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# FIXED POINT APPROXIMATION FOR $\rho$-QUASI NONEXPANSIVE MULTIVALUED MAPPINGS VIA A HYBRID ITERATIVE SCHEME WITH APPLICATIONS

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**Abstract.** We present results on convergence and stability for the Picard-Abbas hybrid iteration for multivalued $\rho$-quasi-nonexpansive mappings in modular function spaces. We also reinforce the results via numerical illustrations. Furthermore, we demonstrate the utilization of this iteration scheme in differential equations.

**Keywords:** Picard-Abbas hybrid iteration; convergence; stability; quasi-nonexpansive mappings; modular function space.

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## 1. Introduction

Nakano [1] generalized the theory of ordered spaces to modular spaces. Musielak and Orlicz [2] further generalized this theory. The initial study of fixed point theory in modular spaces was conducted by Khamsi, Kozlowski, and Reich [3]. Later, via the Mann iteration scheme, Khan and Abbas [4] began the study of approximating fixed points for multivalued nonlinear mappings in modular function spaces. By utilizing a three-step iteration scheme, Khan et al. [5] established convergence results for approximating fixed points for $\rho$-quasi-nonexpansive mappings. Okeke et al. [6] also provided significant results for multivalued $\rho$-quasi-nonexpansive

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mappings by utilizing the hybrid Picard-Krasnoselski iteration scheme. The growing interest in studying quasi-nonexpansive mappings in modular function spaces is fuelled by the rich structural properties of these spaces. These spaces are equipped with modular equivalents of metric and norm concepts. The modular type conditions, offer a more intuitive and verifiable framework compared to the traditional norm-based assumptions, thus contributing to their increasing popularity.

This paper purposes to further the existing work by presenting new results on convergence and stability for $\rho$-quasi-nonexpansive mappings within modular function spaces, utilizing the Picard Abbas-type hybrid iterative approach. In doing so, we contribute to the on-going discourse in fixed point theory, enhancing our understanding of the interplay between modular structures and iterative methods.

## 2. Preliminaries

Consider a nontrivial $\sigma$-algebra $\Sigma$ on $\Omega \neq \emptyset$, and $\mathscr{P}$ a $\delta$-ring of subsets of $\Omega$, with

$$E \cap A \in \mathscr{P} \ \ \forall E \in \mathscr{P}, A \in \Sigma.$$

Let $K_n \in \mathscr{P}$ be an increasing sequence of sets with $\Omega = \cup K_n$. Let $\mathscr{M}_\infty$ be the space of all extended measurable functions. Let $\mathscr{E}$ be the vector space of simple functions with support being contained within $\mathscr{P}$ and $1_A$ be the characteristic function of $A$ in $\Omega$.
We define

$$\mathscr{M}(\Omega, \Sigma, \mathscr{P}, \rho) = \{f \in \mathscr{M}_\infty : |f(\omega)| < \infty \ \ \rho\text{-a.e.}\},$$

For simplicity, we write $\mathscr{M}$ instead of $\mathscr{M}(\Omega, \Sigma, \mathscr{P}, \rho)$.

**Definition 2.1.** [7] *A functional $\rho$ of a vector space $X$ ($\mathbb{R}$ or $\mathbb{C}$) is a modular if for arbitrary $f, g \in X$, the statements below hold:*
*(i)* $\rho(f) = 0 \iff f = 0$
*(ii)* $\rho(\alpha f) = \rho(f)$ *whenever* $|\alpha| = 1$
*(iii)* $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ *whenever* $\alpha, \beta \geq 0, \ \alpha + \beta = 1.$
*If we replace (iii) by*

*(iv)* $\rho(\alpha f+\beta g)\leq\alpha\rho(f)+\beta\rho(g)$ *whenever* $\alpha,\beta\geq 0,\ \alpha+\beta=1.$

*then the modular* $\rho$ *is convex.*

**Definition 2.2.** [7] *The following set is called a modular function space, for a convex modular* $\rho$ *in* $X$*:*

$$L_{\rho}=\{f\in\mathscr{M}:\rho(\lambda f)\to 0\ \ as\ \ \lambda\to 0\}$$

In general, $\rho$ is not sub-additive. We can equip the $L_{\rho}$ with the following F-norm:

$$\|f\|_{\rho}=\inf\left\{\alpha>0:\rho\left(\frac{f}{\alpha}\right)\leq\alpha\right\}.$$

If $\rho$ is convex, then the norm

$$\|f\|_{\rho}=\inf\left\{\alpha>0:\rho\left(\frac{f}{\alpha}\right)\leq 1\right\}$$

is called the Luxemburg Norm on the modular space $L_{\rho}$.

**Definition 2.3.** [7] *The nontrivial, even and convex function* $\rho:\mathscr{M}_{\infty}\to[0,\infty]$ *is called a regular convex function pseudomodular if*

*(i)* $\rho(0)=0$*;*

*(ii)* $|f(\omega)|\leq|g(\omega)|$ *for any* $\omega\in\Omega$ *implies* $\rho(f)\leq\rho(g)$*, where* $f,g\in\mathscr{M}_{\infty}$*. That is,* $\rho$ *is monotone.*

*(iii)* $\rho(f1_{A\cup B})\leq\rho(f1_{A})+\rho(f1_{B})$ *for any* $A,B\in\Sigma$ *such that* $A\cap B\neq\emptyset$*,* $f\in\mathscr{M}_{\infty}$*. That is,* $\rho$ *is orthogonally sub-additive.*

*(iv)* $|f_n(\omega)|\uparrow|f(\omega)|$ *for all* $\omega\in\Omega$ *implies* $\rho(f_n)\uparrow\rho(f)$*, where* $f\in\mathscr{M}_{\infty}$*. That is,* $\rho$ *has Fatou property.*

*(v)* $g_n\in\mathscr{E}$*, and* $|g_n(\omega)|\downarrow 0$ *implies* $\rho(g_n)\downarrow 0$*. That is* $\rho$ *is order continuous in* $\mathscr{E}$*.*

A set $A\in\Sigma$ is $\rho$-null if $\rho(g1_A)=0\ \ \forall g\in\mathscr{E}$. A property $\rho(\omega)$ is $\rho$-almost everywhere ($\rho$-a.e.) if $\{\omega\in\Omega:\rho(\omega)$ does not hold$\}$ is $\rho$-null.

**Definition 2.4.** [7] *A regular function pseudomodular* $\rho$ *is called a regular convex function modular if* $\rho(f)=0\Rightarrow f=0\ \ \rho$*-a.e.*

Let $\mathfrak{R}$ be the class of all non-zero regular convex function modular on $\Omega$.

For $\rho \in \mathfrak{R}$, define $L_\rho^0 = \left\{f \in L_\rho : \rho(f,\cdot) \textit{ is order continuous}\right\}$ and $E_\rho = \left\{f \in L_\rho : \lambda f \in L_\rho^0 \textit{ for every } \lambda > 0\right\}$. Note that $L_\rho \supset L_\rho^0 \supset E_\rho$.

**Definition 2.5.** [5] *A $\rho \in \mathfrak{R}$ satisfy $\Delta_2$-condition if $\sup\limits_{n\geq 1}\rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{D_k\}$ decreases to $\phi$ and $\sup\limits_{n\geq 1}\rho(f_n, D_k) \to 0$ as $k \to \infty$.*

$L_\rho = E_\rho$ holds true if $\rho$ is convex and satisfies $\Delta_2$-condition.

Let $\rho \in \mathfrak{R}$. Let $r > 0,\ \varepsilon > 0$. Define

$$D_1(r,\varepsilon) = \left\{(f,g) : f,g \in L_\rho,\ \rho(f) \leq r,\ \rho(g) \leq r,\ \rho(f-g) \geq \varepsilon r\right\}.$$

Let $\delta_1(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f,g) \in D_1(r,\varepsilon)\right\}$ if $D_1(r,\varepsilon) \neq \emptyset$ and $\delta_1(r,\varepsilon) = 1$ if $D_1(r,\varepsilon) = \emptyset$.

**Definition 2.6.** [8] *A $\rho \in \mathfrak{R}$ satisfy (UC1) if $\forall r > 0,\ \varepsilon > 0$, we have $\delta_1(r,\varepsilon) > 0$. Note that $\forall r > 0,\ D_1(r,\varepsilon) \neq \emptyset$ for $\varepsilon > 0$ small enough.*

**Definition 2.7.** [8] *A $\rho \in \mathfrak{R}$ satisfy (UUC1) if $\forall s \geq 0,\ \varepsilon > 0,\ \exists\ \eta_1(s,\varepsilon) > 0$ depending only upon $s$ and $\varepsilon$ such that $\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0$ for any $r > s$.*

**Definition 2.8.** [7] *Let $\rho \in \mathfrak{R}$ and $D \subset L_\rho$.*

*(i) $\{f_n\} \subset L_\rho$ is $\rho$-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.*

*(ii) $\{f_n\} \subset L_\rho$ is $\rho$-Cauchy, if $\rho(f_n - f_m) \to 0$ as $m,n \to \infty$.*

*(iii) $D$ is $\rho$-closed if for $\{f_n\} \subset D$, $f_n \to f$ implies $f \in D$.*

*(iv) $D$ is $\rho$-compact if for $\{f_n\} \subset D$, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in D$ such that $\rho(f_{n_k} - f) \to 0$ as $k \to \infty$.*

*(v) $D$ is $\rho$-a.e. closed if for $\{f_n\} \subset D$ which is $\rho$-a.e. convergent, $f_n \to f$ as $n \to \infty$ implies $f \in D$.*

*(vi) $D$ is $\rho$-a.e. compact if for $\{f_n\} \subset D$, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in D$ exist such that $\rho(f_n - f) \to 0$ as $k \to \infty$.*

*(vii) $D$ is $\rho$-bounded if the $\rho$-diameter of $D$ is finite, that is*

$$diam_\rho(D) = \sup\{\rho(f-g) : f,g \in D\} < \infty.$$

$\rho$-convergence does not imply $\rho$-Cauchy. If $\rho$ satisfy the $\Delta_2$-condition, then this will be true. The $\rho$-distance from $f \in L_\rho$ to $D \subset L_\rho$ is defined by

$$\mathrm{dist}_\rho(f,D) = \inf\{\rho(f-h) : h \in D\}.$$

**Definition 2.9.** [4] *A set $D \subset L_\rho$ is called $\rho$-proximinal if $\forall f \in L_\rho$, $\exists\, g \in D$ such that $\rho(f-g) = \mathrm{dist}_\rho(f,D)$.*

Let $P_\rho(D)$ be the family of non-empty, $\rho$-bounded, $\rho$-proximinal subsets of $D$ and $C_\rho(D)$ be the family of non-empty, $\rho$-closed, $\rho$-bounded subsets of $D$.

We define $\rho$-Hausdorff distance $H_\rho(\cdot,\cdot)$ on $C_\rho(L_\rho)$ as

$$H_\rho(A,B) = \max\left\{\sup_{f\in A} \mathrm{dist}_\rho(f,B), \sup_{g\in B} \mathrm{dist}_\rho(g,A)\right\},\ A,B \in C_\rho(L_\rho).$$

**Definition 2.10.** [4] *A multivalued mapping $T : D \to C_\rho(D)$ is*

*(i) $\rho$-nonexpansive if $H_\rho(Tf,Tg) \le \rho(f-g)\ \ \forall f,g \in D$.*

*(ii) $\rho$-quasi-nonexpansive if $H_\rho(Tf,p) \le \rho(f-p)\ \ \forall f \in D$ and $p \in F_\rho(T)$.*

**Theorem 2.11.** [9] *Let $\rho \in \Re$ satisfy the $\Delta_2$-condition. Consider the sequences $\{f_n\}$ and $\{g_n\}$ in $L_\rho$. Then*

$$\lim_{n\to\infty} \rho(g_n) = 0 \ \ \textit{implies} \ \ \limsup_{n\to\infty} \rho(f_n+g_n) = \limsup_{n\to\infty} \rho(f_n)$$

*and*

$$\lim_{n\to\infty} \rho(g_n) = 0 \ \ \textit{implies} \ \ \liminf_{n\to\infty} \rho(f_n+g_n) = \liminf_{n\to\infty} \rho(f_n)$$

A sequence $\{t_n\} \subset (0,1)$ is bounded away from 0 if $\exists\, a > 0$ such that $t_n \ge a\ \ \forall n \in \mathbb{N}$, and is bounded away from 1 if $\exists\, b < 1$ such that $t_n \le b\ \ \forall n \in \mathbb{N}$.

**Lemma 2.12.** [8, 10] *Let (UUC1) be satisfied by $\rho \in \Re$, and $\{t_k\} \subset (0,1)$ be bounded away from 0 and 1. If $\exists\, R > 0$ such that*

$$\limsup_{n\to\infty} \rho(f_n) \le R,\ \limsup_{n\to\infty} \rho(g_n) \le R \ \ \textit{and} \ \ \lim_{n\to\infty} \rho\big(t_n f_n + (1-t_n) g_n\big) = R,$$

*then*

$$\lim_{n\to\infty} \rho(f_n - g_n) = 0.$$

**Definition 2.13.** *$f \in D \subset L_\rho$ is a fixed point of $T : D \to P_\rho(D)$ if $f \in Tf$.*

Let $F_\rho(T)$ be the set of all fixed points of $T$.

**Definition 2.14.** [11] *A multivalued mapping $T : D \to P_\rho(D)$ satisfy Condition $(I)$ if there exists a continuous non-decreasing function $l : [0,\infty) \to [0,\infty)$ with $l(0) = 0,\ l(r) > 0\ \forall r \in (0,\infty)$ such that $\operatorname{dist}_\rho(f,Tf) \geq l\big(\operatorname{dist}_\rho(f,F_\rho(T))\big)\ \ \forall f \in D$.*

**Lemma 2.15.** [7] *For a multivalued mapping $T : D \to P_\rho(D)$ with*

$$P_\rho^T(f) = \left\{g \in Tf : \rho(f-g) = \operatorname{dist}_\rho(f,Tf)\right\},$$

*the following statements are equivalent:*

*(i) $f \in F_\rho(T)$, i.e., $f \in Tf$.*

*(ii) $P_\rho^T(f) = \{f\}$, i.e, $f = g$ for each $g \in P_\rho^T(f)$.*

*(iii) $f \in F\big(P_\rho^T(f)\big)$, i.e, $f \in P_\rho^T(f)$. Also, $F_\rho(T) = F\big(P_\rho^T(f)\big)$, with $F\big(P_\rho^T(f)\big)$ being the set of fixed points of $P_\rho^T(f)$.*

**Definition 2.16.** [8] *A set $C \subset L_\rho$ is said to possess the Vitali property if $C \subset E_\rho$, and for any $g \in L_\rho$ and $g_n \in C$ with $\rho(g_n - g) \to 0$, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that for every $\alpha > 0$ the subadditive measures $\rho(\alpha g_{n_k},\cdot)$ are order equicontinuous.*

**Definition 2.17.** [8] *The function modular $\rho \in \Re$ is called separable if $\|f1_{(\cdot)}\|_\rho$ is a separable set function for each $f \in \mathscr{E}$, which means that there exists a countable $\mathscr{A} \in \mathscr{P}$ such that to every $A \in \mathscr{P}$ there corresponds a sequence $\{A_k\}$ of elements of $\mathscr{A}$ with $\rho(\alpha f, A\Delta A_k) \to 0$ for every $\alpha > 0$, where $\Delta$ denotes the symmetric difference.*

## 3. Convergence Analysis

The Picard-Abbas iteration for singlevalued mappings was introduced by Chyne and Kumar [12]. We define the Picard-Abbas iteration for multivalued mapping as follows:

$$f_{n+1} \in P_\rho^T(h_n)$$

$$h_n = (1-\alpha_n)w_n + \alpha_n v_n$$

$$g_n = (1-\beta_n)u_n + \beta_n v_n$$

$$e_n = (1-\gamma_n)f_n + \gamma_n u_n \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $(0,1)$ which are bounded away from 0 and 1, and $u_n \in P_\rho^T(f_n),\ v_n \in P_\rho^T(e_n),\ w_n \in P_\rho^T(g_n)$.

**Theorem 3.1.** *Let $\Delta_2$-condition and (UUC1) be satisfied by $\rho \in \mathfrak{R}$. Let $D \neq \emptyset$ be $\rho$-bounded, $\rho$-closed, and convex in $L_\rho$. Consider a multivalued mapping $T : D \to P_\rho(D)$ with $P_\rho^T$ being a $\rho$-quasi-nonexpansive mapping, and $F_\rho(T) \neq \emptyset$. If $\{f_n\} \subset D$ be defined by (3.1), then $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F_\rho(T)$.*

*Proof.* Let $p \in F_\rho(T)$. Using Lemma 2.15, we get

$$P_\rho^T(p) = \{p\}$$

$$\rho(f_{n+1} - p) \le H_\rho\big(P_\rho^T(h_n), P_\rho^T(p)\big) \le \rho(h_n - p) \tag{3.2}$$

$$\begin{aligned}
\rho(h_n - p) &= \rho\big((1-\alpha_n)w_n + \alpha_n v_n - p\big)\\
&\le (1-\alpha_n)\rho(w_n - p) + \alpha_n(v_n - p) \quad \text{[using convexity of } \rho]\\
&\le (1-\alpha_n)H_\rho\big(P_\rho^T(g_n), P_\rho^T(p)\big) + \alpha_n H_\rho\big(P_\rho^T(e_n), P_\rho^T(p)\big)\\
&\le (1-\alpha_n)\rho(g_n - p) + \alpha_n \rho(e_n - p) \qquad (3.3)
\end{aligned}$$

$$\begin{aligned}
\rho(g_n - p) &= \rho\big((1-\beta_n)u_n + \beta_n v_n - p\big)\\
&\le (1-\beta_n)\rho(u_n - p) + \beta_n(v_n - p) \quad \text{[using convexity of } \rho]\\
&\le (1-\beta_n)H_\rho\big(P_\rho^T(f_n), P_\rho^T(p)\big) + \beta_n H_\rho\big(P_\rho^T(e_n), P_\rho^T(p)\big)\\
&\le (1-\beta_n)\rho(f_n - p) + \beta_n \rho(e_n - p) \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
\rho(e_n - p) &= \rho\big((1-\gamma_n)f_n + \gamma_n u_n - p\big)\\
&\le (1-\gamma_n)\rho(f_n - p) + \gamma_n(u_n - p) \quad \text{[using convexity of } \rho]\\
&\le (1-\gamma_n)\rho(f_n - p) + \gamma_n H_\rho\big(P_\rho^T(f_n), P_\rho^T(p)\big)\\
&\le (1-\gamma_n)\rho(f_n - p) + \gamma_n \rho(f_n - p)
\end{aligned}$$

$$\leq \rho(f_n - p) \tag{3.5}$$

Using (3.5) in (3.4), we get

$$\rho(g_n - p) \leq \rho(f_n - p) \tag{3.6}$$

Using (3.5) and (3.6) in (3.3), we get

$$\rho(h_n - p) \leq \rho(f_n - p) \tag{3.7}$$

From (3.2) and (3.7), we get

$$\rho(f_{n+1} - p) \leq \rho(f_n - p) \tag{3.8}$$

Therefore, the sequence $\{\rho(f_n - p)\}$ is decreasing.
Thus, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F_\rho(T)$. $\square$

**Theorem 3.2.** *Let $\Delta_2$-condition and (UUC1) be satisfied by $\rho \in \mathfrak{R}$. Let $D \neq \emptyset$ be $\rho$-bounded, $\rho$-closed, and convex in $L_\rho$. Consider a multivalued mapping $T : D \to P_\rho(D)$ with $P_\rho^T$ being a $\rho$-quasi-nonexpansive mapping, and $F_\rho(T) \neq \emptyset$. If $\{f_n\} \subset D$ be defined by (3.1), then $\lim_{n\to\infty} \operatorname{dist}_\rho\left(f_n, P_\rho^T(f_n)\right) = 0$.*

*Proof.* By Theorem 3.1, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F_\rho(T)$. Let

$$\lim_{n\to\infty} \rho(f_n - p) = L \tag{3.9}$$

Since $\operatorname{dist}_\rho\left(f_n, P_\rho^T(f_n)\right) \leq \rho(f_n - u_n)$, it suffices to show that $\lim_{n\to\infty} \rho(f_n - u_n) = 0$.
Now

$$\rho(u_n - p) \leq H_\rho\left(P_\rho^T(f_n), P_\rho^T(p)\right) \leq \rho(f_n - p)$$

implies

$$\limsup_{n\to\infty} \rho(u_n - p) \leq \limsup_{n\to\infty} \rho(f_n - p)$$

So, (3.9) gives

$$\limsup_{n\to\infty} \rho(u_n - p) \leq L \tag{3.10}$$

Also, from (3.5), we get

$$\limsup_{n\to\infty} \rho(e_n - p) \leq \limsup_{n\to\infty} \rho(f_n - p)$$

So,

$$\limsup_{n\to\infty} \rho(e_n - p) \leq L \tag{3.11}$$

Similarly, we can show that

$$\limsup_{n\to\infty} \rho(g_n - p) \leq L \tag{3.12}$$

and

$$\limsup_{n\to\infty} \rho(h_n - p) \leq L \tag{3.13}$$

Now

$$\rho(w_n - p) \leq H_\rho\left(P_\rho^T(g_n), P_\rho^T(p)\right) \leq \rho(g_n - p) \leq \rho(f_n - p)$$

So,

$$\limsup_{n\to\infty} \rho(w_n - p) \leq \limsup_{n\to\infty} \rho(f_n - p) \leq L \tag{3.14}$$

Similarly, we can show that

$$\limsup_{n\to\infty} \rho(v_n - p) \leq L \tag{3.15}$$

Again

$$\begin{aligned}
\lim_{n\to\infty} \rho(f_{n+1} - p) &= \lim_{n\to\infty} \rho\left((1-\alpha_n)w_n + \alpha_n v_n - p\right) \\
&= \lim_{n\to\infty} \rho\left((1-\alpha_n)(w_n - p) + \alpha_n(v_n - p)\right) \\
&= L \qquad (3.16)
\end{aligned}$$

Therefore, (3.14), (3.15), (3.16) and Lemma 2.12 gives

$$\lim_{n\to\infty} \rho(w_n - v_n) = 0 \tag{3.17}$$

Let $\varepsilon > 0$ be given. There exists $n_0 \in \mathbb{N}$ such that $\rho(w_n - v_n) < \varepsilon$ for all $n \geq n_0$.

Since $\rho\big(\alpha_n(w_n - v_n)\big) \leq \omega_\rho(\alpha_n)\rho(w_n - v_n) < \varepsilon$, we have

$$\lim_{n\to\infty} \rho\big(\alpha_n(w_n - v_n)\big) = 0.$$

Also,

$$\rho(f_{n+1} - p) = \rho\big((1-\alpha_n)w_n + \alpha_n v_n - p\big) = \rho\big((w_n - p) + \alpha_n(v_n - w_n)\big)$$

From Theorem 2.11, we get

$$\liminf_{n\to\infty} \rho\big((w_n - p) + \alpha_n(v_n - w_n)\big) = \liminf_{n\to\infty} \rho(w_n - p).$$

So,

$$\liminf_{n\to\infty} \rho(w_n - p) = L \tag{3.18}$$

From (3.14) and (3.18), we get

$$\lim_{n\to\infty} \rho(w_n - p) = L \tag{3.19}$$

Using (3.17) and Theorem 2.11, we get

$$L = \liminf_{n\to\infty} \rho(w_n - p) = \liminf_{n\to\infty} \rho\big((w_n - v_n) + (v_n - p)\big) = \liminf_{n\to\infty} \rho(v_n - p).$$

But

$$\rho(v_n - p) \leq H_\rho\big(P_\rho^T(e_n), P_\rho^T(p)\big) \leq \rho(e_n - p).$$

Therefore,

$$L \leq \liminf_{n\to\infty} \rho(e_n - p) \tag{3.20}$$

From (3.11) and (3.20) we get

$$\lim_{n\to\infty} \rho(e_n - p) = L$$

That is,

$$\lim_{n\to\infty} \rho\big((1-\gamma_n)(f_n - p) + \gamma_n(u_n - p)\big) = L.$$

From (3.9), (3.10) and Lemma 2.12, we get

$$\lim_{n\to\infty} \rho(f_n - u_n) = 0.$$

Hence, $\lim\limits_{n\to\infty} \operatorname{dist}_\rho\left(f_n, P_\rho^T(f_n)\right) = 0$. □

**Theorem 3.3.** *Let $\Delta_2$-condition and (UUC1) be satisfied by $\rho \in \mathfrak{R}$. Let $D \neq \emptyset$ be $\rho$-bounded, $\rho$-closed, $\rho$-compact and convex in $L_\rho$. Consider a multivalued mapping $T : D \to P_\rho(D)$ with $P_\rho^T$ being a $\rho$-quasi-nonexpansive mapping, and $F_\rho(T) \neq \emptyset$. If $\{f_n\} \subset D$ be defined by (3.1), then $\{f_n\}$ is $\rho$-convergent to a fixed point of $T$.*

*Proof.* By $D$ being compact, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ exists, such that for some $q \in D$, $\lim\limits_{k\to\infty} \rho(f_{n_k} - q) = 0$. We show that $q \in F_\rho(T)$.
Let $g$ be arbitrary chosen from $P_\rho^T(q)$ and $f$ from $P_\rho^T(f_{n_k})$.
Now,

$$\begin{aligned}
\rho\left(\frac{q-g}{3}\right) &\leq \rho\left(\frac{q-f_{n_k}}{3}\right) + \rho\left(\frac{f_{n_k}-f}{3}\right) + \rho\left(\frac{f-g}{3}\right) \\
&\leq \frac{1}{3}\rho(q-f_{n_k}) + \frac{1}{3}\rho(f_{n_k}-f) + \frac{1}{3}\rho(f-g) \\
&\leq \rho(q-f_{n_k}) + \operatorname{dist}_\rho\left(f_{n_k}, P_\rho^T(f_{n_k})\right) + \operatorname{dist}_\rho\left(P_\rho^T(f_{n_k}), g\right) \\
&\leq \rho(q-f_{n_k}) + \operatorname{dist}_\rho\left(f_{n_k}, P_\rho^T(f_{n_k})\right) + H_\rho\left(P_\rho^T(f_{n_k}), P_\rho^T(q)\right) \\
&\leq \rho(q-f_{n_k}) + \operatorname{dist}_\rho\left(f_{n_k}, P_\rho^T(f_{n_k})\right) + \rho(q-f_{n_k})
\end{aligned}$$

Using Theorems 3.1 and 3.2, we get

$$\rho(q-g) = 0.$$

Thus, $q \in F(P_\rho^T) = F_\rho(T)$. That is, $\{f_n\}$ is $\rho$-convergent to a fixed point of $T$. □

**Theorem 3.4.** *Let $\Delta_2$-condition and (UUC1) be satisfied by $\rho \in \mathfrak{R}$. Let $D \neq \emptyset$ be $\rho$-bounded, $\rho$-closed, and convex in $L_\rho$. Consider a multivalued mapping $T : D \to P_\rho(D)$ satisfying Condition (I) with $P_\rho^T$ being a $\rho$-quasi-nonexpansive mapping, and $F_\rho(T) \neq \emptyset$. If $\{f_n\} \subset D$ be defined by (3.1), then $\{f_n\}$ is $\rho$-convergent to a fixed point of $T$.*

*Proof.* We have shown in Theorem 3.1 that $\lim\limits_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F(P_\rho^T) = F_\rho(T)$.
If $\lim\limits_{n\to\infty} \rho(f_n - p) = 0$, there is nothing to prove. We assume $\lim\limits_{n\to\infty} \rho(f_n - p) = L > 0$.
Again, from Theorem 3.1, we have $\rho(f_{n+1} - p) \leq \rho(f_n - p)$. So,

$$\operatorname{dist}_\rho\left(f_{n+1}, F_\rho(T)\right) \leq \operatorname{dist}_\rho\left(f_n, F_\rho(T)\right).$$

Hence, $\lim_{n\to\infty} \operatorname{dist}_\rho\left(f_n, F_\rho(T)\right)$ exists. We show that $\lim_{n\to\infty} \operatorname{dist}_\rho\left(f_n, F_\rho(T)\right) = 0$.
Using Theorem 3.1 and Condition (I), we get

$$\lim_{n\to\infty} l\big(\operatorname{dist}_\rho(f_n, F_\rho(T))\big) \leq \lim_{n\to\infty} \operatorname{dist}_\rho\left(f_n, F_\rho(T)\right) = 0.$$

That is, $\lim_{n\to\infty} l\big(\operatorname{dist}_\rho(f_n, F_\rho(T))\big) = 0$.
Since $l$ is nondecreasing function and $l(0) = 0$, we have $\lim_{n\to\infty} \big(\operatorname{dist}_\rho(f_n, F_\rho(T))\big) = 0$.
Next, we show that $\{f_n\}$ is a $\rho$-Cauchy sequence in $D$.
Let $\varepsilon > 0$. Since $\lim_{n\to\infty} \operatorname{dist}_\rho\left(f_n, F_\rho(T)\right) = 0$, there is a constant $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, we have

$$\operatorname{dist}_\rho\left(f_n, F_\rho(T)\right) < \frac{\varepsilon}{2}.$$

In particular, $\inf\left\{\rho(f_{n_0}) - p : p \in F_\rho(T)\right\} < \frac{\varepsilon}{2}$. There must exist $p^* \in F_\rho(T)$ such that

$$\rho(f_{n_0} - p^*) < \varepsilon.$$

For $m, n \geq n_0$, we have

$$\rho\left(\frac{f_{n+m} - f_n}{2}\right) \leq \frac{1}{2}\rho(f_{n+m} - p^*) + \frac{1}{2}\rho(f_n - p^*) \leq \rho(f_{n_0} - p^*) < \varepsilon.$$

Therefore, $\{f_n\}$ is a $\rho$-Cauchy in $D \subset L_\rho$. Thus, it is convergent in $D$.
Let $\lim_{n\to\infty} f_n = q$. Using Theorem 3.3, we get $q \in F_\rho(T)$.
Hence, $\{f_n\}$ is $\rho$-convergent to a fixed point of $T$. □

## 4. Stability Analysis

Here we give a stability result for the Picard-Abbas iteration. We begin by stating the stability definition as follows.

**Definition 4.1.** [6] *Let $D \subset L_\rho$ ($D \neq \emptyset$) and operator $T : D \to D$. For a fixed $x_1 \in D$, let $x_{n+1} = f(T, x_n)$ be an iteration generating a sequence $\{x_n\}_{n=1}^\infty \subset D$. Let $\{x_n\}_{n=1}^\infty$ be strongly convergent to $p \in F_\rho(T) \neq \emptyset$. Let the sequence $\{y_n\}_{n=1}^\infty$ be bounded in $D$ and $\varepsilon_n = \rho\big(y_{n+1} - f(T, y_n)\big)$.*
*(i) $\{x_n\}_{n=1}^\infty$ is $T$-stable on $D$ if $\lim_{n\to\infty} \varepsilon_n = 0 \iff \lim_{n\to\infty} y_n = p$*
*(ii) $\{x_n\}_{n=1}^\infty$ is almost $T$-stable on $D$ if $\sum_{n-1}^{\infty} \varepsilon_n < \infty \Rightarrow \lim_{n\to\infty} y_n = p$.*

**Theorem 4.2.** *Let $D \subset L_\rho$ ($D \neq \emptyset$) be convex and bounded. If $T : D \to P_\rho^T(D)$ be a multivalued mapping with $P_\rho^T$ being a $\rho$-quasi-nonexpansive mapping, and $F_\rho(T) \neq \emptyset$, then the iteration (3.1) is $T$-stable.*

*Proof.* Let $\{y_n\} \subset D$. Define $\varepsilon_n = \rho\left(y_{n+1} - f(T, y_n)\right)$. Let $p \in F_\rho(T)$ be unique.
Suppose $\lim_{n\to\infty} y_n = p$. Using (3.1) and the convexity of $\rho$, we have

$$\begin{aligned}
\varepsilon_n &= \operatorname{dist}_\rho\left(P_\rho^T(y_{n+1}), P_\rho^T(h_n)\right) \\
&\leq H_\rho\left(P_\rho^T(y_{n+1}), P_\rho^T(h_n)\right) \\
&\leq \rho(y_{n+1} - h_n) \\
&\leq \rho(y_{n+1} - p) + \rho(p - h_n) \\
&\leq \rho(y_{n+1} - p) + \rho(y_n - p) \qquad (4.1)
\end{aligned}$$

Thus, $\lim_{n\to\infty} \varepsilon_n = 0$.
Conversely, let $\lim_{n\to\infty} \varepsilon_n = 0$. By (4.1), we get $\lim_{n\to\infty} y_n = p$.
Therefore, $\lim_{n\to\infty} y_n = p$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.
Hence the proof. □

## 5. Numerical Illustrations

**Example 5.1.** [5] *Let $\mathbb{R}$ be space modulared as*

$$\rho(f) = |f|^k,\ k \geq 1$$

*Let $D = \left\{f \in L_\rho : 0 \leq f \leq 1\right\}$. We define $T : D \to P_\rho(D)$ as:*

$$Tf = \left[0, \frac{2f+1}{4}\right]$$

*Clearly, $D \neq \emptyset$ is $\rho$-compact, $\rho$-bounded, and $\rho$-convex in $L_\rho = \mathbb{R}$.*
*As $\rho(f) = |f|^k$, $k \geq 1$ is homogeneous of degree k, so (UUC1) holds.*
*Also,*

$$F_\rho(T) = \left[0, \frac{1}{2}\right] \neq \emptyset.$$

*If $f \in \left[0, \frac{1}{2}\right]$, then $P_\rho^T(f) = \{f\}$.*
*Let $f \notin \left[0, \frac{1}{2}\right]$. That is, $f \in \left[\frac{1}{2}, 1\right]$.*

$$\begin{aligned}
P_\rho^T(f) &= \left\{g \in Tf : \rho(f-g) = \mathrm{dist}_\rho(f, Tf)\right\} \\
&= \left\{g \in Tf : |f-g|^k = \mathrm{dist}_\rho\left(f, \left[0, \frac{2f+1}{4}\right]\right)\right\} \\
&= \left\{g \in Tf : |f-g|^k = \left|f - \frac{2f+1}{4}\right|^k\right\} \\
&= \left\{g \in Tf : |f-g| = \left|\frac{2f-1}{4}\right|\right\}
\end{aligned}$$

*Since $f > g \ \ \forall g \in Tf$, we get*

$$|f-g| = f-g$$

*Thus,*

$$\begin{aligned}
f - g &= \frac{2f-1}{4} \\
\Rightarrow g &= \frac{2f+1}{4}
\end{aligned}$$

*Hence,*

$$P_\rho^T(f) = \left\{\frac{2f+1}{4}\right\}.$$

*We now prove that $P_\rho^T(f)$ is $\rho$-quasi nonexpansive for all $f \in D$.*
*For $f \in \left[0, \frac{1}{2}\right]$, the case is trivial.*
*Let $f \in \left(\frac{1}{2}, 1\right]$. We have*

$$H_\rho\left(P_\rho^T(f),\ P_\rho^T(p)\right) = H_\rho\left(\frac{2f+1}{4}, p\right) = \left|\frac{2f+1}{4} - p\right| \leq |f-p| \ \ \forall f \in \left(\frac{1}{2}, 1\right].$$

*We now generate the iterative sequence (3.1) and show that it converges strongly to a fixed point of $T$.*
*Take $f_1 = 1 \in D$, and $\alpha = \beta = \gamma = \frac{1}{2}$.*

***Iteration 1**:*

$$P_\rho^T(f_1) = \left\{\frac{2 \times 1 + 1}{4}\right\} = \left\{\frac{3}{4}\right\} = \left\{\frac{1}{2} + \frac{1}{4}\right\}.$$

$$u_1 \in P_\rho^T(f_1) = \left\{\frac{1}{2} + \frac{1}{4}\right\}. \; So, \; u_1 = \frac{1}{2} + \frac{1}{4}.$$

$$e_1 = \frac{1}{2}f_1 + \frac{1}{2}u_1 = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2} + \frac{3}{8}.$$

$$P_\rho^T(e_1) = \left\{\frac{2e_1+1}{4}\right\} = \left\{\frac{2\left(\frac{1}{2}+\frac{3}{8}\right)+1}{4}\right\} = \left\{\frac{1+\frac{3}{4}+1}{4}\right\} = \left\{\frac{1}{2} + \frac{3}{16}\right\}.$$

$$v_1 \in P_\rho^T(e_1) = \left\{\frac{1}{2} + \frac{3}{16}\right\}. \; So, \; v_1 = \frac{1}{2} + \frac{3}{16}.$$

$$g_1 = \frac{1}{2}u_1 + \frac{1}{2}v_1 = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{3}{16}\right) = \frac{1}{2} + \frac{7}{32}.$$

$$P_\rho^T(g_1) = \left\{\frac{2g_1+1}{4}\right\} = \left\{\frac{2\left(\frac{1}{2}+\frac{7}{32}\right)+1}{4}\right\} = \left\{\frac{1+\frac{7}{16}+1}{4}\right\} = \left\{\frac{1}{2} + \frac{7}{64}\right\}.$$

$$w_1 \in P_\rho^T(g_1) = \left\{\frac{1}{2} + \frac{7}{64}\right\}. \; So, \; w_1 = \frac{1}{2} + \frac{7}{64}.$$

$$h_1 = \frac{1}{2}w_1 + \frac{1}{2}v_1 = \frac{1}{2}\left(\frac{1}{2} + \frac{7}{64}\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{3}{16}\right) = \frac{1}{2} + \frac{19}{128}$$

$$f_2 \in P_\rho^T(h_1) = \left\{\frac{2h_1+1}{4}\right\} = \left\{\frac{2\left(\frac{1}{2}+\frac{19}{128}\right)+1}{4}\right\} = \left\{\frac{1+\frac{19}{64}+1}{4}\right\} = \left\{\frac{1}{2} + \frac{19}{256}\right\}.$$

$So, \; f_2 = \frac{1}{2} + \frac{19}{256} < \frac{1}{2} + \frac{1}{2}$

***Iteration 2***:

$$P_\rho^T(f_2) = \left\{\frac{2f_2+1}{4}\right\} = \left\{\frac{2\left(\frac{1}{2}+\frac{19}{256}\right)+1}{4}\right\} = \left\{\frac{1}{2} + \frac{19}{512}\right\}$$

$$u_2 \in P_\rho^T(f_2) = \left\{\frac{1}{2} + \frac{19}{512}\right\}. \; So, \; u_2 = \frac{1}{2} + \frac{19}{512}.$$

$$e_2 = \frac{1}{2}f_2 + \frac{1}{2}u_2 = \frac{1}{2}\left(\frac{1}{2} + \frac{19}{256}\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{19}{512}\right) = \frac{1}{2} + \frac{57}{1024}.$$

$$P_\rho^T(e_2) = \left\{\frac{2e_2+1}{4}\right\} = \left\{\frac{2\left(\frac{1}{2}+\frac{57}{1024}\right)+1}{4}\right\} = \left\{\frac{1}{2} + \frac{57}{2048}\right\}.$$

$$v_2 \in P_\rho^T(e_2) = \left\{\frac{1}{2} + \frac{57}{2048}\right\}. \; So, \; v_2 = \frac{1}{2} + \frac{57}{2048}.$$

$$g_2 = \frac{1}{2}u_2 + \frac{1}{2}v_2 = \frac{1}{2}\left(\frac{1}{2} + \frac{19}{512}\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{57}{2048}\right) = \frac{1}{2} + \frac{133}{4096}.$$

$$P_\rho^T(g_2) = \left\{\frac{2g_2+1}{4}\right\} = \left\{\frac{2\left(\frac{1}{2}+\frac{133}{4096}\right)+1}{4}\right\} = \left\{\frac{1}{2} + \frac{133}{8192}\right\}.$$

$$w_2 \in P_\rho^T(g_2) = \left\{\frac{1}{2} + \frac{133}{8192}\right\}. \text{ So, } w_2 = \frac{1}{2} + \frac{133}{8192}.$$

$$h_2 = \frac{1}{2}w_2 + \frac{1}{2}v_2 = \frac{1}{2}\left(\frac{1}{2} + \frac{133}{8192}\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{57}{2048}\right) = \frac{1}{2} + \frac{361}{16384}$$

$$f_3 \in P_\rho^T(h_2) = \left\{\frac{2h_2+1}{4}\right\} = \left\{\frac{2\left(\frac{1}{2} + \frac{361}{16384}\right)+1}{4}\right\} = \left\{\frac{1}{2} + \frac{361}{32768}\right\}.$$

*So,* $f_3 = \frac{1}{2} + \frac{361}{32768} < \frac{1}{2} + \frac{1}{3}$

*Similarly,*

$$f_4 < \frac{1}{2} + \frac{1}{4},\ f_5 < \frac{1}{2} + \frac{1}{5}, \ldots, f_n < \frac{1}{2} + \frac{1}{n}.$$

*Thus* $f_n \to \frac{1}{2} \in F_\rho(T) = \left[0, \frac{1}{2}\right]$.

**Example 5.2.** [5] *Consider the modular space* $L_\rho = \mathbb{R}$*, equipped with the norm* $\|\cdot\|$*, that is,* $\rho(f) = |f|$ *and* $D = \{f \in L_\rho : 1 \le f < \infty\}$.

*Clearly,* $D \neq \emptyset$ *is* $\rho$*-closed,* $\rho$*-bounded, and* $\rho$*-convex in* $L_\rho = \mathbb{R}$.

*We define* $T : D \to P_\rho(D)$ *as:*

$$Tf = \left[1, 1 + \frac{f}{2}\right]$$

*Also,*

$$F_\rho(T) = [1,2] \neq \emptyset.$$

*Define a continuous nondecreasing function* $l : [0,\infty) \to [0,\infty)$ *by* $l(r) = \frac{r}{4}$.

*We see that* $\operatorname{dist}_\rho(f, Tf) \ge l\big(\operatorname{dist}_\rho(f, F_\rho(T))\big)\ \ \forall f \in D$.

*If* $f \in F_\rho(T) = [1,2]$*, then clearly*

$$\operatorname{dist}_\rho(f, Tf) = 0 = l\big(\operatorname{dist}_\rho(f, F_\rho(T))\big)$$

*If* $f \in (2,\infty)$*, then*

$$\operatorname{dist}_\rho(f, Tf) = \operatorname{dist}_\rho\left(f, \left[1, 1 + \frac{f}{2}\right]\right) = \left|f - \left(1 + \frac{f}{2}\right)\right| = \frac{f-2}{2}$$

*And*

$$l\big(\operatorname{dist}_\rho(f, F_\rho(T))\big) = l\big(\operatorname{dist}_\rho(f, [1,2])\big) = l(|f-2|) = \frac{f-2}{4}.$$

*If* $f \in [1,2]$, *then* $P_\rho^T(f) = \{f\}$.

*Let* $f \in (2,\infty)$.

$$\begin{aligned}
P_\rho^T(f) &= \left\{g \in Tf : \rho(f-g) = \mathrm{dist}_\rho(f, Tf)\right\} \\
&= \left\{g \in Tf : |f-g| = \mathrm{dist}_\rho\left(f, \left[1, 1+\frac{f}{2}\right]\right)\right\} \\
&= \left\{g \in Tf : |f-g| = \left|f - \left(1+\frac{f}{2}\right)\right|\right\} \\
&= \left\{g \in Tf : |f-g| = \left|\frac{f}{2} - 1\right|\right\}
\end{aligned}$$

*Since* $f > g \ \ \forall g \in Tf$, *where* $f \in (2,\infty)$, *we get*

$$|f-g| = f-g$$

*Thus,*

$$\begin{aligned}
f-g &= \frac{f}{2} - 1 \\
\Rightarrow g &= 1 + \frac{f}{2}
\end{aligned}$$

*Hence,*

$$P_\rho^T(f) = \left\{1 + \frac{f}{2}\right\}.$$

*We now prove that* $P_\rho^T(f)$ *is* $\rho$*-quasi nonexpansive for all* $f > 2$.

*We have*

$$H_\rho\left(P_\rho^T(f),\, P_\rho^T(p)\right) = H_\rho\left(1+\frac{f}{2}, \{p\}\right) = \left|1 + \frac{f}{2} - p\right| \le |f-p| \ \ \forall f > 2.$$

*We now generate the iterative sequence (3.1) and show that it converges strongly to a fixed point of* $T$.

*Take* $f_1 = 3 \in D = [1,\infty)$, *and* $\alpha = \beta = \gamma = \dfrac{1}{2}$.

***Iteration 1***:

$$P_\rho^T(f_1) = \left\{1 + \frac{3}{2}\right\} = \left\{\frac{5}{2}\right\}.$$

$$u_1 \in P_\rho^T(f_1) = \left\{\frac{5}{2}\right\}. \text{ So, } u_1 = \frac{5}{2}.$$

$$e_1 = \frac{1}{2}f_1 + \frac{1}{2}u_1 = \frac{1}{2} \times 3 + \frac{1}{2}\left(\frac{5}{2}\right) = \frac{11}{4}.$$

$$P_\rho^T(e_1) = \left\{1 + \frac{e_1}{2}\right\} = \left\{1 + \frac{1}{2}\left(\frac{11}{4}\right)\right\} = \left\{\frac{19}{8}\right\}.$$

$$v_1 \in P_\rho^T(e_1) = \left\{\frac{19}{8}\right\}. \textit{ So, } v_1 = \frac{19}{8}.$$

$$g_1 = \frac{1}{2}u_1 + \frac{1}{2}v_1 = \frac{1}{2}\left(\frac{5}{2}\right) + \frac{1}{2}\left(\frac{19}{8}\right) = \frac{39}{16}.$$

$$P_\rho^T(g_1) = \left\{1 + \frac{g_1}{2}\right\} = \left\{1 + \frac{1}{2}\left(\frac{39}{16}\right)\right\} = \left\{\frac{71}{32}\right\}.$$

$$w_1 \in P_\rho^T(g_1) = \left\{\frac{71}{32}\right\}. \textit{ So, } w_1 = \frac{71}{32}.$$

$$h_1 = \frac{1}{2}w_1 + \frac{1}{2}v_1 = \frac{1}{2}\left(\frac{71}{32}\right) + \frac{1}{2}\left(\frac{19}{8}\right) = \frac{147}{64}$$

$$f_2 \in P_\rho^T(h_1) = \left\{1 + \frac{h_1}{2}\right\} = \left\{1 + \frac{1}{2}\left(\frac{147}{64}\right)\right\} = \left\{\frac{275}{128}\right\} = \left\{2 + \frac{19}{128}\right\}.$$

*So,* $f_2 = 2 + \frac{19}{128} < 2 + \frac{1}{2}$.

***Iteration 2***:

$$P_\rho^T(f_2) = \left\{1 + \frac{f_2}{2}\right\} = \left\{1 + \frac{1}{2}\left(\frac{275}{128}\right)\right\} = \left\{\frac{531}{256}\right\}$$

$$u_2 \in P_\rho^T(f_2) = \left\{\frac{531}{256}\right\}. \textit{ So, } u_2 = \frac{531}{256}.$$

$$e_2 = \frac{1}{2}f_2 + \frac{1}{2}u_2 = \frac{1}{2}\left(\frac{275}{128}\right) + \frac{1}{2}\left(\frac{531}{256}\right) = \frac{1081}{512}.$$

$$P_\rho^T(e_2) = \left\{1 + \frac{e_2}{2}\right\} = \left\{1 + \frac{1}{2}\left(\frac{1081}{512}\right)\right\} = \left\{\frac{2105}{1024}\right\}.$$

$$v_2 \in P_\rho^T(e_2) = \left\{\frac{2105}{1024}\right\}. \textit{ So, } v_2 = \frac{2105}{1024}.$$

$$g_2 = \frac{1}{2}u_2 + \frac{1}{2}v_2 = \frac{1}{2}\left(\frac{531}{256}\right) + \frac{1}{2}\left(\frac{2105}{1024}\right) = \frac{4229}{2048}.$$

$$P_\rho^T(g_2) = \left\{1 + \frac{g_2}{2}\right\} = \left\{1 + \frac{1}{2}\left(\frac{4229}{2048}\right)\right\} = \left\{\frac{8325}{4096}\right\}.$$

$$w_2 \in P_\rho^T(g_2) = \left\{\frac{8325}{4096}\right\}. \textit{ So, } w_2 = \frac{8325}{4096}.$$

$$h_2 = \frac{1}{2}w_2 + \frac{1}{2}v_2 = \frac{1}{2}\left(\frac{8325}{4096}\right) + \frac{1}{2}\left(\frac{2105}{1024}\right) = \frac{16745}{8192}$$

$$f_3 \in P_\rho^T(h_2) = \left\{1 + \frac{h_2}{2}\right\} = \left\{1 + \frac{1}{2}\left(\frac{16745}{8192}\right)\right\} = \left\{\frac{33129}{16384}\right\}.$$

*So,* $f_3 = \dfrac{33129}{16384} = 2 + \dfrac{361}{16384} < 2 + \dfrac{1}{3}$

*Similarly,*

$$f_4 < 2 + \frac{1}{4},\ f_5 < 2 + \frac{1}{5}, \ldots, f_n < 2 + \frac{1}{n}.$$

*Thus* $f_n \to 2 \in F_\rho(T) = [1,2]$.

## 6. Applications to Differential Equations

Let $\rho \in \mathfrak{R}$. For an unknown function $u : [0,A] \to C$, with $C \subset E_\rho$, consider the initial value problem (IVP)

$$\begin{cases} u(0) = f \\ u'(t) + (I - T)u(t) = 0, \end{cases} \tag{6.1}$$

for fixed $f \in C$, $A > 0$ and $T : C \to C$ with $P_\rho^T$ being $\rho$-quasi-nonexpansive mapping. For any $t > 0$ define

$$K(t) = 1 - e^{-t} = \int_0^t e^{s-t} ds. \tag{6.2}$$

For a function $v : [0,A] \to L_\rho$, $A > 0$, $t \in [0,A]$, define

$$S(v)(t) = \int_0^t e^{s-t} v(s) ds. \tag{6.3}$$

Let us denote

$$S_\tau(v)(t) = \sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i - t} v(t_i), \tag{6.4}$$

for any subdivision $\tau = \{t_{0,} t_1, \ldots, t_n\}$ of $[0,A]$.

**Lemma 6.1.** [8] *Consider a separable* $\rho \in \mathfrak{R}$. *Let* $x, y : [0,A] \to L_\rho$ *be two Bochner-integrable* $\|\cdot\|_\rho$*-bounded functions, with* $A > 0$. *Then* $\forall t \in [0,A]$, *we have*

$$\rho\left(e^{-t} y(t) + \int_0^t e^{s-t} x(s) ds\right) \le e^{-t} \rho\big(y(t)\big) + K(t) \sup_{s \in [0,t]} \rho\big(x(s)\big). \tag{6.5}$$

We now prove our Theorem.

**Theorem 6.2.** *Consider a separable $\rho \in \Re$. Let $D \subset E_\rho$ be non-empty, $\rho$-closed, $\rho$-bounded, convex set having Vitali property, and a multivalued mapping $T : D \to P_\rho(D)$ with $P_\rho^T$ being a $\rho$-quasi-nonexpansive mapping. For fixed $f \in C, A > 0$, we define $u_n : [0,A] \to C$ by*

$$\begin{cases} u_0(t) = f \\ u_{n+1}(t) = e^{-t} f + \int_0^t e^{s-t} T\big(u_n(s)\big) ds. \end{cases} \tag{6.6}$$

*Then $\forall t \in [0,A]$ $\exists u(t) \in C$ such that*

$$\rho\big(u_n(t) - u(t)\big) \to 0 \tag{6.7}$$

*and $u : [0,A] \to C$ defined by (6.6) is a solution of the IVP (6.1).*
*Moreover,*

$$\rho\big(f - u_n(t)\big) \leq K^{n+1}(A)\delta_\rho(C). \tag{6.8}$$

*Proof.* Since $P_\rho^T$ is $\rho$-quasi-nonexpansive mapping, the proof follows the proof of ([8], Theorem 5.28). □

From Theorem 6.2, we obtained the following corollaries:

**Corollary 6.3.** *Consider a separable $\rho \in \Re$. Let $D \subset E_\rho$ be non-empty, $\rho$-closed, $\rho$-bounded, convex set having Vitali property, and a multivalued mapping $T : D \to P_\rho(D)$ with $P_\rho^T$ being a $\rho$-nonexpansive mapping. For fixed $f \in C, A > 0$, we define $u_n : [0,A] \to C$ by*

$$\begin{cases} u_0(t) = f \\ u_{n+1}(t) = e^{-t} f + \int_0^t e^{s-t} T\big(u_n(s)\big) ds. \end{cases} \tag{6.9}$$

*Then $\forall t \in [0,A]$ $\exists u(t) \in C$ such that*

$$\rho\big(u_n(t) - u(t)\big) \to 0 \tag{6.10}$$

*and $u : [0,A] \to C$ defined by (6.9) is a solution of the IVP (6.1).*
*Moreover,*

$$\rho\big(f - u_n(t)\big) \leq K^{n+1}(A)\delta_\rho(C). \tag{6.11}$$

**Corollary 6.4.** *Consider a separable $\rho \in \mathfrak{R}$. Let $D \subset E_\rho$ be non-empty, $\rho$-closed, $\rho$-bounded, convex set having Vitali property, and a multivalued mapping $T : D \to P_\rho(D)$ with $P_\rho^T$ being a $\rho$-contraction mapping. For fixed $f \in C, A > 0$, we define $u_n : [0,A] \to C$ by*

$$\begin{cases} u_0(t) = f \\ u_{n+1}(t) = e^{-t} f + \int_0^t e^{s-t} T\big(u_n(s)\big) ds. \end{cases} \tag{6.12}$$

*Then $\forall t \in [0,A]$ $\exists u(t) \in C$ such that*

$$\rho\big(u_n(t) - u(t)\big) \to 0 \tag{6.13}$$

*and $u : [0,A] \to C$ defined by (6.12) is a solution of the IVP (6.1).*
*Moreover,*

$$\rho\big(f - u_n(t)\big) \leq K^{n+1}(A)\delta_\rho(C). \tag{6.14}$$

## 7. Conclusion

We have proven specific convergence results for the Picard-Abbas iteration for multivalued $\rho$-quasi-nonexpansive mappings in modular function spaces. Furthermore, we proved that the Picard-Abbas iteration exhibits stability. We gave numerical illustrations to reinforce the convergence results, and successfully demonstrated the application of the Picard-Abbas iteration to find solutions of initial value problems in Differential Equations.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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