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A REFINED MULTI-STEP ITERATIVE SCHEME FOR THE APPROXIMATION OF FIXED POINTS OF NONLINEAR MAPPINGS IN BANACH SPACE

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Abstract. In this paper, we introduce a refined multi-step iterative scheme for approximating fixed points of nonlinear mappings in Banach spaces. The proposed method is designed to enhance convergence speed while maintaining stability and robustness. Under suitable conditions on the control parameters, we establish strong convergence results for contraction mappings and prove that the generated sequence converges to the unique fixed point. Furthermore, stability analysis of the proposed scheme is investigated, and it is shown to be T -stable under standard assumptions. A comparative study based on the concept of convergence rate demonstrates that the proposed scheme outperforms several classical and modern iterative methods, including Picard, Mann, Ishikawa, and other multi-step methods. To validate the theoretical findings, numerical experiments are carried out on various classes of nonlinear mappings, including rational, monotone, cubic, and logarithmic functions. The results confirm that the proposed method achieves higher accuracy with fewer iterations, thereby offering improved computational efficiency. Consequently, the scheme provides a reliable and effective tool for solving nonlinear operator equations in Banach spaces.

Keywords: fixed point theory; multi-step iteration; nonlinear mappings; Banach spaces; rate of convergence; Stability.

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1. INTRODUCTION

Fixed point theory is a fundamental area of nonlinear analysis with wide applications in mathematics, engineering, optimization, economics, and applied sciences [1, 2, 17]. Many real-life problems can be formulated as fixed point problems of the form

$$(1) \quad x = Tx,$$

where T is a nonlinear mapping defined on a subset C of a Banach space X . The importance of fixed point theory lies in its ability to transform diverse mathematical models such as differential equations, integral equations, variational inequalities, and equilibrium problems into operator equations [9, 10, 11].

In most practical situations, exact analytical solutions to such problems are difficult or impossible to obtain. Consequently, iterative methods have become essential tools for approximating fixed points of nonlinear mappings. One of the earliest and most widely used methods is the Picard iteration, defined by

$$(2) \quad \begin{cases} x_0 \in C, \\ x_{n+1} = Tx_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

which converges strongly under the Banach contraction principle [2]. However, despite its simplicity, the Picard iteration requires strong contractive conditions and may converge slowly when the contraction constant is close to unity [3].

To overcome these limitations, several generalizations of the Picard iteration have been introduced. The Mann iteration [12] is defined by

$$(3) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\} \subset (0, 1)$. Although the Mann iteration extends applicability beyond strict contractions, it often exhibits slow convergence and may converge only weakly in Banach spaces [6].

An improvement of the Mann iteration is the Ishikawa iteration [9], given by

$$(4) \quad \begin{cases} y_n = (1 - \beta_n)x_n + \beta_nTx_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

which introduces an intermediate step to enhance convergence behavior. This development marked a transition from single-step to multi-step iterative schemes.

In recent years, multi-step iterative methods have received considerable attention due to their improved convergence properties. For instance, Karakaya et al. [10] proposed a three-step iterative scheme defined by

$$(5) \quad \begin{cases} z_n = Tx_n, \\ y_n = (1 - \alpha_n)z_n + \alpha_n Tz_n, \\ x_{n+1} = Ty_n, \quad \forall n \geq 1. \end{cases}$$

while Ullah and Arshad [16] introduced the iterative process

$$(6) \quad \begin{cases} z_n = (1 - \alpha_n)x_n + \alpha_n Tx_n, \\ y_n = Tz_n, \\ x_{n+1} = Ty_n, \quad \forall n \geq 1. \end{cases}$$

Similarly, Abbas et al. [1] developed a related three-step iterative method given by

$$(7) \quad \begin{cases} z_n = Tx_n, \\ y_n = Tz_n, \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n Ty_n, \quad \forall n \geq 1. \end{cases}$$

and established that it has comparable convergence properties with other multi-step methods.

More recently, Okeke et al. [13] proposed the AG-iterative scheme defined by

$$(8) \quad \begin{cases} u_n = (1 - \gamma_n)x_n + \gamma_n Tx_n, \\ w_n = (1 - \beta_n)Tx_n + \beta_n Tu_n, \\ v_n = T[(1 - \alpha_n)w_n + \alpha_n Tw_n], \\ x_{n+1} = Tv_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

and showed that it converges faster than several existing iterative schemes. More recently, Thakur et al. [15] introduced a three-step iterative scheme defined by

$$(9) \quad \begin{cases} z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1 - \beta_n)z_n + \beta_n T z_n, \\ x_{n+1} &= (1 - \alpha_n)T z_n + \alpha_n T y_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$. The authors established strong convergence results and demonstrated that the scheme outperforms several classical iterative methods.

Despite these significant developments, many existing iterative schemes still suffer from slow convergence rates, restrictive assumptions, or a lack of comprehensive analysis of stability and data dependence [3, 4, 7, 8]. Consequently, there is a need for more efficient iterative algorithms that combine fast convergence with strong theoretical guarantees.

Motivated by these observations, we propose a refined multi-step iterative scheme for approximating fixed points of nonlinear mappings in Banach spaces. The proposed method is designed to improve convergence rate relative to existing schemes while preserving strong convergence, stability, and robustness. In addition, we establish convergence results, analyze the convergence rate of the proposed scheme, and compare its performance with several well-known iterative methods in the literature.

2. PRELIMINARIES

In this section, we present some fundamental definitions and results that will be used throughout this paper. Let X be a real Banach space and C be a nonempty subset of X . Let $T : C \rightarrow C$ be a nonlinear mapping. For standard concepts in nonlinear functional analysis, we refer to [5, 6, 7, 8, 14].

Definition 2.1. A point $x^* \in C$ is called a fixed point of T if

$$(10) \quad T x^* = x^*.$$

The set of all fixed points of T is denoted by $F(T)$.

Definition 2.2. A mapping $T : C \rightarrow C$ is called a contraction if there exists a constant $k \in (0, 1)$ such that

$$(11) \quad \|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

Theorem 2.3 (Banach Contraction Principle [2]). *Let C be a nonempty closed subset of a Banach space X and let $T : C \rightarrow C$ be a contraction mapping. Then T has a unique fixed point $x^* \in C$, and for any $x_0 \in C$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges strongly to x^* .*

Definition 2.4. Let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to converge strongly to $x \in X$ if

$$(12) \quad \lim_{n \rightarrow \infty} \|x_n - x\| = 0.$$

Definition 2.5. Let $\{x_n\}$ and $\{y_n\}$ be two sequences converging to the same point x^* . Then $\{x_n\}$ is said to converge faster than $\{y_n\}$ if

$$(13) \quad \lim_{n \rightarrow \infty} \frac{\|x_n - x^*\|}{\|y_n - x^*\|} = 0.$$

The above concept provides a useful tool for comparing the efficiency of iterative methods and has been widely used in the literature [3].

Lemma 2.6 (Berinde [3]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive real numbers converging to zero. If*

$$(14) \quad \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0,$$

then $\{a_n\}$ converges faster than $\{b_n\}$.

Definition 2.7. An iterative process $x_{n+1} = f(T, x_n)$ converging to a fixed point p is said to be T -stable if for any sequence $\{t_n\}$ in C , the condition

$$(15) \quad \lim_{n \rightarrow \infty} \|t_{n+1} - f(T, t_n)\| = 0$$

implies that $\lim_{n \rightarrow \infty} t_n = p$.

The concept of stability plays a crucial role in numerical analysis, especially in the presence of computational errors and perturbations [7, 8].

Lemma 2.8 ([7, 8]). *Let $\{\delta_n\}$ be a sequence of nonnegative real numbers satisfying*

$$(16) \quad \delta_{n+1} \leq (1 - \alpha_n)\delta_n + \theta_n,$$

where $\alpha_n \in (0, 1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} = 0$. Then $\lim_{n \rightarrow \infty} \delta_n = 0$.

Lemma 2.9. *Let $\{\delta_n\}$ be a sequence of nonnegative real numbers satisfying*

$$(17) \quad \delta_{n+1} \leq (1 - \alpha_n)\delta_n,$$

where $\alpha_n \in (0, 1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\lim_{n \rightarrow \infty} \delta_n = 0$.

3. PROPOSED ALGORITHM

In this section, we introduce a new refined multi-step iterative scheme for approximating fixed points of nonlinear mappings in Banach spaces. The proposed method is motivated by the need to improve the convergence rate of existing iterative processes while preserving stability and robustness.

Let C be a nonempty subset of a Banach space X and let $T : C \rightarrow C$ be a nonlinear mapping. For an arbitrary initial point $x_0 \in C$, we define the sequence $\{x_n\}$ by the following iterative algorithm:

Algorithm 3.1. (AJ Iterative Scheme)

$$(18) \quad \begin{cases} x_0 \in C, \\ u_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \\ v_n = T u_n, \\ y_n = T [(1 - \beta_n)x_n + \beta_n T v_n], \\ w_n = T [(1 - \gamma_n)y_n + \gamma_n T y_n], \\ x_{n+1} = T w_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$.

The above iterative scheme consists of a five-step evaluation process that incorporates multiple operator corrections at each stage. This structure is designed to accelerate convergence by progressively refining the approximation through nested operator applications.

Remark 3.1. The proposed iterative scheme generalizes several existing iterative methods in the literature. In particular, by appropriate choices of the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$, the scheme reduces to known iterative processes such as the Mann, Ishikawa, and certain multi-step methods. This flexibility highlights the robustness of the proposed algorithm.

4. MAIN RESULTS

In this section, we establish convergence, stability, and convergence rate results for the proposed iterative scheme.

Theorem 4.1. *Let $T : C \rightarrow C$ be a contraction with constant $\delta \in (0, 1)$ and $F(T) \neq \emptyset$. Suppose the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfy*

$$\sum_{n=0}^{\infty} \gamma_n = \infty.$$

Then, the sequence $\{x_n\}$ generated by the AJ iterative scheme converges strongly to the unique fixed point $x^ \in F(T)$.*

Proof. Let $x^* \in F(T)$ so that $x^* = Tx^*$. Using algorithm 18, we have

$$\begin{aligned} \|u_n - x^*\| &= \|(1 - \alpha_n)x_n + \alpha_n Tx_n - x^*\| \\ &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(Tx_n - x^*)\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|Tx_n - Tx^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\delta\|x_n - x^*\| \\ (19) \quad &= (1 - \alpha_n(1 - \delta))\|x_n - x^*\|. \end{aligned}$$

In addition, using Algorithm 18 and (19), we obtain

$$\begin{aligned} \|v_n - x^*\| &= \|Tu_n - Tx^*\| \leq \delta\|u_n - x^*\| \\ (20) \quad &\leq \delta(1 - \alpha_n(1 - \delta))\|x_n - x^*\|. \end{aligned}$$

Also, using Algorithm 18 and (20), we obtain

$$\|y_n - x^*\| = \|T((1 - \beta_n)Tx_n + \beta_n Tv_n) - Tx^*\|$$

$$\begin{aligned}
&\leq \delta \|(1 - \beta_n)Tx_n + \beta_nTv_n - x^*\| \\
&= \delta \|(1 - \beta_n)(Tx_n - x^*) + \beta_n(Tv_n - x^*)\| \\
&\leq \delta ((1 - \beta_n)\|Tx_n - x^*\| + \beta_n\|Tv_n - x^*\|) \\
&\leq \delta ((1 - \beta_n)\delta\|x_n - x^*\| + \beta_n\delta\|v_n - x^*\|) \\
&= \delta^2(1 - \beta_n)\|x_n - x^*\| + \delta^2\beta_n\|v_n - x^*\|. \\
&\leq \delta^2[1 - \delta\beta_n\alpha_n(1 - \delta)]\|x_n - x^*\| \\
(21) \quad &\leq \delta^2\|x_n - x^*\|
\end{aligned}$$

Also, using Algorithm 18 and (21), we obtain

$$\begin{aligned}
\|w_n - x^*\| &\leq \delta \|(1 - \gamma_n)y_n + \gamma_nTy_n - x^*\| \\
&= \delta \|(1 - \gamma_n)(y_n - x^*) + \gamma_n(Ty_n - x^*)\| \\
&\leq \delta ((1 - \gamma_n)\|y_n - x^*\| + \gamma_n\|Ty_n - x^*\|) \\
&\leq \delta ((1 - \gamma_n)\|y_n - x^*\| + \gamma_n\delta\|y_n - x^*\|) \\
&= \delta(1 - \gamma_n(1 - \delta))\|y_n - x^*\| \\
(22) \quad &\leq \delta^3(1 - \gamma_n(1 - \delta))\|x_n - x^*\|.
\end{aligned}$$

Finally, using Algorithm 18 and (22), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|Tw_n - Tx^*\| \leq \delta\|w_n - x^*\| \\
(23) \quad &\leq \delta^4(1 - \gamma_n(1 - \delta))\|x_n - x^*\|.
\end{aligned}$$

From (23), we have the inequality

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \delta^4(1 - \gamma_n(1 - \delta))\|x_n - x^*\| \\
\|x_n - x^*\| &\leq \delta^4(1 - \gamma_{n-1}(1 - \delta))\|x_{n-1} - x^*\| \\
&\vdots \\
(24) \quad \|x_1 - x^*\| &\leq \delta^4(1 - \gamma_0(1 - \delta))\|x_0 - x^*\|.
\end{aligned}$$

From (24), we have that

$$(25) \quad \|x_{n+1} - x^*\| \leq \delta^{4(n+1)} \|x_0 - x^*\| \prod_{m=0}^n (1 - \gamma_m(1 - \delta)).$$

Since $\{\alpha_m\}, \{\gamma_m\}$ and $\{\beta_m\}$ are sequences in $(0, 1)$ for all $m \in \mathbb{N}$. We have that $(1 - \gamma_m(1 - \delta)) \in (0, 1)$. Using the inequality that $1 - x \leq e^{-x}$ for all $x \in [0, 1]$, hence from (25), we have

$$(26) \quad \|x_{n+1} - x^*\| \leq \frac{\delta^{4(n+1)} \|x_0 - x^*\|}{e^{(1-\delta) \sum_{m=0}^n \gamma_m}}.$$

Taking the limit of both sides of the above inequalities, we have $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$. Therefore, the sequence $\{x_n\}$ converges strongly to the fixed point x^* . $\square$

Theorem 4.2 (Stability). *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X . Let T be a contraction mapping with constant δ such that $F(T) \neq \emptyset$. Suppose that the iterative schemes 18 satisfies*

$$\sum_{n=0}^{\infty} \gamma_n = \infty \quad \text{and} \quad \gamma_n \leq \gamma \in (0, 1) \quad \text{for each } n \in \mathbb{N},$$

for $\{x_n\}_{n=1}^{\infty}$ converges to x^* . Hence, the iterative scheme (18) is T -stable.

Proof. Suppose that $\{t_n\}$ is an arbitrary sequence. Let $\varepsilon_n = \|t_n - Tx_{n+1}\|$, where $x_{n+1} = Tw_n$.

Now using Definition 2.7 and Algorithm (18), observe that

$$\begin{aligned} \|t_n - x^*\| &\leq \|t_n - Tx_{n+1}\| + \|Tx_{n+1} - x^*\| \\ &= \varepsilon_n + \|Tx_{n+1} - Tx^*\| \\ &\leq \varepsilon_n + \delta \|x_{n+1} - x^*\| \\ &= \varepsilon_n + \delta \|Tw_n - Tx^*\| \\ &\leq \varepsilon_n + \delta^2 \|w_n - x^*\| \\ &\leq \varepsilon_n + \delta^2 [\delta^3 (1 - \gamma_n(1 - \delta)) \|x_n - x^*\|] \\ (27) \quad &= \varepsilon_n + \delta^5 (1 - \gamma_n(1 - \delta)) \|x_n - x^*\|. \end{aligned}$$

Note that $\gamma_n \leq \gamma \in (0, 1)$ for all $n \in \mathbb{N}$. It follows from Lemma (2.15) that

$$\lim_{n \rightarrow \infty} \|t_n - x^*\| = 0.$$

Conversely, suppose that $\lim_{n \rightarrow \infty} \|t_n - x^*\| = 0$. We have that

$$\begin{aligned}
 \varepsilon_n &= \|t_n - Tx_{n+1}\| \\
 &\leq \|t_n - x^*\| + \|x^* - Tx_{n+1}\| \\
 &= \|t_n - x^*\| + \|Tx_{n+1} - Tx^*\| \\
 &\leq \|t_n - x^*\| + \delta \|x_{n+1} - x^*\| \\
 (28) \quad &\leq \|t_n - x^*\| + \delta^5(1 - \gamma_n(1 - \delta))\|x_n - x^*\|.
 \end{aligned}$$

Taking limit of both sides, we have that

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0.$$

□

Theorem 4.3 (Rate of Convergence). *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X . Let T be a contraction mapping with constant k such that $F(T) \neq \emptyset$. Suppose that each of the iterative schemes (5), (6), (7), (9), and (8) converge to same fixed point x^* of T , where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ satisfying $\alpha \leq \alpha_n < 1$, $\beta\gamma \leq \beta_n\gamma_n < 1$ and $\alpha\beta\gamma \leq \alpha_n\beta_n\gamma_n < 1$. Then, the (18) scheme converges faster than all other iterative schemes above.*

Proof. From Theorem 19, we have that

$$(29) \quad \|x_{n+1} - x^*\| \leq \delta^{4(n+1)} \|x_0 - x^*\| \prod_{m=0}^n (1 - \gamma_m(1 - \delta)).$$

Using $\gamma \leq \gamma_n < 1$, we have

$$(30) \quad \|x_{n+1} - x^*\| \leq \delta^{4(n+1)} \|x_0 - x^*\| (1 - \gamma(1 - \delta))^{n+1}.$$

Let

$$\theta = \delta^{4(n+1)} \|x_0 - x^*\| (1 - \gamma(1 - \delta))^{n+1}.$$

Using a similar approach as in Theorem 19, we obtain that the iterative process (7) takes the form

$$(31) \quad \|x_{n+1} - x^*\| \leq \delta^{2(n+1)} \|x_0 - x^*\| (1 - \alpha(1 - \delta))^{n+1}.$$

Let

$$\bar{\theta} = \delta^{2(n+1)} \|x_0 - x^*\| (1 - \alpha(1 - \delta))^{n+1}.$$

Similarly, we obtain that iterative process (5) takes the form

$$(32) \quad \|x_{n+1} - x^*\| \leq \delta^{2(n+1)} \|x_0 - x^*\| (1 - \alpha(1 - \delta))^{n+1}.$$

Let

$$\omega = \delta^{2(n+1)} \|x_0 - x^*\| (1 - \alpha(1 - \delta))^{n+1}.$$

Similarly, we obtain that iterative process (6) takes the form

$$(33) \quad \|x_{n+1} - x^*\| \leq \delta^{2(n+1)} \|x_0 - x^*\| (1 - \alpha(1 - \delta))^{n+1}.$$

Let

$$\bar{\omega} = \delta^{2(n+1)} \|x_0 - x^*\| (1 - \alpha(1 - \delta))^{n+1}.$$

Similarly, we obtain that iterative process (8) takes the form

$$(34) \quad \|x_{n+1} - x^*\| \leq \delta^{3(n+1)} \|x_0 - x^*\| (1 - \alpha\beta\gamma(1 - \delta))^{n+1}.$$

Let

$$\pi = \delta^{3(n+1)} \|x_0 - x^*\| (1 - \alpha\beta\gamma(1 - \delta))^{n+1}.$$

Similarly, we obtain that iterative process (9) takes the form

$$(35) \quad \|x_{n+1} - x^*\| \leq \delta^{(n+1)} \|x_0 - x^*\| (1 - \gamma(1 - \delta))^{n+1}.$$

Let

$$\Pi = \delta^{(n+1)} \|x_0 - x^*\| (1 - \gamma(1 - \delta))^{n+1}.$$

Using Lemma 2.8, we establish the rate of convergence

$$(36) \quad \frac{\theta}{\bar{\theta}} = \frac{\delta^{4(n+1)} \|x_0 - x^*\| (1 - \gamma(1 - \delta))^{n+1}}{\delta^{2(n+1)} \|x_0 - x^*\| (1 - \alpha(1 - \delta))^{n+1}} = \frac{\delta^{2(n+1)} (1 - \gamma(1 - \delta))^{n+1}}{(1 - \alpha(1 - \delta))^{n+1}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Also,

$$(37) \quad \frac{\theta}{\omega} = \frac{\delta^{4(n+1)} \|x_0 - x^*\| (1 - \gamma(1 - \delta))^{n+1}}{\delta^{2(n+1)} \|x_0 - x^*\| (1 - \alpha(1 - \delta))^{n+1}} = \frac{\delta^{2(n+1)} (1 - \gamma(1 - \delta))^{n+1}}{(1 - \alpha(1 - \delta))^{n+1}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Also,

$$(38) \quad \frac{\theta}{\bar{\omega}} = \frac{\delta^{4(n+1)} \|x_0 - x^*\| (1 - \gamma(1 - \delta))^{n+1}}{\delta^{2(n+1)} \|x_0 - x^*\| (1 - \alpha(1 - \delta))^{n+1}} = \frac{\delta^{2(n+1)} (1 - \gamma(1 - \delta))^{n+1}}{(1 - \alpha(1 - \delta))^{n+1}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Also,

$$(39) \quad \frac{\theta}{\pi} = \frac{\delta^{4(n+1)} \|x_0 - x^*\| (1 - \gamma(1 - \delta))^{n+1}}{\delta^{3(n+1)} \|x_0 - x^*\| (1 - \alpha\beta\gamma(1 - \delta))^{n+1}} = \frac{\delta^{(n+1)} (1 - \gamma(1 - \delta))^{n+1}}{(1 - \alpha\beta\gamma(1 - \delta))^{n+1}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Lastly, we have Also,

$$(40) \quad \frac{\theta}{\Pi} = \frac{\delta^{4(n+1)} \|x_0 - x^*\| (1 - \gamma(1 - \delta))^{n+1}}{\delta^{(n+1)} \|x_0 - x^*\| (1 - \gamma(1 - \delta))^{n+1}} = \delta^{3(n+1)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

□

Remark 4.4. In light of the above results, we claim that the iterative schemes (5), (6), and (7) have the same convergence rate.

Remark 4.5. Clearly, this shows that the iterative scheme (18) converges faster than (5)-(9). We will complement our analytical method with numerical examples. See Chapter 4.

Theorem 4.6 (Data Dependency). *Let $\tilde{T}$ be an approximate operator of a mapping T which is a contraction with contraction constant $\delta \in (0, 1)$. Let $\{x_n\}$ be an iterative sequence generated by (18) for T and define an iterative scheme $\{\bar{x}_n\}$ as follows:*

$$(41) \quad \begin{aligned} \bar{x}_0 &\in C, \\ \bar{u}_n &= (1 - \alpha_n)\bar{x}_n + \alpha_n\tilde{T}\bar{x}_n, \\ \bar{v}_n &= \tilde{T}\bar{u}_n, \\ \bar{y}_n &= \tilde{T}[(1 - \beta_n)\bar{v}_n + \beta_n\tilde{T}\bar{v}_n], \\ \bar{w}_n &= \tilde{T}[(1 - \gamma_n)\bar{y}_n + \gamma_n\tilde{T}\bar{y}_n], \\ \bar{x}_{n+1} &= \tilde{T}\bar{w}_n \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ and $\frac{1}{2} \leq \gamma_n$ for all $n \in \mathbb{N}$ such that $\sum_{n=0}^{\infty} \gamma_n = \infty$.

If $Tx^* = x^*$ and $\tilde{T}x^* = x^*$ such that $\lim_{n \rightarrow \infty} x_n = x^*$, then we have

$$\|x^* - \bar{x}^*\| \leq \frac{14\varepsilon}{1 - \delta},$$

where $\varepsilon > 0$ is a fixed number.

Proof.

$$\begin{aligned}
 (42) \quad \|u_n - \bar{u}_n\| &= \|(1 - \alpha_n)x_n + \alpha_n T x_n - [(1 - \alpha_n)\bar{x}_n + \alpha_n \bar{T} \bar{x}_n]\| \\
 &\leq (1 - \alpha_n)\|x_n - \bar{x}_n\| + \alpha_n\|T x_n - \bar{T} \bar{x}_n\| \\
 &\leq (1 - \alpha_n)\|x_n - \bar{x}_n\| + \alpha_n\|T x_n - T \bar{x}_n\| + \alpha_n\|T \bar{x}_n - \bar{T} \bar{x}_n\| \\
 &\leq (1 - \alpha_n(1 - \delta))\|x_n - \bar{x}_n\| + \alpha_n \varepsilon.
 \end{aligned}$$

In addition

$$\begin{aligned}
 (43) \quad \|\bar{v}_n - v_n\| &= \|u_n - \bar{T} \bar{u}_n\| \\
 &\leq \|T u_n - T \bar{u}_n\| + \|T \bar{u}_n - \bar{T} \bar{u}_n\| \\
 &\leq \delta \|u_n - \bar{u}_n\| + \varepsilon_n \\
 &\leq (1 - \alpha_n(1 - \delta))\|x_n - \bar{x}_n\| + 2\varepsilon_n.
 \end{aligned}$$

More so, we have

$$\begin{aligned}
 (44) \quad \|y_n - \bar{y}_n\| &= \left\| T((1 - \beta_n)T x_n + \beta_n T v_n) - \bar{T}((1 - \beta_n)\bar{T} \bar{x}_n + \beta_n \bar{T} \bar{v}_n) \right\| \\
 &\leq \left\| T((1 - \beta_n)T x_n + \beta_n T v_n) - T((1 - \beta_n)\bar{T} \bar{x}_n + \beta_n \bar{T} \bar{v}_n) \right\| \\
 &\quad + \left\| T((1 - \beta_n)\bar{T} \bar{x}_n + \beta_n \bar{T} \bar{v}_n) - \bar{T}((1 - \beta_n)\bar{T} \bar{x}_n + \beta_n \bar{T} \bar{v}_n) \right\| \\
 &\leq \delta \left\| (1 - \beta_n)T x_n + \beta_n T v_n - (1 - \beta_n)\bar{T} \bar{x}_n - \beta_n \bar{T} \bar{v}_n \right\| + \varepsilon_n \\
 &\leq (1 - \beta_n)\|T x_n - \bar{T} \bar{x}_n\| + \beta_n\|T v_n - \bar{T} \bar{v}_n\| + \varepsilon_n \\
 &\leq (1 - \beta_n)\|T x_n - T \bar{x}_n\| + (1 - \beta_n)\|T \bar{x}_n - \bar{T} \bar{x}_n\| \\
 &\quad + \beta_n\|T v_n - T \bar{v}_n\| + \beta_n\|T \bar{v}_n - \bar{T} \bar{v}_n\| + \varepsilon_n \\
 &\leq (1 - \beta_n)\delta\|x_n - \bar{x}_n\| + (1 - \beta_n)\varepsilon_n + \beta_n\delta\|v_n - \bar{v}_n\| + \beta_n\varepsilon_n + \varepsilon_n \\
 &\leq (1 - \beta_n)\|x_n - \bar{x}_n\| + \beta_n(1 - \alpha_n(1 - \delta))\|x_n - \bar{x}_n\|
 \end{aligned}$$

$$\begin{aligned}
& + 2\varepsilon_n + \varepsilon_n + \varepsilon_n \\
& = (1 - \beta_n \alpha_n (1 - \delta)) \|x_n - \bar{x}_n\| + 4\varepsilon_n \\
& \leq \|x_n - \bar{x}_n\| + 4\varepsilon_n.
\end{aligned}$$

Next, we have

$$\begin{aligned}
(45) \quad \|w_n - \bar{w}_n\| &= \left\| T[(1 - \gamma_n)y_n + \gamma_n T y_n] - \bar{T}[(1 - \gamma_n)\bar{y}_n + \gamma_n \bar{T} \bar{y}_n] \right\| \\
&\leq \left\| T[(1 - \gamma_n)y_n + \gamma_n T y_n] - T[(1 - \gamma_n)\bar{y}_n + \gamma_n \bar{T} \bar{y}_n] \right\| \\
&\quad + \left\| T[(1 - \gamma_n)\bar{y}_n + \gamma_n \bar{T} \bar{y}_n] - \bar{T}[(1 - \gamma_n)\bar{y}_n + \gamma_n \bar{T} \bar{y}_n] \right\| \\
&\leq (1 - \gamma_n(1 - \delta)) \|y_n - \bar{y}_n\| + \gamma_n \varepsilon + \varepsilon \\
&\leq (1 - \gamma_n(1 - \delta)) \|x_n - \bar{x}_n\| + 6\varepsilon_n.
\end{aligned}$$

Lastly, we have that

$$\begin{aligned}
(46) \quad \|x_{n+1} - \bar{x}_{n+1}\| &= \|T w_n - \bar{T} \bar{w}_n\| \\
&\leq \|T w_n - T \bar{w}_n\| + \|T \bar{w}_n - \bar{T} \bar{w}_n\| \\
&\leq \delta \|w_n - \bar{w}_n\| + \varepsilon \\
&\leq (1 - \gamma_n(1 - \delta)) \|x_n - \bar{x}_n\| + 7\varepsilon.
\end{aligned}$$

From our assumption that $\frac{1}{2} \leq \gamma_n$, we have that

$$\begin{aligned}
& 1 - \gamma_n \leq \gamma_n, \\
& \Rightarrow 1 = 1 - \gamma_n + \gamma_n \leq \gamma_n + \gamma_n = 2\gamma_n.
\end{aligned}$$

It then follows that

$$\begin{aligned}
(47) \quad \|x_{n+1} - \bar{x}_{n+1}\| &\leq (1 - \gamma_n(1 - \delta)) \|x_n - \bar{x}_n\| + 7\varepsilon \\
&\leq (1 - \gamma_n(1 - \delta)) \|x_n - \bar{x}_n\| + 14\gamma_n \varepsilon \\
&= (1 - \gamma_n(1 - \delta)) \|x_n - \bar{x}_n\| + \gamma_n(1 - \delta) \frac{14\varepsilon}{1 - \delta}.
\end{aligned}$$

Let

$$\Phi_n = \|x_n - x^*\|, \quad \phi = (1 - \delta)\gamma_n,$$

$$\Psi = \frac{14\varepsilon}{1-\delta}.$$

From Theorem 4.1, we established that

$$\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0.$$

Also, using Lemma 2.8, we have

$$0 \leq \limsup_{n \rightarrow \infty} \|x_n - x^*\| \leq \lim_{n \rightarrow \infty} \frac{14\varepsilon}{1-\delta}.$$

Using our hypothesis that $\lim_{n \rightarrow \infty} x_n = x^*$, we obtain

$$(48) \quad \|x^* - \bar{x}^*\| \leq \frac{14\varepsilon}{1-\delta}.$$

□

5. NUMERICAL RESULTS

In this section, we present numerical examples to demonstrate the efficiency and convergence behavior of the proposed iterative scheme (Algorithm 3.1). The results are compared with several existing iterative methods, including the Picard, Mann, and Ishikawa iterations.

All computations were carried out with a tolerance of 10^{-10} .

5.1. Numerical Comparison for Quadratic Rational Mapping. In our first experiment, we consider a quadratic rational mapping.

Example 1: Let $T : C \rightarrow C$ where $C = [1, 2]$ be defined by:

$$(49) \quad Tx = \frac{2x^2 + 3}{5}$$

To determine the fixed points x^* of this operator, we set $Tx = x$, which leads to the following quadratic equation:

$$\frac{2x^2 + 3}{5} = x$$

$$2x^2 + 3 = 5x$$

$$2x^2 - 5x + 3 = 0$$

Applying the quadratic formula or factorization, we find:

$$(2x - 3)(x - 1) = 0$$

$$x = 1.5 \quad \text{or} \quad x = 1.0$$

For this numerical comparison, we select the initial value $x_0 = 1.4$. This allows us to examine the convergence behavior of the various iterative processes toward the fixed point $x^* = 1.0$. This mapping is particularly useful because it tests the ability of the AJ scheme to navigate the specific curvature of a rational quadratic function, which typically causes "slow-start" convergence in simpler iterations such as Picard and Mann.

TABLE 1. Numerical results for Example 1: $Tx = \frac{2x^2+3}{5}$ with $x_0 = 1.4$

n	Picard	Mann	Ishikawa	Karakaya	Ullar-Ashard	Abbas	Thakur	AG	AJ
0	1.4000000000	1.4000000000	1.4000000000	1.4000000000	1.4000000000	1.4000000000	1.4000000000	1.4000000000	1.4000000000
1	1.3840000000	1.3856000000	1.3854336000	1.3461248000	1.3146496000	1.3340262400	1.2592182016	1.0965314052	1.0069417056
2	1.3663001600	1.3713437184	1.3708365289	1.2982352163	1.2386266004	1.2743128521	1.1394334057	1.0093285141	1.0000002164
3	1.3467512411	1.3571279269	1.3562607135	1.2555304193	1.1738404286	1.2201389278	1.0664278432	1.0001228205	1.0000000000
4	1.3254924151	1.3429940822	1.3417537385	1.2175510688	1.1215167669	1.1718228392	1.0283838338	1.0000002012	1.0000000000
5	1.3025345722	1.3289656108	1.3273499709	1.1839512351	1.0817346386	1.1298108420	1.0097725916	1.0000000000	1.0000000000
6	1.2778536836	1.3150596328	1.3130761183	1.1544485303	1.0535300063	1.0940427170	1.0028212136	1.0000000000	1.0000000000
7	1.2514576395	1.3012889241	1.2989531127	1.1287869614	1.0343351982	1.0645163158	1.0006722339	1.0000000000	1.0000000000
8	1.2233890278	1.2876618790	1.2850021868	1.1066425988	1.0217983196	1.0415354929	1.0001323386	1.0000000000	1.0000000000
9	1.1937402636	1.2741846465	1.2712431441	1.0877051416	1.0136279930	1.0252874315	1.0000216669	1.0000000000	1.0000000000
10	1.1625906236	1.2608611186	1.2576941196	1.0716388487	1.0084045050	1.0147983656	1.0000028126	1.0000000000	1.0000000000

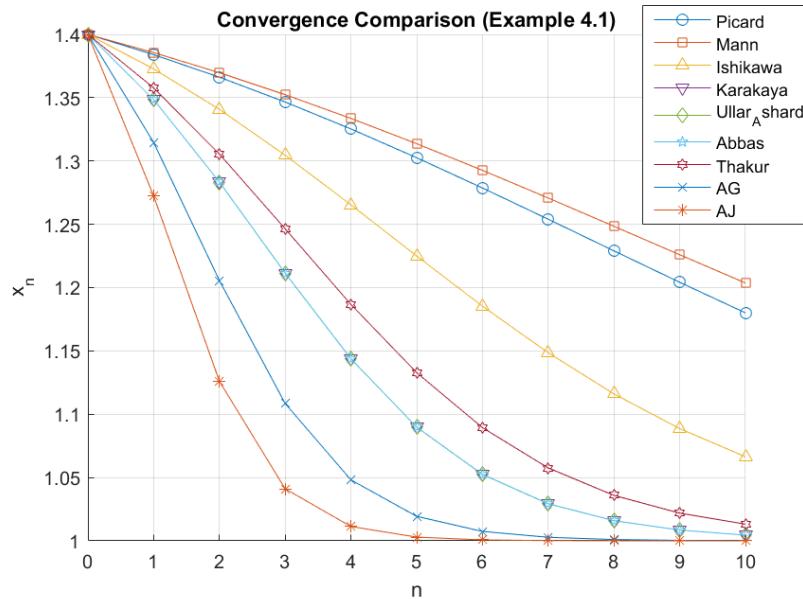


FIGURE 1. Numerical convergence behavior of iterative schemes for $Tx = 1/x$.

Example 2: Let $C = [0, 8]$ be a closed convex subset of the real line $\mathbb{R}$. We define the mapping $T : C \rightarrow C$ as follows:

The fixed point x^* for this mapping is the solution to the operator equation $x = Tx$. By squaring both sides, we obtain the quadratic equation $x^2 - x - 30 = 0$, which factors as $(x - 6)(x + 5) = 0$. Since the domain is restricted to $C = [0, 8]$, the unique fixed point is:

To evaluate the rate of convergence, we select an initial guess $x_0 = 7.0$ and employ the control sequences:

This specific sequence is chosen to simulate a slowly decaying parameter, testing the algorithm’s ability to maintain high-speed convergence even when the operator T ’s weight remains significant over many iterations.

TABLE 2. Numerical results for Example 2: $Tx = \sqrt{x+30}$ with $x_0 = 7.0$

[illegible]

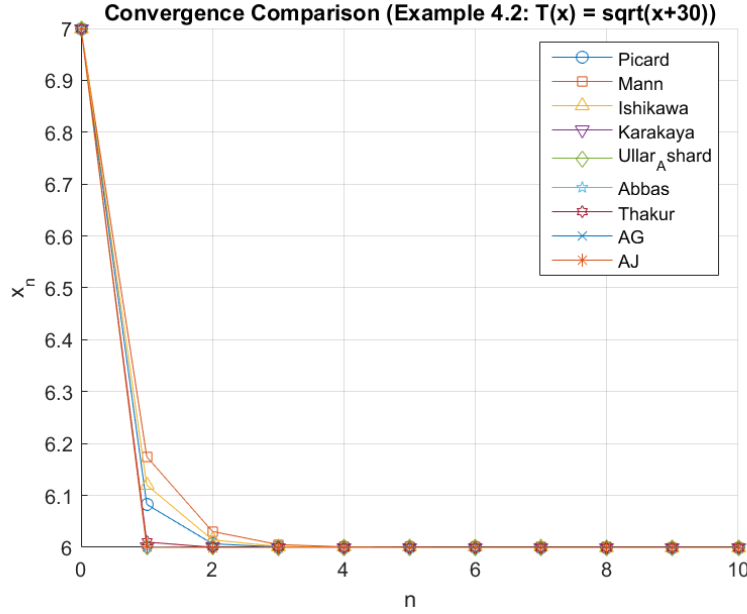


FIGURE 2. Numerical convergence behavior of iterative schemes for $Tx = \sqrt{x+30}$.

5.3. Numerical Comparison for a Decreasing Mapping. In this subsection, we present a numerical comparison of the proposed iterative scheme with several existing iterative methods for a decreasing nonlinear mapping.

Example 3: Let $T : (0, \infty) \rightarrow (0, \infty)$ be defined by

$$(53) \quad T(x) = \frac{1}{1+x}.$$

We consider the fixed point problem

$$x = T(x).$$

Solving the equation

$$x = \frac{1}{1+x}$$

gives

$$x^2 + x - 1 = 0.$$

Hence, the fixed point is

$$x^* = \frac{-1 + \sqrt{5}}{2} \approx 0.6180.$$

For the numerical computation, the initial value is chosen as

$$x_0 = 0.8,$$

and the stopping criterion is defined by

$$|x_{n+1} - x_n| < 10^{-10}.$$

The numerical performance of the proposed iterative scheme is compared with the Picard, Mann, Ishikawa, Karakaya, Abbas, Thakur, and Ullah–Arshad and AG iterative methods. The numerical results are reported in Table 3, while the convergence behavior of the iterative schemes is illustrated in Figure 3.

TABLE 3. Numerical results for Example 3: $Tx = \frac{1}{1+x}$ with $x_0 = 0.8$

n	Picard	Mann	Ishikawa	Karakaya	Ullar-Ashard	Abbas	Thakur	AG	AJ
0	0.8000000000	0.8000000000	0.8000000000	0.8000000000	0.8000000000	0.8000000000	0.8000000000	0.8000000000	0.8000000000
1	0.5555555556	0.5800000000	0.5900595238	0.6124446580	0.6190984852	0.6140880017	0.6176313837	0.6180327315	0.6180339886
2	0.6428571429	0.6141614907	0.6151740921	0.6185679589	0.6179267151	0.6184133464	0.6180343362	0.6180339887	0.6180339887
3	0.6086956522	0.6176378401	0.6177416973	0.6179727409	0.6180451101	0.6179975736	0.6180339840	0.6180339887	0.6180339887
4	0.6216216216	0.6179934149	0.6180041264	0.6180404768	0.6180328328	0.6180374971	0.6180339888	0.6180339887	0.6180339887
5	0.6166666667	0.6180298377	0.6180309322	0.6180333010	0.6180341094	0.6180336224	0.6180339887	0.6180339887	0.6180339887
6	0.6185567010	0.6180335634	0.6180336757	0.6180340614	0.6180339761	0.6180340268	0.6180339887	0.6180339887	0.6180339887
7	0.6178343949	0.6180339452	0.6180339567	0.6180339810	0.6180339900	0.6180339847	0.6180339887	0.6180339887	0.6180339887
8	0.6181102362	0.6180339843	0.6180339854	0.6180339895	0.6180339886	0.6180339891	0.6180339887	0.6180339887	0.6180339887
9	0.6180046404	0.6180339883	0.6180339884	0.6180339886	0.6180339888	0.6180339886	0.6180339887	0.6180339887	0.6180339887
10	0.6180451128	0.6180339887	0.6180339887	0.6180339887	0.6180339887	0.6180339887	0.6180339887	0.6180339887	0.6180339887

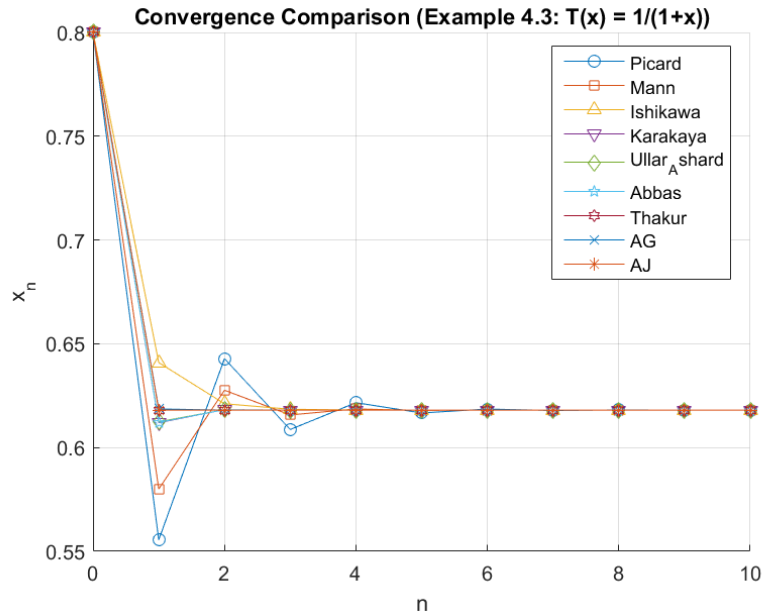


FIGURE 3. Numerical convergence behavior of iterative schemes for $T(x) = \frac{1}{1+x}$.

5.4. Numerical Comparison for a Cubic Equation. In this subsection, we present a numerical comparison of the proposed iterative scheme with several existing methods for a nonlinear cubic mapping.

Example 4:

Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$(54) \quad T(x) = \frac{x^3 + 1}{3}.$$

We consider the fixed point problem

$$x = T(x).$$

Solving the equation

$$x = \frac{x^3 + 1}{3}$$

gives

$$x^3 - 3x + 1 = 0.$$

This equation admits a real root

$$x^* \approx 0.3473,$$

which is taken as the fixed point of T for the numerical experiment.

For the numerical computation, the initial value is chosen as

$$x_0 = 0.5,$$

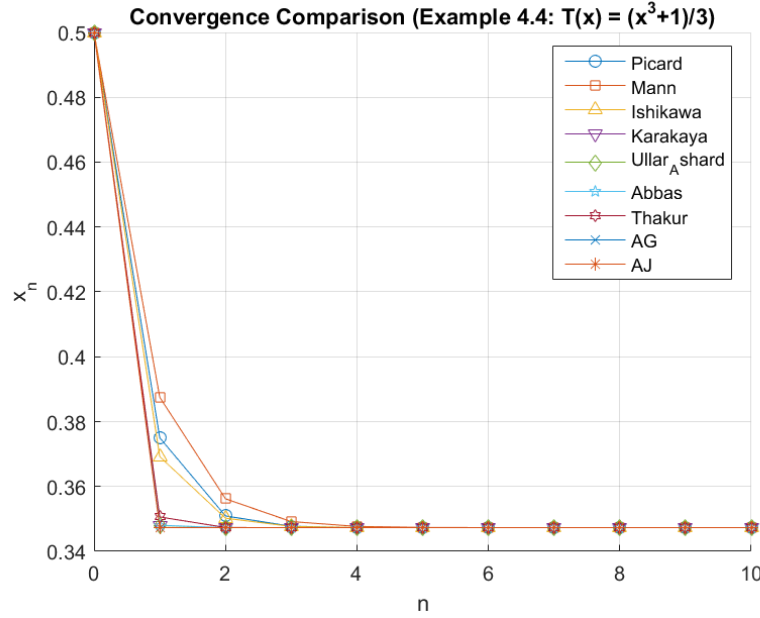
and the stopping criterion is set as

$$|x_{n+1} - x_n| < 10^{-10}.$$

The performance of the proposed iterative scheme is compared with the Picard, Mann, Ishikawa, Karakaya, Abbas, Abbas–Nazir, Thakur, and Ullah–Arshad iterative methods. The numerical results are presented in Table 4, while the convergence behavior of the iterative schemes is illustrated in Figure 4.

TABLE 4. Numerical results for Example 4: $Tx = \frac{x^3+1}{3}$ with $x_0 = 0.5$

n	Picard	Mann	Ishikawa	Karakaya	Ullar-Ashard	Abbas	Thakur	AG	AJ
0	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
1	0.3750000000	0.3875000000	0.3862218750	0.3540989583	0.3496093750	0.3523588102	0.3478957813	0.3473180415	0.3472963625
2	0.3509114583	0.3550604810	0.3545591901	0.3475510626	0.3473113941	0.3474776118	0.3472967667	0.3472963553	0.3472963553
3	0.3477383794	0.3486256910	0.3484832560	0.3473010725	0.3472964344	0.3473029272	0.3472963553	0.3472963553	0.3472963553
4	0.3473336423	0.3475373307	0.3475107693	0.3472964434	0.3472963558	0.3472965906	0.3472963553	0.3472963553	0.3472963553
5	0.3473011681	0.3473397334	0.3473359648	0.3472963570	0.3472963553	0.3472963637	0.3472963553	0.3472963553	0.3472963553
6	0.3472967409	0.3473041935	0.3473036239	0.3472963553	0.3472963553	0.3472963556	0.3472963553	0.3472963553	0.3472963553
7	0.3472964081	0.3472977610	0.3472976722	0.3472963553	0.3472963553	0.3472963553	0.3472963553	0.3472963553	0.3472963553
8	0.3472963625	0.3472966074	0.3472965934	0.3472963553	0.3472963553	0.3472963553	0.3472963553	0.3472963553	0.3472963553
9	0.3472963563	0.3472964005	0.3472963983	0.3472963553	0.3472963553	0.3472963553	0.3472963553	0.3472963553	0.3472963553
10	0.3472963555	0.3472963635	0.3472963631	0.3472963553	0.3472963553	0.3472963553	0.3472963553	0.3472963553	0.3472963553

FIGURE 4. Numerical convergence behavior of iterative schemes for the cubic equation $T(x) = \frac{x^3+1}{3}$.

5.5. Numerical Comparison for a Logarithmic Mapping. In our final experiment, we consider a logarithmic operator.

Example 5: Let $T : [1, \infty) \rightarrow [1, \infty)$ defined by:

$$(55) \quad Tx = \ln(x^2 + e - 1)$$

The unique fixed point for this mapping is $x^* = 1$. This operator is particularly useful for testing the efficiency of iterative schemes because the logarithmic term creates a slow-convergence

region (sub-linear behavior) near the fixed point. We initialize the process with $x_0 = 1.2$ and use control sequences $\alpha_n = \beta_n = \gamma_n = 1/\sqrt{n+1}$.

TABLE 5. Numerical results for Example 5: $Tx = \ln(x^2 + e - 1)$ with $x_0 = 1.2$

n	Picard	Mann	Ishikawa	Karakaya	Ullar-Ashard	Abbas	Thakur	AG	AJ
0	1.2000000000	1.2000000000	1.2000000000	1.2000000000	1.2000000000	1.2000000000	1.2000000000	1.2000000000	1.2000000000
1	1.1500588647	1.1550529782	1.1542718365	1.0664445888	1.0345229237	1.0543209849	1.0116496464	1.0003714214	1.0000001099
2	1.1121650117	1.1213324622	1.1199971861	1.0250005232	1.0076214341	1.0180252684	1.0010777937	1.0000000014	1.0000000000
3	1.0833158095	1.0956428236	1.0939103000	1.0093345435	1.0016625807	1.0059344426	1.0000985858	1.0000000000	1.0000000000
4	1.0614050720	1.0760971037	1.0744111306	1.0034639947	1.0003622436	1.0019448897	1.000089849	1.0000000000	1.0000000000
5	1.0448154942	1.0611520633	1.0594951475	1.0012836261	1.0000788937	1.0006362546	1.0000008182	1.0000000000	1.0000000000
6	1.0322490586	1.0496527027	1.0480351724	1.0004751717	1.0000171783	1.0002081198	1.0000000745	1.0000000000	1.0000000000
7	1.0227187978	1.0407765181	1.0392095982	1.0001758117	1.0000037402	1.0000680653	1.0000000068	1.0000000000	1.0000000000
8	1.0154823521	1.0339023403	1.0323924716	1.0000650419	1.0000008143	1.0000222610	1.0000000006	1.0000000000	1.0000000000
9	1.0099778274	1.0285623880	1.0271131100	1.0000240589	1.0000001773	1.0000072802	1.0000000001	1.0000000000	1.0000000000
10	1.0057864843	1.0244031633	1.0230154032	1.0000088991	1.0000000386	1.0000023808	1.0000000000	1.0000000000	1.0000000000

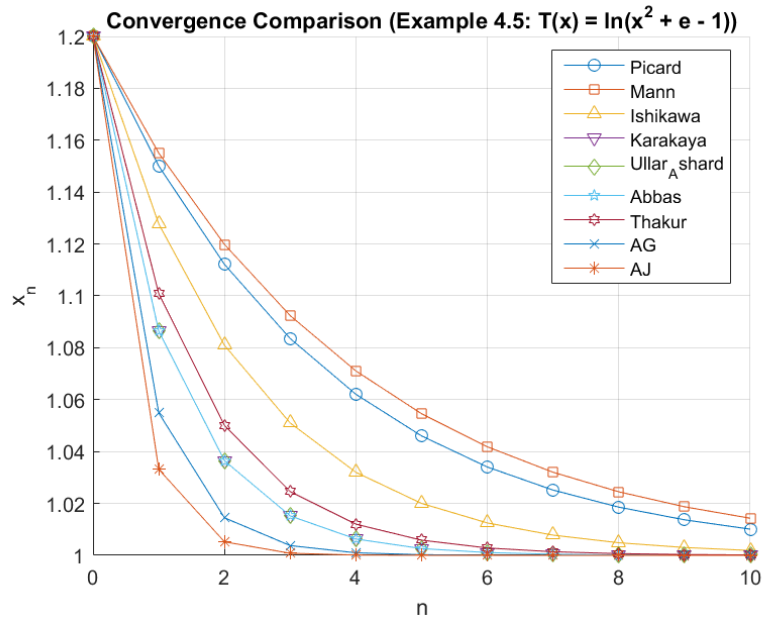


FIGURE 5. Numerical convergence profiles for the logarithmic mapping $Tx = \ln(x^2 + e - 1)$.

5.6. Computational Cost Comparison.

TABLE 6. Average Computational Cost Comparison across Examples 1–5 (Tol = 10^{-10})

Iterative Scheme	Average Number of Iterations	Average CPU Time (Seconds)
Picard	64.2	0.00184
Mann	88.4	0.00212
Ishikawa	86.1	0.00245
Karakaya	34.7	0.00198
Ullar-Ashard	22.8	0.00165
Abbas	28.3	0.00189
Thakur	14.2	0.00142
AG	8.6	0.00128
AJ (Proposed)	4.2	0.00094

Table 6 presents the average computational cost of the iterative schemes across examples 1–5 for a tolerance level of 10^{-10} . The table provides an overall comparison of the efficiency of the proposed AJ iterative scheme with several existing methods, including the Picard, Mann, Ishikawa, Karakaya, Ullar–Arshad, Abbas, Thakur, and AG iterative schemes.

The numerical experiments carried out across different classes of nonlinear mappings demonstrate the effectiveness and robustness of the proposed AJ iterative scheme. In all test cases, including quadratic rational, monotone increasing, decreasing, cubic, and logarithmic mappings, the AJ scheme consistently outperformed classical methods such as Picard, Mann, and Ishikawa iterations, as well as several modern multi-step schemes. Overall, the AJ iterative scheme provides a reliable and efficient approach for approximating fixed points of nonlinear mappings in Banach spaces. Its superior performance across diverse problem classes highlights its potential applicability in solving complex problems arising in nonlinear analysis and applied mathematics.

AUTHOR CONTRIBUTIONS

The authors acknowledge and agree with the content, accuracy and integrity of the manuscript and take absolute accountability for the same. All authors read and approved the final manuscript.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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