_1)\big)}{\varphi\big(\rho_m(\omega_0,\omega_1)\big)}.\end{aligned}$$

Then, we deduce

$$\varphi\big(\rho_m(\omega_1,\omega_2)\big)\le k\,\varphi\big(\rho_m(\omega_0,\omega_1)\big),\quad\text{where } k=\frac{\lambda_1}{1-\lambda_2}<1.\tag{3.2}$$

Iterating in a similar fashion, we obtain

$$\varphi\big(\rho_m(\omega_n,\omega_{n+1})\big)\le k^n\,\varphi\big(\rho_m(\omega_0,\omega_1)\big),\quad n\ge 0.$$

Proceeding as in Theorem (2.1), for all $n \geq 0$ and $p \geq 1$, we have

$$\rho_m(\omega_n, \omega_{n+p}) \leq e^{\frac{\lambda^n}{1-\lambda}\left\{\frac{\varphi(\rho_m(\omega_0,\omega_1))}{b}\right\}^{\frac{1}{c}}} \to 1 \ when\ n \to \infty.$$

Therefore, $\{\omega_n\}$ is a multiplicative Cauchy sequence in $\mathbf{W}$. On the other hand, since $\bigcup_{i=1}^{p} S_i$ is a closed subset of the complete metric space $(\mathbf{W}, \rho_m)$. It follows that $\left(\bigcup_{i=1}^{p} S_i, \rho_m\right)$ is itself a complete metric space, which means it is an $\mathscr{F}$-orbitally complete. There exists $\omega^* \in \bigcup_{i=1}^{p} S_i$ such that

$$\lim_{n\to\infty} \rho_m(\omega_n, \omega^*) = 1. \tag{3.3}$$

Furthermore, for each $1 \leq i \leq p$, there exists a subsequence $\{\omega_{k_n}\}$ of $\{\omega_n\}$ such that

$$\{\omega_{k_n}\} \subset S_i.$$

From (3.3), it then follows that

$$\lim_{n\to\infty} \rho_m(\omega_{k_n}, \omega^*) = 1, \text{ for all } 1 \leq i \leq p. \tag{3.4}$$

Since each S_i is closed, we deduce that

$$\omega^* \in \bigcap_{i=1}^{p} S_i,$$

which shows, in particular, that

$$\bigcap_{i=1}^{p} S_i \neq \phi$$

Consider the restricted mapping

$$\mathscr{F}_{\bigcap_{i=1}^{p} S_i} : \bigcap_{i=1}^{p} S_i \to \bigcap_{i=1}^{p} S_i.$$

Also, from (3.4) and orbital continuity of $\mathscr{F}$,we have

$$\lim_{k\to\infty} \rho_m(\mathscr{F}\omega_{n_k}, \mathscr{F}\omega^*) = 1.$$

It follows that

$$\lim_{k\to\infty} \omega_{n_{k+1}} = \omega^* = \mathscr{F}\omega^*.$$

It is clear that $\mathscr{F}_{\bigcap_{i=1}^{p} S_i}$ satisfies all the hypotheses of Theorem 2.1. Hence, $\mathscr{F}_{\bigcap_{i=1}^{p} S_i}$ possesses a fixed point

$$\omega^* \in \bigcap_{i=1}^{p} S_i.$$

Now, suppose $\varphi \in \Phi_0$ and let $\varpi^* \in \mathbf{W}$ be such that $\mathscr{F}\varpi^* = \varpi^*$. From (2.1), we have

$$\varphi\big(\rho_m(\omega^*, \varpi^*)\big) = \varphi\big(\rho_m(\mathscr{F}\omega^*, \mathscr{F}\varpi^*)\big) \leq \lambda_1 \varphi\big(\rho_m(\omega^*, \varpi^*)\big).$$

Thus,

$$(1-\lambda_1)\,\varphi\big(\rho_m(\omega^*, \varpi^*)\big) \leq 0.$$

Since $0 \leq \lambda_1 < 1$ and $\varphi \in \Phi_0$, it follows that

$$\varphi\big(\rho_m(\omega^*, \varpi^*)\big) = 0 \quad \Longrightarrow \quad \rho_m(\omega^*, \varpi^*) = 1,$$

which yields $\omega^* = \varpi^*$.

Hence, $\mathscr{F}$ has a unique fixed point in $\bigcap_{i=1}^{p} S_i$. □

4. Boundary Value Problem

Consider the boundary value problem of second order

$$u''(t) = -f\Big(t, u(t)\Big), u(t) \in I, u(0) = u(1) = 0, \tag{4.1}$$

where $I = [0,1]$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In the literature, the existence of solution of such type of the problem(4.1) has been considered by providing certain conditions on f. One can verify that the BVP (4.1) is equivalent to following integral equation

$$u(t) = \int_0^1 \mathscr{G}(t,s) f\Big(s, u(s)\Big) ds, \tag{4.2}$$

for all $t \in I$, where $\mathscr{G}(t,s)$ is the Green's function defined by

$$\mathscr{G}(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1; \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Therefore, $u(t)$ is a solution of BVP (4.1) if and only if it is a solution of the integral equation (4.2).

Consider $\mathscr{W} = \mathscr{C}(I)$ be a space of all real continuous functions on I equipped

$$\rho(u,v) = \sup_{t\in[0,1]} |u(t) - v(t)|.$$

Then, $(\mathscr{W}, \rho)$ is a complete metric space. Also, if $\rho_m(u,v) = e^{\rho(u,v)}$, then $\mathscr{W}$ equipped with ρ_m is a multiplicative metric space.

Theorem 4.1. *Aside from the above given conditions, suppose for all $t \in I, a, b \in \mathbb{R}$ if $\frac{1}{8}|f(t,a) - f(t,b)| \leq g(|a-b|)$ where $g : [0,\infty) \to [0,\infty)$ is an increasing function and $\varphi(e^{g(t)}) \leq \varphi(e^t) - \varphi(\psi(e^t))$, where $t \geq 0$ and $\varphi \in \Phi^{\uparrow}$. Then equation (3.1) has a unique solution $u \in \mathscr{C}^2(I)$.*

Proof. An operator $\mathscr{F} : \mathscr{C}(I) \to \mathscr{C}(I)$ is defined such that

$$\mathscr{F}u(t) = \int_0^1 G(t,s) f(s,u(s)) ds, for\ all\ t \in\ I.$$

Here, $\mathscr{F}$ is a continuous mapping. Hence, it is orbitally continuous. Also, $\mathscr{C}(I)$ is a complete metric space with respect to the metric ρ. Consequently, $\mathscr{C}(I)$ is $\mathscr{F}$-orbitally complete. Let $u, v \in \mathscr{C}(I)$. We have,

$$\begin{aligned}
\rho(\mathscr{F}u, \mathscr{F}v) &= \sup_{t\in[0,1]} |\int_0^1 \mathscr{G}(t,s)[f(s,u(s)) - f(s,v(s))] ds| \\
&\leq \sup_{t\in[0,1]} \int_0^1 G(t,s)|f(s,u(s)) - f(s,v(s))| ds \\
&\leq 8 \sup_{t\in[0,1]} \int_0^1 G(t,s) g\Big(|u(t) - v(t)|\Big) ds \\
&\leq 8 \Big(\sup_{t\in[0,1]} \int_0^1 G(t,s) ds\Big) g\Big(d(u,v)\Big) \\
&\leq g\Big(\rho(u,v)\Big)
\end{aligned}$$

Where, for all $t \in I$,

$$\int_0^1 G(t,s) ds = \frac{t(1-t)}{2}\ .\ \text{Then,}\ \sup_{t\in[0,1]} \int_0^1 G(t,s) ds = \frac{1}{8}.$$

Thus, we can conclude that

$$\rho_m(\mathscr{F}u, \mathscr{F}v) = e^{\rho(\mathscr{F}u, \mathscr{F}v)} \leq e^{g(\rho(u,v))}$$

Again as $\varphi \in \Phi^{\uparrow}$, we have

$$\begin{aligned}\varphi(\rho_m(Fu,Fv)) \leq& \varphi(e^{g(\rho(u,v))}) \\ &\leq \varphi(e^{\rho(u,v)}) - \varphi(\psi(e^{\rho(\varpi,\omega)})) \\ &\leq \varphi(\rho_m(u,v)) - \varphi(\psi(\rho_m(u,v))).\end{aligned}$$

So, the criteria of Theorem (2.5) are satisfied. Therefore, F has a fixed point in $\mathscr{C}(I)$.

Example 4.2. We consider a boundary value problem

$$u'' = t + u,\ u(0) = u(1) = 0 \tag{4.3}$$

which is equivalent to the integral equation

$$u(t) = \int_0^t t(1-s)(u(s)+s)ds + \int_t^1 s(1-t)(u(s)+s)ds$$

Then,for all $t \in I, a, b \in \mathbb{R}$

$$\frac{1}{8}|f(t,a) - f(t,b)| = \frac{1}{8}|b-a| \leq g(|b-a|)$$

Taking,

$$\begin{aligned}\varphi(t) &= ln\, t \\ g(t) &= \frac{t}{8} \\ \psi(t) &= 2 - \frac{2}{t+1} \\ Also, e^{\frac{15t}{8}} &\geq e^t \\ \implies e^{\frac{15t}{8}} + e^t &\geq e^t + e^t \\ \implies e^{\frac{7t}{8}}(e^t+1) &\geq 2 + 2e^t - 2 = 2(1+e^t) - 2 \\ \implies e^{\frac{7t}{8}} &\geq 2 - \frac{2}{(e^t+1)} \\ \implies \frac{7t}{8} &\geq ln(2 - \frac{2}{1+e^t}) \\ \implies t - ln(2 - \frac{2}{1+e^t}) &\geq \frac{t}{8} \\ \implies \varphi(e^{g(t)}) &\leq \varphi(e^t) - \varphi(\psi(e^t)), for\ all\ t \geq 0\end{aligned}$$

$$i.e., \varphi(\rho_m(\mathscr{F}u, \mathscr{F}v)) \leq \varphi(\rho_m(u,v)) - \varphi(\psi(\rho_m(u,v))).$$

Thus, all the criteria of Theorem 4.1 is satisfied. Consequently, the BVP (4.1) has a unique solution which is found to be $u^*(t) = \frac{e^2 t - t + e^{1-t} - e^{1+t}}{1-e^2}$.

□

5. Conclusion

In this paper, we establish fixed point theorems for Jaggi and weak contractions within the setting of multiplicative metric spaces. The results are obtained using a function $\varphi : [1,+\infty) \to [0,\infty)$ satisfying $\varphi(s) \geq b(\ln s)^c$ for $s > 1$, where b and c are positive constants. Notably, the multiplicative metric framework considered here is distinct from the classical metric setting. We provide illustrative examples to support our theoretical results and further demonstrate applications of the weak contraction principle, including the existence of solutions to boundary value problems for second-order differential equations and the determination of a unique fixed point for cyclic mappings.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] M. Abbas, M. De la Sen, T. Nazir, Common Fixed Points of Generalized Rational Type Cocyclic Mappings in Multiplicative Metric Spaces, Discret. Dyn. Nat. Soc. 2015 (2015), 532725. https://doi.org/10.1155/2015/532725.

[2] A.A.N. Abdou, Fixed Point Theorems for Generalized Contraction Mappings in Multiplicative Metric Spaces, J. Nonlinear Sci. Appl. 09 (2016), 2347–2363. https://doi.org/10.22436/jnsa.009.05.36.

[3] R.P. Agarwal, E. Karapınar, B. Samet, An Essential Remark on Fixed Point Results on Multiplicative Metric Spaces, Fixed Point Theory Appl. 2016 (2016), 21. https://doi.org/10.1186/s13663-016-0506-7.

[4] Y.I. Alber, S. Guerre-Delabriere, Principle of Weakly Contractive Maps in Hilbert Spaces, in: New Results in Operator Theory and Its Applications, Birkhäuser, Basel, 1997: pp. 7–22. https://doi.org/10.1007/978-3-0348-8910-0_2.

[5] S. Banach, Sur les Opérations dans les Ensembles Abstraits et Leur Application aux Équations Intégrales, Fundam. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[6] G.V.R. Babu, P.D. Sailaja, A Fixed Point Theorem of Generalized Weakly Contractive Maps in Orbitally Complete Metric Spaces, Thai J. Math. 9 (2011), 1–10.

[7] A.E. Bashirov, E.M. Kurpınar, A. Özyapıcı, Multiplicative Calculus and Its Applications, J. Math. Anal. Appl. 337 (2008), 36–48. https://doi.org/10.1016/j.jmaa.2007.03.081.

[8] S. Chandok, M.S. Khan, M. Abbas, Common Fixed-Point Theorems for Nonlinear Weakly Contractive Mappings, Ukr. Math. J. 66 (2014), 594–601. https://doi.org/10.1007/s11253-014-0956-1.

[9] S.K. Chatterjea, Fixed-Point Theorems, C. R. Acad. Bulg. Sci. 25 (1972), 727–730.

[10] L.B. Ćirić, On Contraction Type Mappings, Math. Balk. 1 (1971), 52–57.

[11] T. Došenović, M. Postolache, S. Radenović, On Multiplicative Metric Spaces: Survey, Fixed Point Theory Appl. 2016 (2016), 92. https://doi.org/10.1186/s13663-016-0584-6.

[12] M. Grossman, R. Katz, Non-Newtonian Calculus, Lee Press, 1972.

[13] D.S. Jaggi, Some Unique Fixed Point Theorems, Indian J. Pure Appl. Math. 8 (1977), 223–230.

[14] R. Kannan, Some Results on Fixed Points, Bull. Calcutta Math. Soc. 60 (1968), 71–76.

[15] W.A. Kirk, P.S. Srinivasan, P. Veeramani, Fixed Points for Mappings Satisfying Cyclical Contractive Conditions, Fixed Point Theory 4 (2003), 79–89.

[16] E. Karapinar, I.M. Erhan, Cyclic Contractions and Fixed Point Theorems, Filomat 26 (2012), 777–782. https://doi.org/10.2298/FIL1204777K.

[17] M.S. Khan, M. Swaleh, S. Sessa, Fixed Point Theorems by Altering Distances between the Points, Bull. Aust. Math. Soc. 30 (1984), 1–9. https://doi.org/10.1017/S0004972700001659.

[18] M. Özavşar, Fixed Points of Multiplicative Contraction Mappings on Multiplicative Metric Spaces, J. Eng. Technol. Appl. Sci. 2 (2017), 65–79. https://doi.org/10.30931/jetas.338608.

[19] A. Petrușel, Cyclic Representations and Periodic Points, Stud. Univ. Babeș-Bolyai Math. 50 (2005), 107–112.

[20] S. Radenovic, B. Samet, New Directions in Fixed Point Theory in Multiplicative Metric Spaces, Filomat 39 (2025), 3083–3097. https://doi.org/10.2298/FIL2509083R.

[21] T. Dosenovic, S. Radenovic, Rational Contraction in Multiplicative Metric Spaces, Adv. Theory Nonlinear Anal. Appl. 2 (2018), 195–201. https://doi.org/10.31197/atnaa.481995.

[22] N. Sharma, K. Kumar, S. Sharma, R. Jha, Rational Contractive Condition in Multiplicative Metric Space and Common Fixed Point Theorem, Int. J. Innov. Res. Sci. Eng. Technol. 5 (2016), 10473–10480.

[23] B.E. Rhoades, Some Theorems on Weakly Contractive Maps, Nonlinear Anal. Theory Methods Appl. 47 (2001), 2683–2693. https://doi.org/10.1016/S0362-546X(01)00388-1.

[24] I.A. Rus, Cyclic Representations and Fixed Points, An. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convex. 3 (2005), 171–178.

[25] S. Shukla, Some Critical Remarks on Multiplicative Metric Spaces and Fixed Point Results, J. Adv. Math. Stud. 9 (2016), 454–458.

[26] I.S. Singh, Y.M. Singh, Some Results in the New Direction of Multiplicative Metric Space, Bol. Soc. Paranaense Mat. 43 (2025), 1–16. https://doi.org/10.5269/bspm.77365.