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GENERALIZED LARGE TRIANGLE–PERIMETER CONTRACTIONS WITH TIME SCALING IN FUZZY b-METRIC SPACES

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Abstract. In this paper, we propose a new class of generalized large triangle–perimeter contractions for single-valued mappings in fuzzy b-metric spaces. The proposed contraction is based on a hybrid geometric functional involving three points and incorporates a time-scaling mechanism. This approach allows us to study mappings that may fail to be contractive at a fixed time parameter but exhibit contractive behavior when observed at varying time scales. We establish a fixed point result confirming the existence and uniqueness of fixed points in complete fuzzy b-metric spaces. The proof technique significantly differs from classical approaches by utilizing time iteration and fuzzy geometric decay mechanism. An application to a nonlinear Volterra-type integral equation with delay is also presented.

Keywords: fuzzy b-metric space; fixed point theorem; generalized geometric contraction; time-scaling method; multi-point mapping; fractional integral equation.

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1. Introduction

Fixed point theory $(\widetilde{FPT})$ is a fundamental tool in nonlinear analysis, with wide-ranging applications in differential equations, optimization, and applied sciences. The classical Banach contraction principle [14] guarantees the existence and uniqueness of fixed points in complete

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metric spaces and serves as a cornerstone for many developments in this area. To address uncertainty and vagueness inherent in real-world problems, fuzzy metric spaces $(\widetilde{FMS})$ were introduced by Kramosil and Michálek [6], and later refined by George and Veeramani [2]. These spaces incorporate a time parameter to describe the degree of nearness between elements, providing a more flexible framework for analysis.

On the other hand, b-metric spaces, introduced by Bakhtin [5] and further developed by Czerwik [15], generalize classical metric spaces by relaxing the triangle inequality via a constant $b \geq 1$. Over the years, numerous generalizations of contraction mappings have been proposed, including those due to Ćirić [8] and Meir–Keeler [3]. More recent developments involve various contractive conditions in both metric and fuzzy settings [13]. In addition, several authors have established important results in extended fuzzy metric spaces (see, e.g., [10]-[12], [17]).

Recently, attention has shifted toward multi-point contractions. In particular, Petrov [4] introduced contraction conditions based on triangle perimeters, opening new directions in $\widetilde{FPT}$. The concept of large contraction, introduced by T. A. Burton, generalizes the classical contraction principle by relaxing the strict Lipschitz condition, thereby enabling the study of a broader class of nonlinear mappings. Moreover, fixed point techniques have proven to be powerful tools in the analysis of integral and fractional differential equations [16, 1, 7], further highlighting their applicability in diverse mathematical models.

Motivated by the above developments, we introduce a new class of generalized large triangle–perimeter contractions in fuzzy b-metric spaces $(\widetilde{FbMS})$. Our approach integrates the fuzzy metric structure [2], the b-metric generalization [15], and triangle–perimeter contraction techniques [4], along with a novel time-scaling mechanism. This framework enables the analysis of mappings that may not exhibit contractive behavior at a fixed time but become contractive under variable time scales, thereby significantly extending existing results in the literature.

2. Preliminaries

In this section, we recall some basic definitions and results used throughout the paper.

Definition 2.1. [6] A $\widetilde{FMS}$ $(\grave{X}, \grave{M}, \maltese)$ where $\grave{X} \neq \phi$, $\maltese$ is a continuous t- norm and $\grave{M} : \grave{X} \times \grave{X} \times (0, \infty) \to [0, 1]$ satisfies:

- $\grave{M}(\varpi,\gamma,t) > 0$;
- $\grave{M}(\varpi,\gamma,t) = 1$ if and only if $\varpi = \gamma$;
- $\grave{M}(\varpi,\gamma,t) = \grave{M}(\gamma,\varpi,t)$;
- $\grave{M}(\varpi,\omega,t+s) \geq \grave{M}(\varpi,\gamma,t) \maltese \grave{M}(\gamma,\omega,s)$;
- $\grave{M}(\varpi,\gamma,\cdot) : (0,\infty) \to [0,1]$ is left continuous.

for all $\varpi,\gamma,\omega \in \grave{X}$ and $t,s > 0$.

We now recall the definition of $\widetilde{FMS}$ in the sense of George and Veeramani.

Definition 2.2. [2] A $\widetilde{FMS}$ $(\grave{X},\grave{M},\maltese)$ where $\grave{X} \neq \phi$, $\maltese$ is a continuous t- norm and $\grave{M} : \grave{X} \times \grave{X} \times (0,\infty) \to [0,1]$ satisfies:

- $\grave{M}(\varpi,\gamma,t) > 0$;
- $\grave{M}(\varpi,\gamma,t) = 1$ if and only if $\varpi = \gamma$;
- $\grave{M}(\varpi,\gamma,t) = \grave{M}(\gamma,\varpi,t)$;
- $\grave{M}(\varpi,\omega,t+s) \geq \grave{M}(\varpi,\gamma,t) \maltese \grave{M}(\gamma,\omega,s)$;
- $\grave{M}(\varpi,\gamma,\cdot) : (0,\infty) \to [0,1]$ is continuous.

for all $\varpi,\gamma,\omega \in \grave{X}$ and $t,s > 0$.

The concept of b-metric space was introduced by I.A. Bakhtin and extensively used by S. Czerwik.

Definition 2.3. [15] Let $\grave{X}$ be a nonempty set and $k \geq 1$ be a given real number. A function $d : \grave{X} \times X \to [0,\infty)$ is a b-metric on $\grave{X}$ if, for all $\varpi,\gamma,\omega \in \grave{X}$, the following conditions hold:

- $d(\varpi,\gamma) = 0$ if and only if $\varpi = \gamma$;
- $d(\varpi,\gamma) = d(\gamma,\varpi)$;
- $d(\varpi,\omega) \leq k[d(\varpi,\gamma) + d(\gamma,\omega)]$.

The triple $(\grave{X},d,k)$ will be called b-metric space.

A $\widetilde{FbMS}$ is a generalization of fuzzy metric and b-metric spaces, and has been investigated by various authors (see, for example, [9]).

Definition 2.4. [9] Let $\grave{X}$ be a nonempty set. Let $k \geq 1$ be a given real number and $\maltese$ be a continuous t-norm. A fuzzy set $\grave{M}$ in $\grave{X} \times \grave{X} \times [0,\infty)$ is called fuzzy b-metric if, for all $\varpi, \gamma, \omega \in \grave{X}$, the following conditions hold:

- $\grave{M}(\varpi,\gamma,0) = 0$;
- $\grave{M}(\varpi,\gamma,t) = 1 \forall t > 0$ if and only if $\varpi = \gamma$;
- $\grave{M}(\varpi,\gamma,t) = \grave{M}(\gamma,\varpi,t) \forall t > 0$;
- $\grave{M}(\varpi,\omega,k(t+s)) \geq \grave{M}(\varpi,\gamma,t) \maltese \grave{M}(\gamma,\omega,s) \forall t,s > 0$;
- $\grave{M}(\varpi,\gamma,\cdot) : [0,\infty) \to [0,1]$ is left continuous and $\lim_{t\to\infty} \grave{M}(\varpi,\gamma,t) = 1$.

The quadruple $(\grave{X},\grave{M},\maltese,k)$ is said to be a $\widetilde{FbMS}$.

Definition 2.5. [9] A sequence $\{\varpi_n\}$ in a $\widetilde{FbMS}$ is said to converge to $\varpi \in X$ if $\grave{M}(\varpi_n,\varpi,t) \to 1$ as $n \to \infty$ for all $t > 0$.

Definition 2.6. [9] A sequence $\{\varpi_n\}$ is called Cauchy if $\grave{M}(\varpi_n,\varpi_m,t) \to 1$ as $n,m \to \infty$.

Definition 2.7. [9] A $\widetilde{FbMS}$ is said to be complete if every Cauchy sequence converges in $\grave{X}$.

Definition 2.8. [4] A mapping $T : \grave{X} \to \grave{X}$ is called a *triangle–perimeter contraction* if there exists a constant $\alpha \in [0,1)$ such that

$$P(T\varpi, T\gamma, T\omega) \leq \alpha P(\varpi,\gamma,\omega),$$

for all pairwise distinct points $\varpi, \gamma, \omega \in X$.

Theorem 2.9. [4] *Let $(\grave{X},d)$ be a complete metric space. If $T : \grave{X} \to \grave{X}$ is a triangle–perimeter contraction, then T has a unique fixed point. Moreover, for every $\varpi_0 \in \grave{X}$, the sequence $\{T^n\varpi_0\}$ converges to this fixed point.*

Definition 2.10. [16] Let $(\grave{X},d)$ be a complete metric space. A mapping $T : \grave{X} \to \grave{X}$ is called a *large contraction* if the following conditions hold:

(1) $d(T\varpi, T\gamma) < d(\varpi,\gamma)$ for all $\varpi \neq \gamma$,
(2) For every $\varepsilon > 0$, there exists a constant $\delta(\varepsilon) \in (0,1)$ such that

$$d(\varpi,\gamma) \geq \varepsilon \;\Rightarrow\; d(T\varpi,T\gamma) \leq \delta(\varepsilon)\, d(\varpi,\gamma), \quad \text{for all } \varpi,\gamma \in \grave{X}.$$

Theorem 2.11. [16] *Let $(\grave{X},d)$ be a complete metric space and let $T:\grave{X}\to\grave{X}$ be a large contraction. If there exist $\varpi_0\in\grave{X}$ and $L>0$ such that*

$$d(\varpi_0,T^n\varpi_0)\le L \quad \text{for all } n\in\mathbb{N},$$

then T has a unique fixed point in $\grave{X}$.

3. Main Results

Definition 3.1. Define the generalized triangle-perimeter functional:

$$\mathscr{P}_{\grave{M}}(\varpi,\gamma,\omega,t)=\max\{1-\grave{M}(\varpi,\gamma,t),1-\grave{M}(\gamma,\omega,t),1-\grave{M}(\omega,\varpi,t)\}+\lambda\sum_{cyc}(1-\grave{M}(\varpi,\gamma,t)),$$

where $\lambda\in(0,1)$.

A mapping $T:\grave{X}\to\grave{X}$ is called a generalized large triangle-perimeter contraction if:

(1) $\mathscr{P}_{\grave{M}}(T\varpi,T\gamma,T\omega,t)<\mathscr{P}_{\grave{M}}M(\varpi,\gamma,\omega,t)$,

(2) For every $\varepsilon>0$, there exists $\delta(\varepsilon)\in(0,1)$ and $\phi(t)>t$ such that

$$\mathscr{P}_{\grave{M}}(T\varpi,T\gamma,T\omega,t)\le\delta(\varepsilon)\mathscr{P}_{\grave{M}}(\varpi,\gamma,\omega,\phi(t)).$$

Theorem 3.2. *Let $(\grave{X},\grave{M},\maltese,k)$ be a complete $\widetilde{FbMS}$. Then every generalized large triangle–perimeter contraction $T:\grave{X}\to\grave{X}$ has a unique fixed point.*

Proof. Let $\varpi_0\in\grave{X}$ and define the sequence $\{\varpi_n\}$ by

$$\varpi_{n+1}=T\varpi_n,\quad n\ge 0.$$

Define

$$P_n(t)=P_{\grave{M}}(\varpi_n,\varpi_{n+1},\varpi_{n+2},t).$$

Step 1: Strict decrease.

If for some n, $\varpi_n=T\varpi_n$, then ϖ_n is a fixed point. Otherwise, assume $\varpi_n\ne T\varpi_n$ for all n. By condition (i),

$$P_{n+1}(t)=P_{\grave{M}}(T\varpi_n,T\varpi_{n+1},T\varpi_{n+2},t)<P_{\grave{M}}(\varpi_n,x_{n+1},\varpi_{n+2},t)=P_n(t).$$

Thus $\{P_n(t)\}$ is strictly decreasing and bounded below by 0, hence convergent. Let

$$\lim_{n\to\infty}P_n(t)=\ell\ge 0.$$

Step 2: Time-scaled contraction.

From condition (ii), for every $\varepsilon > 0$, there exists $\delta(\varepsilon) \in (0,1)$ and $\phi(t) > t$ such that

$$P_{n+1}(t) \leq \delta(\varepsilon)\, P_n(\phi(t)).$$

Iterating this inequality, we obtain

$$P_n(t) \leq \delta(\varepsilon)^n\, P_0(\phi^n(t)).$$

Since $\delta(\varepsilon) \in (0,1)$ and $P_0(\phi^n(t))$ is bounded, it follows that

$$P_n(t) \to 0 \quad \text{as } n \to \infty.$$

Hence $\ell = 0$.

Step 3: Cauchy property.

Since $P_n(t) \to 0$, we have

$$1 - \grave{M}(\varpi_n, x_{n+1}, t) \to 0,$$

which implies

$$\grave{M}(\varpi_n, \varpi_{n+1}, t) \to 1 \quad \text{as } n \to \infty.$$

Now, using the fuzzy b-metric inequality repeatedly, for any $m > n$, we obtain

$$\grave{M}(\varpi_n, \varpi_m, t) \geq \grave{M}(\varpi_n, \varpi_{n+1}, t_1) \maltese \grave{M}(\varpi_{n+1}, \varpi_{n+2}, t_2) \maltese \cdots \maltese \grave{M}(\varpi_{m-1}, \varpi_m, t_{m-n}),$$

where $t_1 + t_2 + \cdots + t_{m-n} = t/b^k$ for suitable k.

Since each term tends to 1, it follows that

$$\grave{M}(\varpi_n, \varpi_m, t) \to 1 \quad \text{as } n, m \to \infty.$$

Thus $\{\varpi_n\}$ is a Cauchy sequence in $\grave{X}$.

Step 4: Existence of fixed point.

Since $(\grave{X}, \grave{M}, \maltese, k)$ is complete, there exists $\varpi^* \in \grave{X}$ such that

$$\varpi_n \to \varpi^*.$$

Now,

$$\grave{M}(\varpi_n, T\varpi_n, t) = \grave{M}(\varpi_n, \varpi_{n+1}, t) \to 1.$$

Using the continuity of M and the convergence $\varpi_n \to \varpi^*$, we obtain

$$\grave{M}(\varpi^*, T\varpi^*, t) = \lim_{n\to\infty} \grave{M}(\varpi_n, T\varpi_n, t) = 1.$$

Hence,

$$T\varpi^* = \varpi^*.$$

Step 5: Uniqueness.

Assume that $\gamma^* \in \grave{X}$ is another fixed point with $\gamma^* \neq \varpi^*$.

Then, using condition (ii), for sufficiently large n, we have

$$P_{\grave{M}}(T\varpi^*, T\gamma^*, T\varpi_n, t) \leq \delta P_{\grave{M}}(\varpi^*, \gamma^*, \varpi_n, \phi(t)).$$

Since $T\varpi^* = \varpi^*$ and $T\gamma^* = \gamma^*$, this reduces to

$$P_{\grave{M}}(\varpi^*, \gamma^*, \varpi_{n+1}, t) \leq \delta P_{\grave{M}}(\varpi^*, \gamma^*, \varpi_n, \phi(t)).$$

Iterating,

$$P_{\grave{M}}(\varpi^*, \gamma^*, \varpi_n, t) \leq \delta^n P_{\grave{M}}(\varpi^*, \gamma^*, \varpi_0, \phi^n(t)) \to 0.$$

Taking limit as $n \to \infty$, we get

$$\grave{M}(\varpi^*, \gamma^*, t) = 1,$$

which implies

$$\varpi^* = \gamma^*,$$

a contradiction.

Thus the fixed point is unique. □

Corollary 3.3. *Let $(\grave{X}, d)$ be a complete metric space. Suppose that the fuzzy b-metric reduces to a standard metric (i.e., $b = 1$ and $\grave{M}(\varpi, \gamma, t) = \frac{t}{t+d(\varpi,\gamma)}$) and the time-scaling function satisfies $\phi(t) = t$. If $T : \grave{X} \to \grave{X}$ satisfies the triangle–perimeter contraction condition*

$$P(T\varpi, T\gamma, T\omega) \leq \alpha P(\varpi, \gamma, \omega), \quad \alpha \in [0, 1),$$

for all pairwise distinct $\varpi, \gamma, \omega \in \grave{X}$, then T has a unique fixed point in $\grave{X}$.

Remark 3.4. The obtained result generalizes the Banach contraction principle and triangle–perimeter contractions to fuzzy b-metric spaces. The incorporation of a time-scaling function $\phi(t)$ and a multi-point functional allows the treatment of mappings that are not contractive in the classical sense. Thus, the result provides a more flexible and unified framework for fixed point theory.

4. Illustrative Example

Let $\grave{X} = [0,1]$ and define the fuzzy b-metric

$$\grave{M}(\varpi,\gamma,t) = \frac{t}{t+|\varpi-\gamma|^p}, \quad t>0,\ p>1,$$

with the product t-norm $a \maltese b = ab$.

Then $(\grave{X},\grave{M},\maltese)$ is a complete fuzzy b-metric space with $b = 2^{p-1} > 1$.

Define the mapping $T : \grave{X} \to \grave{X}$ by

$$T(\varpi) = \frac{\varpi}{1+\varpi} + \frac{1}{3}\varpi^2 \sin(\varpi).$$

Step 1: T maps $\grave{X}$ into $\grave{X}$

For $\varpi \in [0,1]$, we have

$$0 \le \frac{\varpi}{1+\varpi} \le \frac{1}{2}, \quad 0 \le \varpi^2 \sin(\varpi) \le 1.$$

Thus,

$$0 \le T(\varpi) \le \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1.$$

Hence, $T(\grave{X}) \subset \grave{X}$.

Step 2: T is not a Banach contraction

Compute the derivative:

$$T'(\varpi) = \frac{1}{(1+\varpi)^2} + \frac{1}{3}\big(2\varpi\sin\varpi + \varpi^2\cos\varpi\big).$$

At $\varpi = 1$,

$$T'(1) = \frac{1}{4} + \frac{1}{3}\big(2\sin 1 + \cos 1\big).$$

Since $\sin 1 > 0$ and $\cos 1 > 0$, we get $T'(1) > 1$.

Hence, T is not a Banach contraction.

Step 3: Uniform Lipschitz-type estimate

Using standard inequalities:

$$\left|\frac{\varpi}{1+\varpi}-\frac{\gamma}{1+\gamma}\right|=\frac{|\varpi-\gamma|}{(1+\varpi)(1+\gamma)}\leq|\varpi-\gamma|,$$

and

$$|\varpi^2\sin\varpi-\gamma^2\sin\gamma|\leq|\varpi^2\sin\varpi-\gamma^2\sin\varpi|+|\gamma^2(\sin\varpi-\sin\gamma)|.$$

Now,

$$|\varpi^2-\gamma^2|=|\varpi-\gamma||\varpi+\gamma|\leq 2|\varpi-\gamma|,$$

and

$$|\sin\varpi-\sin\gamma|\leq|\varpi-\gamma|.$$

Thus,

$$|\varpi^2\sin\varpi-\gamma^2\sin\gamma|\leq 2|\varpi-\gamma|+|\varpi-\gamma|=3|\varpi-\gamma|.$$

Therefore,

$$|T(\varpi)-T(\gamma)|\leq|\varpi-\gamma|+\frac{1}{3}\cdot 3|\varpi-\gamma|=2|\varpi-\gamma|.$$

Step 4: Key estimate in fuzzy b-metric

We compute:

$$1-\grave{M}(T\varpi,T\gamma,t)=\frac{|T(\varpi)-T(\gamma)|^p}{t+|T(\varpi)-T(\gamma)|^p}.$$

Using Step 3:

$$|T(\varpi)-T(\gamma)|\leq 2|\varpi-\gamma|,$$

so

$$|T(\varpi)-T(\gamma)|^p\leq 2^p|\varpi-\gamma|^p.$$

Thus,

$$1-\grave{M}(T\varpi,T\gamma,t)\leq\frac{2^p|\varpi-\gamma|^p}{t+2^p|\varpi-\gamma|^p}.$$

Step 5: Time scaling

Define

$$\phi(t)=2t.$$

Then,

$$1-\grave{M}(\varpi,\gamma,\phi(t))=\frac{|\varpi-\gamma|^p}{2t+|\varpi-\gamma|^p}.$$

Step 6: Construction of $\delta(\varepsilon)$

Assume $|\varpi-\gamma|\geq\varepsilon>0$. Then

$$1-\grave{M}(T\varpi,T\gamma,t)\leq\frac{2^p|\varpi-\gamma|^p}{t+2^p\varepsilon^p}.$$

Also,

$$1-\grave{M}(\varpi,\gamma,\phi(t))=\frac{|\varpi-\gamma|^p}{2t+|\varpi-\gamma|^p}\geq\frac{\varepsilon^p}{2t+1}.$$

Thus,

$$\frac{1-\grave{M}(T\varpi,T\gamma,t)}{1-\grave{M}(\varpi,\gamma,\phi(t))}\leq 2^p\cdot\frac{2t+1}{t+2^p\varepsilon^p}.$$

Define

$$\delta(\varepsilon)=\sup_{t>0}2^p\cdot\frac{2t+1}{t+2^p\varepsilon^p}.$$

Since the function is bounded for $t>0$, we can choose $\varepsilon>0$ sufficiently large such that $\delta(\varepsilon)\in(0,1)$.

Hence,

$$1-\grave{M}(T\varpi,T\gamma,t)\leq\delta(\varepsilon)\big(1-\grave{M}(\varpi,\gamma,\phi(t))\big).$$

Step 7: Triangle–perimeter contraction

Applying the inequality obtained in Step 6 to the pairs (ϖ,γ), (γ,ω), and (ω,ϖ), we obtain

$$P_{\grave{M}}(T\varpi,T\gamma,T\omega,t)\leq\delta(\varepsilon)P_{\grave{M}}(\varpi,\gamma,\omega,\phi(t)).$$

Step 8: Conclusion

Thus, T satisfies the generalized large triangle-perimeter contraction condition in the complete $\widetilde{FbMS}$ $(\grave{X},\grave{M},\maltese)$ with $b=2^{p-1}>1$.

Hence by Theorem 3.2, T has a unique fixed point in $\grave{X}$.

5. Application to a Nonlinear Volterra Integral Equation with State-Dependent Delay

In this section, we apply our main result to a nonlinear Volterra-type integral equation with state-dependent delay, which arises in advanced models of biological systems, neural networks, and control processes with feedback.

Consider the equation:

$$f(\varpi) = h(\varpi) + \int_0^{\varpi} K\big(\varpi, s, f(s), f(\psi(s, f(s)))\big)\, ds, \quad \varpi \in [0,1], \tag{1}$$

where $\psi(s, f(s))$ represents a *state-dependent delay* satisfying $\psi(s, f(s)) \in [0,1]$.

Real-world interpretation

This model describes systems where:

- The present state depends on accumulated past states (integral term),
- The delay is not constant but depends on the current system state,
- The system exhibits nonlinear feedback.

Such models arise in:

- Neural networks with adaptive delay,
- Population dynamics with environment-dependent maturation time,
- Engineering systems with state-based control delays.

Step 1: Functional framework

Let $\grave{X} = C([0,1])$ with norm $\|f\| = \sup_{\varpi \in [0,1]} |f(\varpi)|$.

Define the fuzzy metric:

$$\grave{M}(f, g, t) = \frac{t}{t + \|f - g\|^p}, \quad t > 0,\ p > 1.$$

Then $(\grave{X}, \grave{M}, *)$ is a complete fuzzy b-metric space with $b = 2^{p-1} > 1$.

Step 2: Operator formulation

Define $T : \grave{X} \to \grave{X}$ by

$$(Tf)(\varpi) = h(\varpi) + \int_0^{\varpi} K\big(x, s, f(s), f(\psi(s, f(s)))\big)\, ds.$$

Then the solution of the equation corresponds to a fixed point of T.

Step 3: Assumptions on the kernel

Assume:

(1) K is continuous,

(2) There exists $L > 0$ such that

$$|K(\varpi, s, u_1, v_1) - K(\varpi, s, u_2, v_2)| \leq L(|u_1 - u_2| + |v_1 - v_2|),$$

(3) The delay function ψ satisfies

$$|\psi(s, u_1) - \psi(s, u_2)| \leq \lambda |u_1 - u_2|, \quad \lambda \in (0, 1).$$

Step 4: Lipschitz-type estimate

For $f, g \in \grave{X}$, we estimate:

$$|Tf(\varpi) - Tg(\varpi)| \leq \int_0^{\varpi} |K\big(\varpi, s, f(s), f(\psi(s, f(s)))\big) - K\big(\varpi, s, g(s), g(\psi(s, g(s)))\big)| ds.$$

Using the Lipschitz condition:

$$\leq L \int_0^{\varpi} \Big(|f(s) - g(s)| + |f(\psi(s, f(s))) - g(\psi(s, g(s)))|\Big) ds.$$

Now,

$$|f(\psi(s, f(s))) - g(\psi(s, g(s)))| \leq |f(\psi(s, f(s))) - f(\psi(s, g(s)))| + |f(\psi(s, g(s))) - g(\psi(s, g(s)))|.$$

Using continuity and Lipschitz property:

$$\leq L_1 |\psi(s, f(s)) - \psi(s, g(s))| + \|f - g\| \leq L_1 \lambda |f(s) - g(s)| + \|f - g\|.$$

Thus,

$$|Tf(\varpi) - Tg(\varpi)| \leq C \|f - g\|,$$

for some constant $C > 0$.

Hence,

$$\|Tf - Tg\| \leq C \|f - g\|.$$

Step 5: Fuzzy contraction estimate

We have:

$$1-\grave{M}(Tf,Tg,t)=\frac{\|Tf-Tg\|^p}{t+\|Tf-Tg\|^p}\leq\frac{C^p\|f-g\|^p}{t+C^p\|f-g\|^p}.$$

Define $\phi(t)=2t$. Then:

$$1-\grave{M}(f,g,\phi(t))=\frac{\|f-g\|^p}{2t+\|f-g\|^p}.$$

Proceeding as in the main theorem, we obtain:

$$1-\grave{M}(Tf,Tg,t)\leq\delta(\varepsilon)\,(1-\grave{M}(f,g,\phi(t))),$$

for some $\delta(\varepsilon)\in(0,1)$.

Step 6: Triangle-perimeter contraction

Applying the above inequality to $f,g,h\in\grave{X}$ and summing, we obtain:

$$P_{\grave{M}}(Tf,Tg,Th,t)\leq\delta(\varepsilon)P_{\grave{M}}(f,g,h,\phi(t)).$$

Step 7: Application of main theorem

All the conditions of Theorem 3.2 are satisfied. Hence, T admits a unique fixed point in $\grave{X}$.

Conclusion and Interpretation

The nonlinear integral equation with state-dependent delay admits a unique solution.

This implies:

- Stability of systems with adaptive delay,
- Robust convergence under nonlinear feedback,
- Applicability to complex real-world systems where delay depends on system state.

6. CONCLUSION

In this paper, we proposed a new class of generalized contractions based on triangle–perimeter geometry and time-scaling in the setting of $\widetilde{FbMS}$. The developed framework extends both classical and modern fixed point results, including Banach-type contractions and large triangle–perimeter contractions, by incorporating a time-dependent notion of nearness.

The inclusion of a time-scaling function enables the contractive condition to be satisfied at varying time levels, thereby significantly enlarging the class of admissible mappings. This approach provides a flexible and effective tool for the analysis of nonlinear problems under uncertainty.

The applicability of the obtained results has been illustrated through a nontrivial example, demonstrating that mappings which are not contractions in the classical sense can still satisfy the proposed conditions.

Future research may focus on extending the present framework to multivalued and hybrid mappings, as well as to more general settings such as fuzzy S-metric, partial metric, and modular metric spaces. It would also be of interest to investigate analogous results under weaker or implicit contractive conditions, and to explore stability analysis, data dependence, and well-posedness of the associated fixed point problems. Furthermore, potential applications to dynamical systems, optimization theory, and control problems offer promising directions for further study.

Conflict of Interests

The authors declare that there is no conflict of interests.

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