Available online at http://scik.org
Adv. Fixed Point Theory, 2026, 16:26
https://doi.org/10.28919/afpt/9985
ISSN: 1927-6303

ORDERED PARTIAL MODULAR METRIC SPACES: STABILITY, COINCIDENCE POINT, AND FIXED POINT RESULTS WITH APPLICATION TO NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

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Abstract. In this article, we introduce the notion of ordered partial modular metric spaces and derive novel coincidence point and fixed point results for generalized (φ,ψ)-weak contractive type mappings, incorporating both with and without continuity assumptions on the involved mappings. To demonstrate the applicability of our theoretical result, we provide illustrative examples. As an application, we analyze the existence and uniqueness of solutions to nonlinear Volterra integro-differential equation, supported by numerical simulation. Moreover, we establish Hyers-Ulam-Rassias stability for the nonlinear Volterra integro-differential equation by employing the fixed point technique and a numerical example is illustrated to verify the result.

Keywords: generalized (φ,ψ)-weak contraction; Hyers-Ulam-Rassias stability; ordered partial metric space; partial modular metric space; Volterra integro-differential equation.

2020 AMS Subject Classification: 34K20, 45D05, 45M10, 47G20, 47H10, 54H25.

1. Introduction

Nonlinear analysis fundamentally relies on fixed point ($\mathcal{FP}$) theory, which provides powerful tools for establishing the existence and uniqueness of solutions to diverse nonlinear problems.

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Received May 16, 2026

Over time, $\mathcal{FP}$ results have been extensively extended and developed within generalized metric spaces ($\mathcal{MS}$), leading to significant applications in nonlinear integral and differential equations.

The theory of generalized $\mathcal{MS}$ has undergone significant development over the past few decades. Matthews [14] introduced partial $\mathcal{MS}$ ($\mathcal{PMS}$), allowing non-zero self-distances, while Chistyakov [5] developed modular $\mathcal{MS}$ ($\mathcal{MMS}$), thereby generalizing classical $\mathcal{MS}$. Later, partial modular $\mathcal{MS}$ ($\mathcal{PMMS}$) were modified by Das et al. [7] that generalizes both $\mathcal{PMS}$ and $\mathcal{MMS}$. Building on this refined framework, recent developments include $\mathcal{PM}\mathrm{b}-\mathcal{MS}$ [13] and C^*-algebra valued $\mathcal{PMMS}$ [18], further extending the theoretical framework.

The theory of $\mathcal{FP}$ has been substantially extended to partially ordered metric type structures. Abbas and Jungck [1] established the presence of common $\mathcal{FP}$, coincidence points ($\mathcal{CP}$), and unique point of coincidence ($\mathfrak{poc}$) for suitable contractive mappings, while Ran and Reurings [22] established Banach $\mathcal{FP}$ theorem in partially ordered set (poset). Subsequently, Nieto and Lopez [20], Samet et al. [24], and Mutlu [17] extended these results to ordered $\mathcal{MS}$, ordered $\mathcal{PMS}$, and partially ordered $\mathcal{MMS}$, respectively. In parallel, weak contraction theory evolved from the work of Alber and Guerre-Delabriere [2] and Rhoades [23] to later generalizations in ordered metric frameworks by Doric [9], Radenovic and Kadelburg [21], Aydi [3] and others [6, 8, 19, 24]. The $\mathcal{FP}$ method was utilized to find the $\mathcal{HU}$ stability by Cadariu and Radu [4] in their investigation of the Cauchy additive functional equation. Extensive research has been directed towards determining the stability of nonlinear Volterra integro-differential equation ($\mathcal{VIDE}$) using the $\mathcal{FP}$ technique (see, [11, 25]).

Motivated by the significance of these concepts, we investigate how existing $\mathcal{FP}$ theoretical frameworks can be generalized. The present work develops the theory of generalized weak contractive mappings in ordered $\mathcal{PMMS}$ ($\mathcal{OPMMS}$), a generalization of classical $\mathcal{MS}$. This theoretical advancement enables the application of $\mathcal{FP}$ methods to nonlinear $\mathcal{VIDE}$, expanding the scope of nonlinear analysis techniques.

The paper consists of eight sections, arranged as follows: Introduction, Preliminaries, Continuity in $\mathcal{PMMS}$, $\mathcal{OPMMS}$, Main results, Application to nonlinear $\mathcal{VIDE}$, Hyers-Ulam-Rassias ($\mathcal{HUR}$) stability, Conclusion.

2. PRELIMINARIES

This section outlines the core definitions and auxiliary results that will be utilized throughout the paper.

Definition 2.1. [5] *A modular metric* $(\mathcal{MM})$ *on a set* $\mathcal{V} \neq \emptyset$ *is a function* $\mathsf{s}^{\omega} : (0,+\infty) \times \mathcal{V} \times \mathcal{V} \to [0,+\infty]$ *such that, for all* $\mathfrak{y}, \mathfrak{z}, \mathfrak{w} \in \mathcal{V}$ *and for all* $\beta, \gamma > 0$:

s_1 : $\mathsf{s}^{\omega}_{\beta}(\mathfrak{y},\mathfrak{z}) = 0 \Leftrightarrow \mathfrak{y} = \mathfrak{z}$;

s_2 : $\mathsf{s}^{\omega}_{\beta}(\mathfrak{y},\mathfrak{z}) = \mathsf{s}^{\omega}_{\beta}(\mathfrak{z},\mathfrak{y})$;

s_3 : $\mathsf{s}^{\omega}_{\beta+\gamma}(\mathfrak{y},\mathfrak{z}) \leq \mathsf{s}^{\omega}_{\beta}(\mathfrak{y},\mathfrak{w}) + \mathsf{s}^{\omega}_{\gamma}(\mathfrak{w},\mathfrak{z})$.

The pair $(\mathcal{V}, \mathsf{s}^{\omega})$ *is then called* $\mathcal{MMS}$.

Definition 2.2. [5] *Suppose* $\mathcal{V} \neq \emptyset$. *For any* $\mathfrak{y}_0 \in \mathcal{V}$, *define*

$$\mathcal{V}_{\mathsf{s}^{\omega}}(\mathfrak{y}_0) = \{\mathfrak{z} \in \mathcal{V} : \lim_{\beta \to +\infty} \mathsf{s}^{\omega}_{\beta}(\mathfrak{y}_0,\mathfrak{z}) = 0\},$$

then $\mathcal{V}_{\mathsf{s}^{\omega}}$ *is called modular space centered at* $\mathfrak{y}_0$.

The set $\mathcal{V}_{\mathsf{s}^{\omega}}$ is said to be $\mathcal{MMS}$, since s^{ω} is a $\mathcal{MM}$ in $\mathcal{V}$.

Definition 2.3. [7] *A partial modular metric* $(\mathcal{PMM})$ *on* $\mathcal{V} \neq \emptyset$ *is a function* $\wp^{\omega} : (0,+\infty) \times \mathcal{V} \times \mathcal{V} \to [0,+\infty)$ *such that, for all* $\mathfrak{y}, \mathfrak{z}, \mathfrak{w} \in \mathcal{V}$ *and for all* $\beta, \gamma > 0$:

$\wp_1$: $\wp^{\omega}_{\beta}(\mathfrak{y},\mathfrak{y}) = \wp^{\omega}_{\gamma}(\mathfrak{y},\mathfrak{y})$; $\wp^{\omega}_{\beta}(\mathfrak{y},\mathfrak{z}) = \wp^{\omega}_{\beta}(\mathfrak{y},\mathfrak{y}) = \wp^{\omega}_{\beta}(\mathfrak{z},\mathfrak{z}) \Leftrightarrow \mathfrak{y} = \mathfrak{z}$;

$\wp_2$: $\wp^{\omega}_{\beta}(\mathfrak{y},\mathfrak{y}) \leq \wp^{\omega}_{\beta}(\mathfrak{y},\mathfrak{z})$;

$\wp_3$: $\wp^{\omega}_{\beta}(\mathfrak{y},\mathfrak{z}) = \wp^{\omega}_{\beta}(\mathfrak{z},\mathfrak{y})$;

$\wp_4$: $\wp^{\omega}_{\beta+\gamma}(\mathfrak{y},\mathfrak{z}) \leq \wp^{\omega}_{\beta}(\mathfrak{y},\mathfrak{w}) + \wp^{\omega}_{\gamma}(\mathfrak{w},\mathfrak{z}) - \wp^{\omega}_{\beta}(\mathfrak{w},\mathfrak{w})$.

The pair $(\mathcal{V}, \wp^{\omega})$ *is then called* $\mathcal{PMMS}$.

Definition 2.4. [7] *Suppose* $\mathcal{V} \neq \emptyset$. *For any* $\mathfrak{y}_0 \in \mathcal{V}$, *define*

$$\mathcal{V}_{\wp^{\omega}}(\mathfrak{y}_0) = \{\mathfrak{z} \in \mathcal{V} : \lim_{\beta \to +\infty} \wp^{\omega}_{\beta}(\mathfrak{y}_0,\mathfrak{z}) = c\}, \textit{for some } c \geq 0,$$

then $\mathcal{V}_{\wp^{\omega}}$ *is called partial modular space centered at* $\mathfrak{y}_0$.

The set $\mathcal{V}_{\wp^{\omega}}$ is said to be $\mathcal{PMMS}$, since $\wp^{\omega}$ is a $\mathcal{PMM}$ in $\mathcal{V}$.

Lemma 2.1. [7] *Suppose $\mathcal{V} \neq \emptyset$ and $\mathcal{V}_{\wp^\omega}$ is a $\mathcal{PMMS}$. If*

$$\mathsf{s}^{\omega}_{\beta}(\mathfrak{y},\mathfrak{z}) = 2\wp^{\omega}_{\beta}(\mathfrak{y},\mathfrak{z}) - \wp^{\omega}_{\beta}(\mathfrak{y},\mathfrak{y}) - \wp^{\omega}_{\beta}(\mathfrak{z},\mathfrak{z}),$$

for all $\mathfrak{y},\mathfrak{z} \in \mathcal{V}$ and for all $\beta > 0$. Then, $\mathcal{V}_{\mathsf{s}^\omega}$ is a $\mathcal{MMS}$.

Definition 2.5. [7] *Let $\{\mathfrak{x}_\mathfrak{m}\}_{\mathfrak{m}\in\mathbb{N}}$ be a sequence in $\mathcal{V}_{\wp^\omega}$. Then,*

(i) *$\{\mathfrak{x}_\mathfrak{m}\}$ is said to converge to $\boldsymbol{\varpi}$ in $\mathcal{V}_{\wp^\omega}$, if and only if for every $\delta > 0$, there exists $\mathcal{B}_0 \in \mathbb{W}$ such that*

$$\left|\wp^{\omega}_{\beta}(\mathfrak{x}_\mathfrak{m},\boldsymbol{\varpi}) - \wp^{\omega}_{\beta}(\boldsymbol{\varpi},\boldsymbol{\varpi})\right| \leq \delta, \forall\, \mathfrak{m} \geq \mathcal{B}_0, \forall\, \beta > 0.$$

Equivalently, $\lim_{\mathfrak{m}\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{x}_\mathfrak{m},\boldsymbol{\varpi}) = \wp^{\omega}_{\beta}(\boldsymbol{\varpi},\boldsymbol{\varpi})$, for all $\beta > 0$.

(ii) *$\{\mathfrak{x}_\mathfrak{m}\}$ is a Cauchy sequence $(\mathcal{CS})$ in $\mathcal{V}_{\wp^\omega}$, if $\lim_{\mathfrak{m},\mathfrak{j}\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{x}_\mathfrak{m},\mathfrak{x}_\mathfrak{j}) = \mathfrak{c}$, for all $\beta > 0$ and for some $\mathfrak{c} \geq 0$.*
Equivalently, $\lim_{\mathfrak{m}\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{x}_\mathfrak{m},\mathfrak{x}_\mathfrak{m}) = \lim_{\mathfrak{j}\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{x}_\mathfrak{j},\mathfrak{x}_\mathfrak{j}) = \mathfrak{c}$, for all $\beta > 0$. Therefore, $\mathfrak{c} = 0$ if $\{\mathfrak{x}_\mathfrak{m}\}$ is a $\mathcal{CS}$ in $\mathcal{V}_{\mathsf{s}^\omega}$.

(iii) *$\mathcal{V}_{\wp^\omega}$ is Complete, if every $\mathcal{CS}$ converges to $\boldsymbol{\varpi}$ in $\mathcal{V}_{\wp^\omega}$ such that*

$$\lim_{\mathfrak{m},\mathfrak{j}\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{x}_\mathfrak{m},\mathfrak{x}_\mathfrak{j}) = \wp^{\omega}_{\beta}(\boldsymbol{\varpi},\boldsymbol{\varpi}), \forall\, \beta > 0.$$

Lemma 2.2. [7] *Let $\{\mathfrak{x}_\mathfrak{m}\}_{\mathfrak{m}\in\mathbb{N}}$ be a sequence in $\mathcal{V}_{\wp^\omega}$. Then,*

(i) *$\{\mathfrak{x}_\mathfrak{m}\}$ is a $\mathcal{CS}$ in $\mathcal{V}_{\wp^\omega}$, provided $\{\mathfrak{x}_\mathfrak{m}\}$ is a $\mathcal{CS}$ in $\mathcal{V}_{\mathsf{s}^\omega}$.*

(ii) *$\mathcal{V}_{\wp^\omega}$ is Complete, provided $\mathcal{V}_{\mathsf{s}^\omega}$ is Complete. Additionally, for all $\beta > 0$,*

$$\lim_{\mathfrak{m}\to+\infty} \mathsf{s}^{\omega}_{\beta}(\mathfrak{x}_\mathfrak{m},\boldsymbol{\varpi}) = 0 \Leftrightarrow \lim_{\mathfrak{m}\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{x}_\mathfrak{m},\boldsymbol{\varpi}) = \lim_{\mathfrak{m}\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{x}_\mathfrak{m},\mathfrak{x}_\mathfrak{m}) = \wp^{\omega}_{\beta}(\boldsymbol{\varpi},\boldsymbol{\varpi}).$$

Lemma 2.3. [16, 15] *A sequence $\{\mathfrak{x}_\mathfrak{m}\}_{\mathfrak{m}\in\mathbb{N}}$ in $\mathcal{V}_{\mathsf{s}^\omega}$ is a $\mathcal{CS}$ if for all $\beta > 0$ and for each $\varepsilon > 0$, there exists a positive integer $N_0 \in \mathbb{N}$ such that $\mathsf{s}^{\omega}_{\beta}(\mathfrak{x}_\mathfrak{m},\mathfrak{x}_\mathfrak{j}) < \varepsilon$, for all $\mathfrak{m},\mathfrak{j} \geq N_0$.*

Theorem 2.4. [16, 15] *Let $(\mathcal{V},\mathsf{s}^\omega)$ be a $\mathcal{MMS}$ and assume that $\mathcal{V}_{\mathsf{s}^\omega}$ is complete. If $\mathfrak{f} : \mathcal{V}_{\mathsf{s}^\omega} \to \mathcal{V}_{\mathsf{s}^\omega}$ is such that $\mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{w}) \leq \kappa \mathsf{s}^{\omega}_{\beta}(\mathfrak{y},\mathfrak{w})$, for all $\mathfrak{y},\mathfrak{w} \in \mathcal{V}_{\mathsf{s}^\omega}$, and for all $\beta > 0$, where $0 \leq \kappa < 1$ and assume that there exists $\mathfrak{y}_0 \in \mathcal{V}_{\mathsf{s}^\omega}$ such that $\mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_0,\mathfrak{y}_0) \leq +\infty$, then $\mathfrak{f}$ has a unique $\mathcal{FP}$ $\mathfrak{x} \in \mathcal{V}_{\mathsf{s}^\omega}$. In addition, $\{\mathfrak{f}^\mathfrak{j}\mathfrak{x}\}$ converges to this $\mathcal{FP}$.*

Definition 2.6. [1] *Suppose $\mathcal{V} \neq \emptyset$ and $\mathfrak{f}, \mathfrak{g} : \mathcal{V} \to \mathcal{V}$ are self-maps. Whenever $\mathfrak{f}\mathfrak{x} = \mathfrak{g}\mathfrak{x}$ for some $\mathfrak{x} \in \mathcal{V}$, then $\mathfrak{x}$ is called a $\mathcal{CP}$ of $\mathfrak{f}$ and $\mathfrak{g}$. For the $\mathcal{CP}$ $\mathfrak{x}$, if $\mathfrak{w} = \mathfrak{f}\mathfrak{x} = \mathfrak{g}\mathfrak{x}$, then $\mathfrak{w}$ is said to be the $\mathfrak{poc}$ of $\mathfrak{f}$ and $\mathfrak{g}$.*

Definition 2.7. [1] *Suppose $\mathcal{V} \neq \emptyset$ and $\mathfrak{f}, \mathfrak{g} : \mathcal{V} \to \mathcal{V}$ are self-maps. The mappings $\mathfrak{f}$ and $\mathfrak{g}$ are said to be weakly compatible provided that $\mathfrak{f}\mathfrak{g}\mathfrak{x} = \mathfrak{g}\mathfrak{f}\mathfrak{x}$ whenever $\mathfrak{x} \in \mathcal{V}$ is a $\mathcal{CP}$ of $\mathfrak{f}$ and $\mathfrak{g}$.*

Proposition 2.5. [1] *Suppose $\mathcal{V} \neq \emptyset$ and $\mathfrak{f}, \mathfrak{g} : \mathcal{V} \to \mathcal{V}$ are self-maps and assume that the mappings are weakly compatible on $\mathcal{V}$. If $\mathfrak{f}$ and $\mathfrak{g}$ possess a unique $\mathfrak{poc}$, where $\mathfrak{w} = \mathfrak{f}\mathfrak{x} = \mathfrak{g}\mathfrak{x}$ for some $\mathfrak{x} \in \mathcal{V}$, then $\mathfrak{w}$ is the unique common $\mathcal{FP}$ of $\mathfrak{f}$ and $\mathfrak{g}$.*

Throughout this paper, we assume that $\mathcal{PMM}$ is bounded, that is $\wp^{\omega}_{\beta}(\mathfrak{y}, \mathfrak{z}) < +\infty$ for all $\mathfrak{y}, \mathfrak{z} \in \mathcal{V}$ and for all $\beta > 0$. Additionally, $\mathbb{N}$, $\mathbb{W}$, $\mathbb{R}$ and $\mathbb{R}_+$ denotes the sets of natural numbers, whole numbers, real numbers, and positive real numbers, respectively.

3. Continuity in Partial Modular Metric Spaces

Definition 3.1. *Let $(\mathcal{V}, \wp^{\omega})$ be a $\mathcal{PMMS}$ and $\mathfrak{f} : \mathcal{V}_{\wp^{\omega}} \to \mathcal{V}_{\wp^{\omega}}$ be a mapping. Then $\mathfrak{f}$ is continuous at a point $\mathfrak{y}_0 \in \mathcal{V}_{\wp^{\omega}}$ if for a given $\varepsilon > 0$, there exists $\Bbbk > 0$ such that for $\mathfrak{y} \in \mathcal{V}_{\wp^{\omega}}$,*

$$\wp^{\omega}_{\beta}(\mathfrak{y}, \mathfrak{y}_0) < \wp^{\omega}_{\beta}(\mathfrak{y}_0, \mathfrak{y}_0) + \Bbbk \Rightarrow \wp^{\omega}_{\beta}(\mathfrak{f}(\mathfrak{y}), \mathfrak{f}(\mathfrak{y}_0)) < \wp^{\omega}_{\beta}(\mathfrak{f}(\mathfrak{y}_0), \mathfrak{f}(\mathfrak{y}_0)) + \varepsilon, \forall \beta > 0.$$

Hence, $\mathfrak{f}$ is continuous in $\mathcal{V}_{\wp^{\omega}}$ if it is continuous at all points of $\mathcal{V}_{\wp^{\omega}}$.

Proposition 3.1. *Let $(\mathcal{V}, \wp^{\omega})$ be a $\mathcal{PMMS}$ and $\mathfrak{f} : \mathcal{V}_{\wp^{\omega}} \to \mathcal{V}_{\wp^{\omega}}$ be a mapping. The mapping $\mathfrak{f}$ is said to be continuous at a point $\mathfrak{y} \in \mathcal{V}_{\wp^{\omega}}$ if and only if whenever a sequence $\{\mathfrak{y}_{\mathrm{j}}\}$ is convergent to $\mathfrak{y}$ in $\mathcal{V}_{\wp^{\omega}}$, the sequence $\{\mathfrak{f}(\mathfrak{y}_{\mathrm{j}})\}$ is convergent to $\mathfrak{f}(\mathfrak{y})$ in $\mathcal{V}_{\wp^{\omega}}$; that is $\mathfrak{f}$ is sequentially continuous at $\mathfrak{y}$ in $\mathcal{V}_{\wp^{\omega}}$. Analogously,*

$$\lim_{\mathrm{j} \to +\infty} \wp^{\omega}_{\beta}(\mathfrak{y}_{\mathrm{j}}, \mathfrak{y}) = \wp^{\omega}_{\beta}(\mathfrak{y}, \mathfrak{y}) \Rightarrow \lim_{\mathrm{j} \to +\infty} \wp^{\omega}_{\beta}(\mathfrak{f}(\mathfrak{y}_{\mathrm{j}}), \mathfrak{f}(\mathfrak{y})) = \wp^{\omega}_{\beta}(\mathfrak{f}(\mathfrak{y}), \mathfrak{f}(\mathfrak{y})), \forall \beta > 0.$$

Example 3.1. *Suppose that $\mathcal{V} = [0, 1]$ and $\wp^{\omega}_{\beta}(\xi, \zeta) = \frac{1}{\beta}|\xi - \zeta| + \max\{\xi, \zeta\}$, for all $\xi, \zeta \in \mathcal{V}$ and for all $\beta > 0$, then $(\mathcal{V}, \wp^{\omega})$ is a $\mathcal{PMMS}$ and **Figure** (1) illustrates the case for $\beta = 7$. Assume that $\mathcal{B} : \mathcal{V}_{\wp^{\omega}} \to \mathcal{V}_{\wp^{\omega}}$ is such that $\mathcal{B}\xi = \exp(-\ell)\frac{\xi^2}{(1+\xi)}$, for $\xi \in \mathcal{V}_{\wp^{\omega}}$ and $\ell > 0$, which is depicted in **Figure** (2). Let $\{\xi_{\mathrm{j}}\}$ be a non-decreasing sequence converging to ξ in $\mathcal{V}_{\wp^{\omega}}$ such* *that* $\xi_j \leq \xi$, *for all* $j \in \mathbb{W}$. *Also,* $\mathcal{B}$ *is non-decreasing in* $\mathcal{V}_{\wp^\omega}$. *Therefore,* $\mathcal{B}(\xi_j) \leq \mathcal{B}(\xi)$*, for all* $j \in \mathbb{W}$. *Now,* $\xi_j \to \xi$ *in* $\mathcal{V}_{\wp^\omega}$. *Thus, there exists* $\mathbb{k} > 0$ *such that,* $\wp^\omega_\beta(\xi_j, \xi) - \wp^\omega_\beta(\xi, \xi) < \mathbb{k}$*, for all* $\beta > 0$

$$\Rightarrow \qquad \frac{1}{\beta}|\xi_j - \xi| + \max\{\xi_j, \xi\} - \frac{1}{\beta}|\xi - \xi| - \max\{\xi, \xi\} < \mathbb{k}$$

$$\Rightarrow \qquad \frac{1}{\beta}|\xi_j - \xi| < \mathbb{k}. \tag{1}$$

Now,
$$\begin{aligned}\wp^\omega_\beta(\mathcal{B}(\xi_j), \mathcal{B}(\xi)) - \wp^\omega_\beta(\mathcal{B}(\xi), \mathcal{B}(\xi)) &= \frac{1}{\beta}|\mathcal{B}(\xi_j) - \mathcal{B}(\xi)| + \max\{\mathcal{B}(\xi_j), \mathcal{B}(\xi)\} \\ &\quad - \frac{1}{\beta}|\mathcal{B}(\xi) - \mathcal{B}(\xi)| - \max\{\mathcal{B}(\xi), \mathcal{B}(\xi)\} \\ &= \frac{1}{\beta}\left|\frac{\exp(-\ell)\xi_j^2}{(1+\xi_j)} - \frac{\exp(-\ell)\xi^2}{(1+\xi)}\right| + \mathcal{B}(\xi) - \mathcal{B}(\xi) \\ &= \frac{1}{\beta}\left|\frac{\exp(-\ell)(\xi_j - \xi)(\xi + \xi_j + \xi\xi_j)}{(1 + \xi + \xi_j + \xi\xi_j)}\right| \\ &< \frac{1}{\beta}\exp(-\ell)|(\xi_j - \xi)| < \exp(-\ell)\mathbb{k}, \ (\textit{From } (1)).\end{aligned}$$

$$\Rightarrow \qquad \wp^\omega_\beta(\mathcal{B}(\xi_j), \mathcal{B}(\xi)) - \wp^\omega_\beta(\mathcal{B}(\xi), \mathcal{B}(\xi)) < \varepsilon, \textit{ where } \varepsilon = \exp(-\ell)\mathbb{k} > 0.$$

Hence, $\mathcal{B}$ *is continuous in* $\mathcal{V}_{\wp^\omega}$.

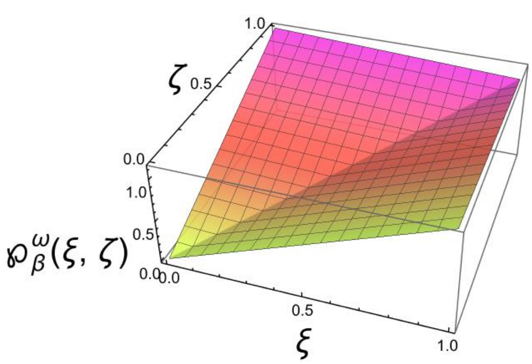

FIGURE 1. $\wp^\omega_\beta(\xi, \zeta)$ for $\beta = 7$.

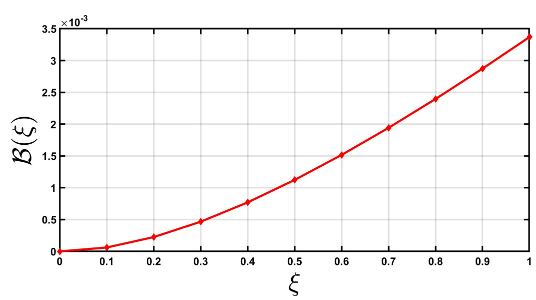

FIGURE 2. $\mathcal{B}\xi = \frac{\exp(-\ell)\xi^2}{(1+\xi)}$ for $\ell = 5$.

4. Ordered Partial Modular Metric Spaces

Definition 4.1. [20] *A poset* $(\mathcal{V}, \preceq)$ *consists of a set* $\mathcal{V} \neq \emptyset$ *together with a partial order relation* $\preceq$*, that is, a binary relation satisfying the following conditions:*

(i) Reflexive: $\xi \preceq \xi$, $\forall \xi \in \mathcal{V}$;

(ii) Antisymmetric: $\xi \preceq \zeta, \zeta \preceq \xi \Rightarrow \xi = \zeta, \forall \xi, \zeta \in \mathcal{V}$;

(iii) Transitive: $\xi \preceq \zeta, \zeta \preceq \mathfrak{j} \Rightarrow \xi \preceq \mathfrak{j}, \forall \xi, \zeta, \mathfrak{j} \in \mathcal{V}$.

Definition 4.2. [20] *Let* $(\mathcal{V}, \preceq)$ *be a poset, then two elements* $\xi, \zeta \in \mathcal{V}$ *are said to be comparable if* $\xi \preceq \zeta$ *or* $\zeta \preceq \xi$.

Definition 4.3. [6] *Let* $(\mathcal{V}, \preceq)$ *denote a poset. Consider a self mapping* $\mathfrak{f} : \mathcal{V} \to \mathcal{V}$. *Then,* $\mathfrak{f}$ *is said to be non-decreasing if*

$$\xi \preceq \zeta \Rightarrow \mathfrak{f}\xi \preceq \mathfrak{f}\zeta, \textit{ for } \xi, \zeta \in \mathcal{V}.$$

Definition 4.4. [6] *Let* $(\mathcal{V}, \preceq)$ *denote a poset. Consider the self mappings* $\mathfrak{f}, \mathfrak{g} : \mathcal{V} \to \mathcal{V}$. *Then,* $\mathfrak{f}$ *is said to be* $\mathfrak{g}$*-non-decreasing if*

$$\mathfrak{g}\xi \preceq \mathfrak{g}\zeta \Rightarrow \mathfrak{f}\xi \preceq \mathfrak{f}\zeta, \textit{ for } \xi, \zeta \in \mathcal{V}.$$

Definition 4.5. *Let* $(\mathcal{V}, \preceq)$ *be a poset and let* $\wp^{\omega} : (0, +\infty) \times \mathcal{V} \times \mathcal{V} \to [0, +\infty)$ *be a* $\mathcal{PMM}$ *on the set* $\mathcal{V}$. *The triple* $(\mathcal{V}, \preceq, \wp^{\omega})$ *is then referred to as an* $\mathcal{OPMMS}$.

Example 4.1. *Let* $\mathcal{V} = [0, +\infty)$ *and define* $\wp^{\omega}_{\beta}(\xi, \zeta) = \exp(-\beta)|\xi - \zeta| + |\xi| + |\zeta|$ *for all* $\xi, \zeta \in \mathcal{V}$ *and* $\beta > 0$. *Let* $\preceq$ *be the order such that* $\xi \preceq \zeta$, *if and only if* $\exp(\xi) \leq \exp(\zeta)$, *then* $(\mathcal{V}, \preceq)$ *is a poset and* $(\mathcal{V}, \preceq, \wp^{\omega})$ *is* $\mathcal{OPMMS}$. ***Figure*** *(3) illustrates a 3D surface plot of* $\wp^{\omega}_{\beta}(\xi, \zeta)$ *for* $\beta = 7$.

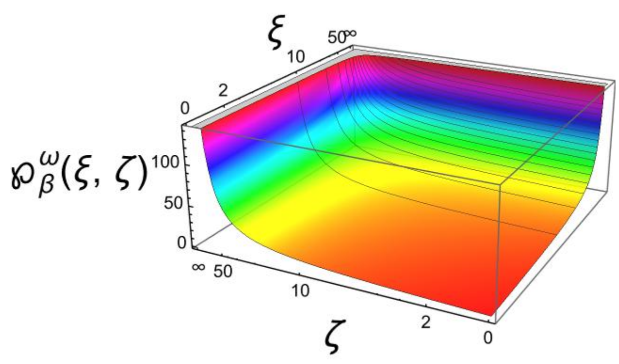

FIGURE 3. Graph of $\wp^{\omega}_{\beta}(\xi, \zeta) = \exp(-\beta)|\xi - \zeta| + |\xi| + |\zeta|$, for $\beta = 7$.

Definition 4.6. *Let* $(\mathcal{V}, \preceq)$ *be a poset and let* $\wp^{\omega} : (0, +\infty) \times \mathcal{V} \times \mathcal{V} \to [0, +\infty)$ *be a* $\mathcal{PMM}$ *on the set* $\mathcal{V}$. *If* $(\mathcal{V}, \wp^{\omega})$ *is complete* $\mathcal{PMMS}$, *then* $(\mathcal{V}, \preceq, \wp^{\omega})$ *referred to as complete* $\mathcal{OPMMS}$.

5. Main Result

We start this section by introducing the definition of compatibility in the framework of $\mathcal{PMMS}$ that generalizes the notion of compatibility introduced by Jungck [12] and Samet et al. [24].

Definition 5.1. *Consider the $\mathcal{PMMS}$, $\mathcal{V}_{\wp^\omega}$ and $\mathfrak{f}, \mathfrak{g} : \mathcal{V}_{\wp^\omega} \to \mathcal{V}_{\wp^\omega}$ are the self-mappings. The pair $\{\mathfrak{f}, \mathfrak{g}\}$ is called partial modular compatible if:*

(i) $\wp^\omega_\beta(\mathfrak{y},\mathfrak{y}) = 0 \Rightarrow \wp^\omega_\beta(\mathfrak{g}\mathfrak{y},\mathfrak{g}\mathfrak{y}) = 0, \forall \beta > 0$;

(ii) whenever a sequence $\{\mathfrak{y}_\mathfrak{j}\}$ in $\mathcal{V}_{\wp^\omega}$ is such that, both $\{\mathfrak{f}\mathfrak{y}_\mathfrak{j}\}$ and $\{\mathfrak{g}\mathfrak{y}_\mathfrak{j}\}$ converges to $\mathfrak{w}$, for some $\mathfrak{w}$ in $\mathcal{V}_{\wp^\omega}$, then

$$\lim_{\mathfrak{j}\to+\infty} \wp^\omega_\beta(\mathfrak{f}\mathfrak{g}\mathfrak{y}_\mathfrak{j}, \mathfrak{g}\mathfrak{f}\mathfrak{y}_\mathfrak{j}) = 0, \ \forall \beta > 0.$$

Consider the sets of functions:

(a) $\Phi = \{\varphi : [0,+\infty) \to [0,+\infty) \mid \varphi$ is non-decreasing and continuous and $\varphi(0) = 0\}$, and

(b) $\Psi = \{\psi : [0,+\infty) \to [0,+\infty) \mid \psi$ is lower semi-continuous and $\psi(0) = 0\}$.

Motivated by the work of Radenovic and Kadelburg [21], Aydi [3], and Samet [24], we introduce the notion of generalized (φ,ψ)-weak contraction in the framework of $\mathcal{OPMMS}$.

Definition 5.2. *(Generalized (φ,ψ)-weak contraction). Consider the $\mathcal{OPMMS}$, $(\mathcal{V},\preceq,\wp^\omega)$. A pair of self-maps $\mathfrak{f}, \mathfrak{g} : \mathcal{V}_{\wp^\omega} \to \mathcal{V}_{\wp^\omega}$ is termed as generalized (φ,ψ)-weak contraction, whenever $\mathfrak{g}\mathfrak{z} \preceq \mathfrak{g}\mathfrak{q}$ for all $\mathfrak{q},\mathfrak{z} \in \mathcal{V}_{\wp^\omega}$ with*

$$\varphi(\wp^\omega_\beta(\mathfrak{f}\mathfrak{q},\mathfrak{f}\mathfrak{z})) \le \varphi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{q},\mathfrak{z})) - \psi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{q},\mathfrak{z})), \tag{2}$$

where $\varphi \in \Phi$, $\psi \in \Psi$ and

$$\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{q},\mathfrak{z}) = \max\left\{\wp^\omega_\beta(\mathfrak{g}\mathfrak{q},\mathfrak{g}\mathfrak{z}), \wp^\omega_\beta(\mathfrak{g}\mathfrak{q},\mathfrak{f}\mathfrak{q}), \wp^\omega_\beta(\mathfrak{g}\mathfrak{z},\mathfrak{f}\mathfrak{z}), \frac{\wp^\omega_{2\beta}(\mathfrak{g}\mathfrak{q},\mathfrak{f}\mathfrak{z}) + \wp^\omega_{2\beta}(\mathfrak{g}\mathfrak{z},\mathfrak{f}\mathfrak{q})}{2}\right\}.$$

Lemma 5.1. *Consider the $\mathcal{OPMMS}$ $(\mathcal{V},\preceq,\wp^\omega)$. Suppose a pair of self-mappings $\mathfrak{f}, \mathfrak{g} : \mathcal{V}_{\wp^\omega} \to \mathcal{V}_{\wp^\omega}$ is a generalized (φ,ψ)-weak contraction satisfying the following conditions:*

(i) $\mathfrak{f}$ is $\mathfrak{g}$-non-decreasing mapping with respect to $\preceq$;

(ii) $\mathfrak{f}\mathcal{V}_{\wp^\omega} \subseteq \mathfrak{g}\mathcal{V}_{\wp^\omega}$;

(iii) $\mathfrak{g}\mathfrak{y}_0 \preceq \mathfrak{f}\mathfrak{y}_0$ *for some* $\mathfrak{y}_0 \in \mathcal{V}_{\wp^\omega}$.

Then, a sequence $\{\mathfrak{f}\mathfrak{y}_\mathrm{j}\}$, $\mathrm{j} \in \mathbb{W}$ *in* $\mathcal{V}_{\wp^\omega}$ *is a* $\mathcal{CS}$ *in* $\mathcal{V}_{\wp^\omega}$.

Proof. Step 1: Consider an arbitrary point $\mathfrak{y}_0$ in $\mathcal{V}_{\wp^\omega}$ such that $\mathfrak{g}\mathfrak{y}_0 \preceq \mathfrak{f}\mathfrak{y}_0$. As $\mathfrak{f}\mathcal{V}_{\wp^\omega} \subseteq \mathfrak{g}\mathcal{V}_{\wp^\omega}$ we select $\mathfrak{y}_1 \in \mathcal{V}_{\wp^\omega}$ such that $\mathfrak{g}\mathfrak{y}_1 = \mathfrak{f}\mathfrak{y}_0$. Similarly, we choose $\mathfrak{y}_2 \in \mathcal{V}_{\wp^\omega}$ such that $\mathfrak{g}\mathfrak{y}_2 = \mathfrak{f}\mathfrak{y}_1$. Continuing this process we have a sequence $\{\mathfrak{y}_\mathrm{j}\}$, $\mathrm{j} \in \mathbb{W}$ in $\mathcal{V}_{\wp^\omega}$ such that,

$$\mathfrak{g}\mathfrak{y}_{\mathrm{j}+1} = \mathfrak{f}\mathfrak{y}_\mathrm{j}, \forall \mathrm{j} \in \mathbb{W}.$$

Since, $\mathfrak{g}\mathfrak{y}_0 \preceq \mathfrak{f}\mathfrak{y}_0 = \mathfrak{g}\mathfrak{y}_1$ and $\mathfrak{f}$ is $\mathfrak{g}$-non-decreasing. Therefore, $\mathfrak{f}\mathfrak{y}_0 \preceq \mathfrak{f}\mathfrak{y}_1$. Similarly, $\mathfrak{g}\mathfrak{y}_1 = \mathfrak{f}\mathfrak{y}_0 \preceq \mathfrak{f}\mathfrak{y}_1 = \mathfrak{g}\mathfrak{y}_2$ and so $\mathfrak{f}\mathfrak{y}_1 \preceq \mathfrak{f}\mathfrak{y}_2$. Continuing this process we get, $\mathfrak{g}\mathfrak{y}_0 \preceq \mathfrak{g}\mathfrak{y}_1 \preceq \mathfrak{g}\mathfrak{y}_2 \preceq \cdots \preceq \mathfrak{g}\mathfrak{y}_\mathrm{j} \preceq \mathfrak{g}\mathfrak{y}_{\mathrm{j}+1} \preceq \cdots$. Consider the case where $\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}_0}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}_0+1}) = 0$, for all $\beta > 0$ and for some index j_0. Under this condition, by ($\wp_2$) we have, $\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}_0}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}_0}) \leq \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}_0}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}_0+1}) = 0$.

$$\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}_0}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}_0}) = 0 \Rightarrow \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}_0+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}_0+1}) = 0.$$

by ($\wp_1$)

$$\mathfrak{f}\mathfrak{y}_{\mathrm{j}_0} = \mathfrak{f}\mathfrak{y}_{\mathrm{j}_0+1} \Rightarrow \mathfrak{g}\mathfrak{y}_{\mathrm{j}_0+1} = \mathfrak{f}\mathfrak{y}_{\mathrm{j}_0+1}.$$

Hence, $\mathfrak{y}_{\mathrm{j}_0+1}$ is a $\mathcal{CP}$. Assume now that $\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_\mathrm{j}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}) > 0$, $\forall \beta > 0$ and for all $\mathrm{j} \in \mathbb{W}$ and we show that

$$\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+2}) \leq \mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{y}_{\mathrm{j}+2}) = \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_\mathrm{j}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}), \ \forall \beta > 0. \tag{3}$$

Utilizing equation (2), which is valid, since $\mathfrak{g}\mathfrak{y}_{\mathrm{j}+1}$ and $\mathfrak{g}\mathfrak{y}_{\mathrm{j}+2}$ are comparable, we conclude that

$$\varphi(\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+2})) \leq \varphi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{y}_{\mathrm{j}+2})) - \psi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{y}_{\mathrm{j}+2})) \tag{4}$$

and since φ is non-decreasing, it yields $\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+2}) \leq \mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{y}_{\mathrm{j}+2})$.

$$\begin{aligned}\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{y}_{\mathrm{j}+2}) = \max\Big\{&\wp^\omega_\beta(\mathfrak{g}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{g}\mathfrak{y}_{\mathrm{j}+2}), \wp^\omega_\beta(\mathfrak{g}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}),\\ &\wp^\omega_\beta(\mathfrak{g}\mathfrak{y}_{\mathrm{j}+2}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+2}), \frac{\wp^\omega_{2\beta}(\mathfrak{g}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+2}) + \wp^\omega_{2\beta}(\mathfrak{g}\mathfrak{y}_{\mathrm{j}+2}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+1})}{2}\Big\}\\ = \max\Big\{&\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_\mathrm{j}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}), \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+2}), \frac{\wp^\omega_{2\beta}(\mathfrak{f}\mathfrak{y}_\mathrm{j}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+2}) + \wp^\omega_{2\beta}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+1})}{2}\Big\}.\end{aligned}$$

By ($\wp_1$),

$$\wp^\omega_{2\beta}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}) = \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}), \ \forall \beta > 0.$$

By ($\wp_4$),

$$\wp^\omega_{2\beta}(\mathfrak{f}\mathfrak{y}_\mathrm{j}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+2}) \leq \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_\mathrm{j}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}) + \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}, \mathfrak{f}\mathfrak{y}_{\mathrm{j}+2})$$

$$- \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+1}),\ \forall \beta > 0.$$

Thus,
$$\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2}) \leq \mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{j+1}, \mathfrak{y}_{j+2})$$
$$= \max\{\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_{j+1}), \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2})\}. \tag{5}$$

When $\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2}) \geq \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_{j+1}) > 0$, for all $\beta > 0$, then it follows that, $\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{j+1}, \mathfrak{y}_{j+2}) = \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2})$, $\forall \beta > 0$, and condition (4) implies that, for all $\beta > 0$

$$\varphi(\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2})) \leq \varphi(\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2})) - \psi(\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2})),$$

which can occur only if $\psi(\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2})) = 0$. Thus, $\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2}) = 0$, for all $\beta > 0$. Hence a contradiction. Therefore, by (5), $\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2}) \leq \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_{j+1})$, for all $\beta > 0$. In the same manner, we obtain that $\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+2}, \mathfrak{f}\mathfrak{y}_{j+3}) \leq \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+2})$, for all $\beta > 0$. Therefore, (3) is satisfied for every $j \in \mathbb{N}$. This implies that the sequence $\{\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_{j+1})\}$ is non-increasing, so the limit

$$\lim_{j\to+\infty} \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_{j+1}) = \lim_{j\to+\infty} \mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{j+1}, \mathfrak{y}_{j+2}) = \Bbbk \geq 0,\ \forall \beta > 0,$$

must exists. Assuming $\Bbbk > 0$, using the inequality (4), and by passing the upper limit as j approaches infinity and applying the properties of φ and ψ, we get

$$\varphi(\Bbbk) \leq \varphi(\Bbbk) - \liminf_{j\to+\infty} \psi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{j+1}, \mathfrak{y}_{j+2})) \leq \varphi(\Bbbk) - \psi(\Bbbk),$$

from which it follows that $\psi(\Bbbk) \leq 0$, which is a contradiction. Hence, $\Bbbk = 0$.

(6) Therefore,
$$\lim_{j\to+\infty} \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_{j+1}) = 0, \text{ for all } \beta > 0, \text{ for all } j \in \mathbb{W}.$$

(7) By $(\wp_2)$
$$\wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_j) \leq \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_{j+1}) \Rightarrow \lim_{j\to+\infty} \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_j) = 0.$$

Similarly,
$$\lim_{j\to+\infty} \wp_\beta^\omega(\mathfrak{f}\mathfrak{y}_{j+1}, \mathfrak{f}\mathfrak{y}_{j+1}) = 0,\ \forall \beta > 0.$$

Step 2: To establish that the sequence $\{\mathfrak{f}\mathfrak{y}_j\}$ is a $\mathcal{CS}$ in $\mathcal{V}_{\wp^\omega}$. Hence, it will be enough to show that $\{\mathfrak{f}\mathfrak{y}_j\}$ is a $\mathcal{CS}$ in $\mathcal{V}_{\mathfrak{s}^\omega}$. Assume, contrary to our goal, that $\{\mathfrak{f}\mathfrak{y}_j\}$ is not a $\mathcal{CS}$ in $\mathcal{V}_{\mathfrak{s}^\omega}$. For some fixed $\flat > 0$, there exists two subsequences $\{\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}\}$ and $\{\mathfrak{f}\mathfrak{y}_{j(\theta)}\}$ such that $j(\theta) > \mathfrak{m}(\theta) > \theta$,

(8) and
$$\mathfrak{s}_\beta^\omega(\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{y}_{j(\theta)}) > \flat,\ \forall \beta > 0.$$

We then select $\mathfrak{j}(\theta)$ corresponding to $\mathfrak{m}(\theta)$ as the smallest even integer with $\mathfrak{j}(\theta) > \mathfrak{m}(\theta)$, for which inequality (8) holds. Consequently,

$$\mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}) \leq \flat, \ \forall \beta > 0. \tag{9}$$

Using the triangle inequality of (s_3) together with (9), we obtain

$$\begin{aligned} \mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) &\leq \mathsf{s}^{\omega}_{\frac{\beta}{2}}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}) + \mathsf{s}^{\omega}_{\frac{\beta}{2}}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) \\ &\leq \flat + \mathsf{s}^{\omega}_{\frac{\beta}{2}}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}). \end{aligned} \tag{10}$$

Applying Lemma 2.1 and for all $\beta > 0$, yields

$$\begin{aligned} \mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) = 2\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) &- \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}) \\ &- \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}). \end{aligned} \tag{11}$$

Taking $\theta \to +\infty$ in (11) and invoking equations (6), (7), it follows that

$$\lim_{\theta \to +\infty} \mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) = 0, \ \forall \beta > 0. \tag{12}$$

Taking $\theta \to +\infty$ in (10) and invoking relations (8) and (12), it follows that,

$$\lim_{\theta \to +\infty} \mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) = \flat, \ \forall \beta > 0. \tag{13}$$

Analogously,

$$\begin{cases} \lim_{\theta \to +\infty} \mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) = \flat, \lim_{\theta \to +\infty} \mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}) = \flat, \\ \lim_{\theta \to +\infty} \mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}) = \flat, \ \forall \beta > 0. \end{cases} \tag{14}$$

Again, applying Lemma 2.1 and for arbitrary $\beta > 0$, yields

$$\begin{aligned} \mathsf{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) = 2\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) &- \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}) \\ &- \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}). \end{aligned} \tag{15}$$

Letting $\theta \to +\infty$ in (15) and invoking equations (7) and (13), it follows that

$$\lim_{\theta \to +\infty} \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) = \frac{\flat}{2} = \mathfrak{k}\,(\text{say}). \tag{16}$$

Similarly, applying Lemma 2.1 and invoking (7) and (14), we get

$$\begin{cases} \lim_{\theta\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) = \mathfrak{k}, \lim_{\theta\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}) = \mathfrak{k}, \\ \lim_{\theta\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}) = \mathfrak{k}, \ \forall \beta > 0. \end{cases} \tag{17}$$

Now, as $\mathfrak{g}\mathfrak{y}_{\mathfrak{m}(\theta)+1}$ and $\mathfrak{g}\mathfrak{y}_{\mathfrak{j}(\theta)}$ are comparable. Thus, from (2), we have

$$\varphi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)})) \le \varphi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{y}_{\mathfrak{j}(\theta)})) - \psi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{y}_{\mathfrak{j}(\theta)})), \tag{18}$$

$$\begin{aligned} \mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{y}_{\mathfrak{j}(\theta)}) &= \max\Big\{ \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{g}\mathfrak{y}_{\mathfrak{j}(\theta)}), \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)+1}), \\ &\quad \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y}_{\mathfrak{j}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}), \frac{\wp^{\omega}_{2\beta}(\mathfrak{g}\mathfrak{y}_{\mathfrak{m}(\theta)+1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) + \wp^{\omega}_{2\beta}(\mathfrak{g}\mathfrak{y}_{\mathfrak{j}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)+1})}{2} \Big\} \\ &= \max\Big\{ \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}), \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)+1}), \\ &\quad \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}), \frac{\wp^{\omega}_{2\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) + \wp^{\omega}_{2\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)-1}, \mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)+1})}{2} \Big\}. \end{aligned} \tag{19}$$

Taking $\theta \to +\infty$ in (18) and (19) and using (6), (7), (16) and (17), we get

$$\begin{aligned} & \varphi(\mathfrak{k}) \le \varphi(\mathfrak{k}) - \psi(\mathfrak{k}) \Rightarrow \psi(\mathfrak{k}) \le 0 \Rightarrow \mathfrak{k} = 0 \\ \Rightarrow\ & \lim_{\theta\to+\infty} \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) = 0 \\ \Rightarrow\ & \flat = 0 \Rightarrow \lim_{\theta\to+\infty} \mathfrak{s}^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}_{\mathfrak{m}(\theta)}, \mathfrak{f}\mathfrak{y}_{\mathfrak{j}(\theta)}) = 0, \forall \beta > 0. \end{aligned} \tag{20}$$

Which is a contradiction. Thus, $\{\mathfrak{f}\mathfrak{y}_{\mathfrak{j}}\}$ is a $\mathcal{CS}$ in $\mathcal{V}_{\mathfrak{s}^{\omega}}$. Therefore, $\{\mathfrak{f}\mathfrak{y}_{\mathfrak{j}}\}$ is a $\mathcal{CS}$ in $\mathcal{V}_{\wp^{\omega}}$. □

Theorem 5.2. *Consider the complete $\mathcal{OPMMS}$ $(\mathcal{V}, \preceq, \wp^{\omega})$. Suppose a pair of self-mappings $\mathfrak{f}, \mathfrak{g} : \mathcal{V}_{\wp^{\omega}} \to \mathcal{V}_{\wp^{\omega}}$ is a generalized (φ, ψ)-weak contraction satisfying the following conditions:*

- *(i) $\mathfrak{f}$ is $\mathfrak{g}$-non-decreasing mapping with respect to $\preceq$;*
- *(ii) $\mathfrak{f}\mathcal{V}_{\wp^{\omega}} \subseteq \mathfrak{g}\mathcal{V}_{\wp^{\omega}}$;*
- *(iii) $\mathfrak{g}\mathfrak{y}_0 \preceq \mathfrak{f}\mathfrak{y}_0$ for some $\mathfrak{y}_0 \in \mathcal{V}_{\wp^{\omega}}$;*
- *(iv) the pair $\{\mathfrak{f}, \mathfrak{g}\}$ is partial modular compatible and continuous;*

Then $\mathfrak{f}$ and $\mathfrak{g}$ possess a $\mathcal{CP}$. Furthermore, $\wp^{\omega}_{\beta}(\mathfrak{y}, \mathfrak{y}) = \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}, \mathfrak{f}\mathfrak{y}) = \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y}, \mathfrak{g}\mathfrak{y}) = 0$, for all $\beta > 0$.

Proof. By Lemma 5.1, $\{\mathfrak{f}\mathfrak{y}_{\mathrm{j}}\}$ is a $\mathcal{CS}$ in $\mathcal{V}_{\wp^\omega}$. Since $\mathcal{V}_{\wp^\omega}$ is complete, the sequence $\{\mathfrak{f}\mathfrak{y}_{\mathrm{j}}\}$ converges in $\mathcal{V}_{\wp^\omega}$. Hence, there exists $\mathfrak{y}$ in $\mathcal{V}_{\wp^\omega}$, and by Lemma 2.2 together with relation (20), we obtain

$$\begin{aligned}\lim_{\mathrm{j},\mathfrak{m}\to+\infty}\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}},\mathfrak{f}\mathfrak{y}_{\mathfrak{m}}) &= \lim_{\mathrm{j}\to+\infty}\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}},\mathfrak{f}\mathfrak{y}_{\mathrm{j}}) = \lim_{\mathrm{j}\to+\infty}\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{\mathrm{j}},\mathfrak{y})\\ &= \wp^\omega_\beta(\mathfrak{y},\mathfrak{y}) = 0, \forall\beta>0. \end{aligned} \tag{21}$$

Suppose, $\mathfrak{f}$ is continuous. Therefore,

$$\lim_{\mathrm{j}\to+\infty}\wp^\omega_\beta(\mathfrak{f}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}}),\mathfrak{f}\mathfrak{y}) = \wp^\omega_\beta(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}), \forall\beta>0. \tag{22}$$

Similarly, $\mathfrak{g}$ is continuous and $\lim_{\mathrm{j}\to+\infty}\wp^\omega_\beta(\mathfrak{y},\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}) = \wp^\omega_\beta(\mathfrak{y},\mathfrak{y})$ for all $\beta>0$.

(23) Therefore, $$\lim_{\mathrm{j}\to+\infty}\wp^\omega_\beta(\mathfrak{g}\mathfrak{y},\mathfrak{g}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1})) = \wp^\omega_\beta(\mathfrak{g}\mathfrak{y},\mathfrak{g}\mathfrak{y}), \forall\beta>0.$$

Now by, $(\wp_4)$ $$\begin{aligned}\wp^\omega_\beta(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}) &\le \wp^\omega_{\frac{\beta}{2}}(\mathfrak{g}\mathfrak{y},\mathfrak{f}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}})) + \wp^\omega_{\frac{\beta}{2}}(\mathfrak{f}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}}),\mathfrak{f}\mathfrak{y}) - \wp^\omega_{\frac{\beta}{2}}(\mathfrak{f}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}}),\mathfrak{f}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}}))\\ &\le \wp^\omega_{\frac{\beta}{2}}(\mathfrak{g}\mathfrak{y},\mathfrak{f}(\mathfrak{g}\mathfrak{y}_{\mathrm{j}+1})) + \wp^\omega_{\frac{\beta}{2}}(\mathfrak{f}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}}),\mathfrak{f}\mathfrak{y})\\ &\le \wp^\omega_{\frac{\beta}{4}}(\mathfrak{g}\mathfrak{y},\mathfrak{g}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1})) + \wp^\omega_{\frac{\beta}{4}}(\mathfrak{g}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}),\mathfrak{f}(\mathfrak{g}\mathfrak{y}_{\mathrm{j}+1}))\\ &\quad + \wp^\omega_{\frac{\beta}{2}}(\mathfrak{f}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}}),\mathfrak{f}\mathfrak{y}), \forall\beta>0.\end{aligned} \tag{24}$$

Since, $\{\mathfrak{f},\mathfrak{g}\}$ is partial modular compatible, continuous, and $\wp^\omega_\beta(\mathfrak{y},\mathfrak{y})=0$. Therefore,

$$\wp^\omega_\beta(\mathfrak{g}\mathfrak{y},\mathfrak{g}\mathfrak{y}) = 0, \forall\beta>0. \tag{25}$$

$$\lim_{\mathrm{j}\to+\infty}\wp^\omega_\beta(\mathfrak{g}(\mathfrak{f}\mathfrak{y}_{\mathrm{j}+1}),\mathfrak{f}(\mathfrak{g}\mathfrak{y}_{\mathrm{j}+1})) = 0, \forall\beta>0. \tag{26}$$

Therefore, letting $\mathrm{j}\to+\infty$ in (24) and using equation (22), (23), (25) and (26), we get for all $\beta>0$, $\wp^\omega_\beta(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}) \le \wp^\omega_\beta(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y})$. Also by $(\wp_2)$, for all $\beta>0$,

$$\wp^\omega_\beta(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}) \le \wp^\omega_\beta(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}) \Rightarrow \wp^\omega_\beta(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}) = \wp^\omega_\beta(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}). \tag{27}$$

As $\mathfrak{g}\mathfrak{y}$ is comparable to itself, by (2) we have

$$\varphi(\wp^\omega_\beta(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y})) \le \varphi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y},\mathfrak{y})) - \psi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y},\mathfrak{y})), \tag{28}$$

where,

$$\begin{aligned}\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y},\mathfrak{y}) &= \max\left\{\wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{g}\mathfrak{y}), \wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}), \wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}), \frac{\wp_{2\beta}^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})+\wp_{2\beta}^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})}{2}\right\}\\ &= \max\{\wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{g}\mathfrak{y}), \wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})\}\\ &= \max\{0, \wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})\}, \text{ (using (25))}\\ &= \wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}).\end{aligned}$$

Hence, from (28) $\quad \varphi(\wp_\beta^\omega(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y})) \le \varphi(\wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})) - \psi(\wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})),$

Using (27), $\quad \varphi(\wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})) \le \varphi(\wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})) - \psi(\wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})),$

$\Rightarrow \quad \psi(\wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})) \le 0 \Rightarrow \wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}) = 0$

Also, $\quad \wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{g}\mathfrak{y}) = \wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}) = \wp_\beta^\omega(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y})$

By $(\wp_1)$, $\quad \mathfrak{g}\mathfrak{y} = \mathfrak{f}\mathfrak{y}.$

Therefore, $\mathfrak{y}$ is the $\mathcal{CP}$. The proof of the theorem is thus complete. □

Theorem 5.3. *In addition to the hypothesis of Theorem 5.2, if*

(i) $\mathfrak{f}$ *and* $\mathfrak{g}$ *are weakly compatible,*

(ii) $\mathfrak{poc}(\mathfrak{f},\mathfrak{g}) \neq \emptyset$ *and* $\mathfrak{g}\mathfrak{y} \preceq \mathfrak{g}(\mathfrak{g}\mathfrak{y})$ *for all* $\mathfrak{y} \in \mathcal{V}_{\wp^\omega}$,

then $\mathfrak{f}$ *and* $\mathfrak{g}$ *possess a common* $\mathcal{FP}$.

Proof. From the Theorem 5.2 $\mathfrak{f}$ and $\mathfrak{g}$ possess a $\mathcal{CP}$ $\mathfrak{y}$(say). Since, $\mathfrak{poc}(\mathfrak{f},\mathfrak{g}) \neq \emptyset$. Thus, there exists $\mathfrak{u} \in \mathcal{V}_{\wp^\omega}$ such that $\mathfrak{u} = \mathfrak{g}\mathfrak{y} = \mathfrak{f}\mathfrak{y}$. Also, $\mathfrak{f}$ and $\mathfrak{g}$ are weakly compatible and therefore,

$$\mathfrak{f}\mathfrak{u} = \mathfrak{f}(\mathfrak{g}\mathfrak{y}) = \mathfrak{g}(\mathfrak{f}\mathfrak{y}) = \mathfrak{g}\mathfrak{u}. \tag{29}$$

By hypothesis, $\mathfrak{g}\mathfrak{y} \preceq \mathfrak{g}(\mathfrak{g}\mathfrak{y}) = \mathfrak{g}\mathfrak{u}$. If $\mathfrak{g}\mathfrak{y} = \mathfrak{g}\mathfrak{u}$, then $\mathfrak{u} = \mathfrak{g}\mathfrak{u} = \mathfrak{f}\mathfrak{u}$, which completes the proof. Now, assume $\mathfrak{g}\mathfrak{y} \prec \mathfrak{g}\mathfrak{u}$. Thus, from (2), we have

$$\varphi(\wp_\beta^\omega(\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{y})) \le \varphi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{u},\mathfrak{y})) - \psi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{u},\mathfrak{y})), \tag{30}$$

where

$$\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{u},\mathfrak{y}) = \max\left\{\wp_\beta^\omega(\mathfrak{g}\mathfrak{u},\mathfrak{g}\mathfrak{y}), \wp_\beta^\omega(\mathfrak{g}\mathfrak{u},\mathfrak{f}\mathfrak{u}), \wp_\beta^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}), \frac{\wp_{2\beta}^\omega(\mathfrak{g}\mathfrak{u},\mathfrak{f}\mathfrak{y})+\wp_{2\beta}^\omega(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{u})}{2}\right\},$$

$$= \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y}).$$

From (30), we get $$\varphi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})) \leq \varphi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})) - \psi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})),$$

$$\Rightarrow \quad \psi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})) \leq 0, \forall \beta > 0.$$

Which is possible only if $\psi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y})) = 0$. Thus, $\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y}) = 0$, for all $\beta > 0$.

Also by $(\wp_2)$, $$\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u}) \leq \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y}) = 0, \forall \beta > 0.$$

$$\Rightarrow \quad \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y}, \mathfrak{f}\mathfrak{y}) = \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{y}) = \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u}) = 0, \forall \beta > 0.$$

$$\Rightarrow \quad \mathfrak{f}\mathfrak{u} = \mathfrak{f}\mathfrak{y} = \mathfrak{u}.$$

Therefore, by (29), we get $\mathfrak{u} = \mathfrak{f}\mathfrak{u} = \mathfrak{g}\mathfrak{u}$. Hence, $\mathfrak{u}$ is a common $\mathcal{FP}$ of $\mathfrak{f}$ and $\mathfrak{g}$. The proof of the theorem is thus complete. □

Corollary 5.4. *In addition to the hypothesis of Theorem 5.3, if* $\mathfrak{poc}(\mathfrak{f}, \mathfrak{g})$ *is unique, then the mappings* $\mathfrak{g}$ *and* $\mathfrak{f}$ *possess a unique common* $\mathcal{FP}$.

In the subsequent theorem, we drop the condition of continuity from the mappings $\mathfrak{f}$ and $\mathfrak{g}$.

Theorem 5.5. *Consider the complete* $\mathcal{OPMMS}$ $(\mathcal{V}, \preceq, \wp^{\omega})$. *Suppose a pair of self-mappings* $\mathfrak{f}, \mathfrak{g} : \mathcal{V}_{\wp^{\omega}} \to \mathcal{V}_{\wp^{\omega}}$ *satisfies*

$$\varphi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{q}, \mathfrak{f}\mathfrak{z})) \leq \varphi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{q}, \mathfrak{z})) - \psi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{q}, \mathfrak{z})), \tag{31}$$

where $\varphi \in \Phi$, $\psi \in \Psi$ *and*

$$\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{q}, \mathfrak{z}) = \max\left\{\wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{q}, \mathfrak{g}\mathfrak{z}), \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{q}, \mathfrak{f}\mathfrak{q}), \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{z}, \mathfrak{f}\mathfrak{z}), \frac{\wp^{\omega}_{2\beta}(\mathfrak{g}\mathfrak{q}, \mathfrak{f}\mathfrak{z}) + \wp^{\omega}_{2\beta}(\mathfrak{g}\mathfrak{z}, \mathfrak{f}\mathfrak{q})}{2}\right\}, \tag{32}$$

whenever $\mathfrak{g}\mathfrak{z} \prec \mathfrak{g}\mathfrak{q}$ *for all* $\mathfrak{q}, \mathfrak{z} \in \mathcal{V}_{\wp^{\omega}}$, $\mathfrak{q} \neq \mathfrak{z}$ *and assume the following conditions:*

(i) $\mathfrak{g}$*-non-decreasing mapping with respect to* $\preceq$*;*

(ii) $\mathfrak{f}\mathcal{V}_{\wp^{\omega}} \subseteq \mathfrak{g}\mathcal{V}_{\wp^{\omega}}$ *and* $\mathfrak{g}\mathcal{V}_{\wp^{\omega}}$ *is closed;*

(iii) $\mathfrak{g}\mathfrak{y}_0 \preceq \mathfrak{f}\mathfrak{y}_0$ *for some* $\mathfrak{y}_0 \in \mathcal{V}_{\wp^{\omega}}$*;*

(iv) if $\{\mathfrak{g}\mathfrak{y}_{\mathfrak{j}}\}$ *is a non-decreasing sequence converges to* $\mathfrak{g}\mathfrak{y}$ *in* $\mathfrak{g}\mathcal{V}_{\wp^{\omega}}$, *then* $\mathfrak{g}\mathfrak{y}_{\mathfrak{j}} \prec \mathfrak{g}\mathfrak{y}$ *and* $\mathfrak{g}\mathfrak{y} \preceq \mathfrak{g}(\mathfrak{g}\mathfrak{y})$ *for all* $\mathfrak{j} \in \mathbb{W}$. (33)

Then $\mathfrak{f}$ and $\mathfrak{g}$ possess a $\mathcal{CP}$. Furthermore, whenever $\mathfrak{f}$ and $\mathfrak{g}$ are weakly compatible, they possess a common $\mathcal{FP}$.

Proof. As in Lemma 5.1, we can construct a sequence $\{\mathfrak{y}_j\}$, $j \in \mathbb{W}$ by

$$\mathfrak{g}\mathfrak{y}_0 \prec \mathfrak{g}\mathfrak{y}_1 \prec \mathfrak{g}\mathfrak{y}_2 \prec \cdots \prec \mathfrak{g}\mathfrak{y}_j \prec \mathfrak{g}\mathfrak{y}_{j+1} \prec \cdots. \tag{34}$$

By assuming, $\mathfrak{f}\mathfrak{y}_j \neq \mathfrak{f}\mathfrak{y}_{j+1}$ and proceeding in the same way as Lemma 5.1, we can prove that $\{\mathfrak{f}\mathfrak{y}_j\}$ is a $\mathcal{CS}$ in $\mathcal{V}_{\wp^\omega}$ and since $\mathcal{V}_{\wp^\omega}$ is complete, so there exists $\mathfrak{w}$ in $\mathcal{V}_{\wp^\omega}$ such that, $\{\mathfrak{f}\mathfrak{y}_j\}$ converges to $\mathfrak{w}$ in $\mathcal{V}_{\wp^\omega}$ such that,

$$\lim_{j,\mathfrak{m}\to+\infty} \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_\mathfrak{m}) = \lim_{j\to+\infty} \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_j, \mathfrak{w}) = \wp^\omega_\beta(\mathfrak{w}, \mathfrak{w}) = 0, \forall \beta > 0. \tag{35}$$

Now, given that $\{\mathfrak{f}\mathfrak{y}_j\} \subseteq \mathfrak{g}\mathcal{V}_{\wp^\omega}$ and $\mathfrak{g}\mathcal{V}_{\wp^\omega}$ is closed, so there is some $\mathfrak{y} \in \mathcal{V}_{\wp^\omega}$ for which $\mathfrak{w} = \mathfrak{g}\mathfrak{y}$. Invoking (34) alongside hypothesis (33) yields

$$\mathfrak{g}\mathfrak{y}_j \prec \mathfrak{g}\mathfrak{y}, \forall j, \mathfrak{g}\mathfrak{y} \preceq \mathfrak{g}(\mathfrak{g}\mathfrak{y}). \tag{36}$$

Next, we demonstrate that $\mathfrak{y}$ serves as a $\mathcal{CP}$ for $\mathfrak{f}$ and $\mathfrak{g}$.

Now, from (36) and using (31) and (32), we get

$$\varphi(\wp^\omega_\beta(\mathfrak{f}\mathfrak{y}, \mathfrak{f}\mathfrak{y}_j)) \leq \varphi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}, \mathfrak{y}_j)) - \psi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}, \mathfrak{y}_j)), \tag{37}$$

where,

$$\begin{aligned}
\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{y}, \mathfrak{y}_j) = \max\Bigg\{ & \wp^\omega_\beta(\mathfrak{g}\mathfrak{y}, \mathfrak{g}\mathfrak{y}_j), \wp^\omega_\beta(\mathfrak{g}\mathfrak{y}, \mathfrak{f}\mathfrak{y}), \wp^\omega_\beta(\mathfrak{g}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y}_j), \\
& \frac{\wp^\omega_{2\beta}(\mathfrak{g}\mathfrak{y}, \mathfrak{f}\mathfrak{y}_j) + \wp^\omega_{2\beta}(\mathfrak{g}\mathfrak{y}_j, \mathfrak{f}\mathfrak{y})}{2} \Bigg\}, \\
= \max\Bigg\{ & \wp^\omega_\beta(\mathfrak{w}, \mathfrak{f}\mathfrak{y}_{j-1}), \wp^\omega_\beta(\mathfrak{g}\mathfrak{y}, \mathfrak{f}\mathfrak{y}), \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{j-1}, \mathfrak{f}\mathfrak{y}_j), \\
& \frac{\wp^\omega_{2\beta}(\mathfrak{w}, \mathfrak{f}\mathfrak{y}_j) + \wp^\omega_{2\beta}(\mathfrak{f}\mathfrak{y}_{j-1}, \mathfrak{f}\mathfrak{y})}{2} \Bigg\}, \\
\leq \max\Bigg\{ & \wp^\omega_\beta(\mathfrak{w}, \mathfrak{f}\mathfrak{y}_{j-1}), \wp^\omega_\beta(\mathfrak{g}\mathfrak{y}, \mathfrak{f}\mathfrak{y}), \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{j-1}, \mathfrak{f}\mathfrak{y}_j), \\
& \frac{\wp^\omega_{2\beta}(\mathfrak{w}, \mathfrak{f}\mathfrak{y}_j) + \wp^\omega_\beta(\mathfrak{f}\mathfrak{y}_{j-1}, \mathfrak{g}\mathfrak{y}) + \wp^\omega_\beta(\mathfrak{g}\mathfrak{y}, \mathfrak{f}\mathfrak{y}) - \wp^\omega_\beta(\mathfrak{g}\mathfrak{y}, \mathfrak{g}\mathfrak{y})}{2} \Bigg\}.
\end{aligned} \tag{38}$$

Taking limit as $\mathfrak{j} \to +\infty$ in (37) and (38) and invoking (35), we get

$$\varphi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y},\mathfrak{w})) \leq \varphi(\wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})) - \psi(\wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})),$$

$$\Rightarrow \quad \varphi(\wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})) \leq \varphi(\wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})) - \psi(\wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})), \text{ (using } (\wp_3) \text{ and } \mathfrak{g}\mathfrak{y} = \mathfrak{w})$$

$$\Rightarrow \quad \psi(\wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})) \leq 0, \forall \beta > 0.$$

Which is possible only if $\psi(\wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y})) = 0$ that is $\wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}) = 0$, for all $\beta > 0$.

Also by $(\wp_2)$, $\quad \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}) \leq \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}), \forall \beta > 0.$

$$\Rightarrow \quad \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}) = \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}) = \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{g}\mathfrak{y}) = 0, \forall \beta > 0.$$

$$\Rightarrow \quad \mathfrak{g}\mathfrak{y} = \mathfrak{f}\mathfrak{y}.$$

Hence, $\mathfrak{y}$ is a $\mathcal{CP}$ of $\mathfrak{f}$ and $\mathfrak{g}$. Hence, $\mathfrak{w} = \mathfrak{g}\mathfrak{y} = \mathfrak{f}\mathfrak{y}$ that is $\mathfrak{poc}(\mathfrak{f},\mathfrak{g}) \neq \emptyset$. Now, since $\mathfrak{f}$ and $\mathfrak{g}$ are weakly compatible, so $\mathfrak{f}(\mathfrak{g}\mathfrak{y}) = \mathfrak{g}(\mathfrak{f}\mathfrak{y})$. Hence,

$$\mathfrak{f}\mathfrak{w} = \mathfrak{f}(\mathfrak{g}\mathfrak{y}) = \mathfrak{g}(\mathfrak{f}\mathfrak{y}) = \mathfrak{g}\mathfrak{w}. \tag{39}$$

By hypothesis (33), $\mathfrak{g}\mathfrak{y} \preceq \mathfrak{g}(\mathfrak{g}\mathfrak{y}) = \mathfrak{g}\mathfrak{w}$. If $\mathfrak{g}\mathfrak{y} = \mathfrak{g}\mathfrak{w}$, then $\mathfrak{w} = \mathfrak{g}\mathfrak{w} = \mathfrak{f}\mathfrak{w}$, which completes the proof. Now, assume $\mathfrak{g}\mathfrak{y} \prec \mathfrak{g}\mathfrak{w}$. Thus, from (31), we have

$$\varphi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y})) \leq \varphi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{w},\mathfrak{y})) - \psi(\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{w},\mathfrak{y})), \tag{40}$$

where
$$\begin{aligned}\mathcal{W}^{\mathfrak{f},\mathfrak{g}}(\mathfrak{w},\mathfrak{y}) &= \max\left\{\wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{w},\mathfrak{g}\mathfrak{y}), \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{w},\mathfrak{f}\mathfrak{w}), \wp^{\omega}_{\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{y}), \right. \\ &\qquad \left. \frac{\wp^{\omega}_{2\beta}(\mathfrak{g}\mathfrak{w},\mathfrak{f}\mathfrak{y}) + \wp^{\omega}_{2\beta}(\mathfrak{g}\mathfrak{y},\mathfrak{f}\mathfrak{w})}{2}\right\} \\ &= \max\{\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y}), \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{w})\} = \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y}).\end{aligned}$$

From (40), we get

$$\varphi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y})) \leq \varphi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y})) - \psi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y})) \Rightarrow \psi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y})) \leq 0, \forall \beta > 0.$$

Which is possible only if $\psi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y})) = 0$ that is $\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y}) = 0, \forall \beta > 0$.

Also by $(\wp_2)$, $\quad \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{w}) \leq \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y}) = 0, \forall \beta > 0.$

$$\Rightarrow \quad \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{y},\mathfrak{f}\mathfrak{y}) = \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{y}) = \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{w},\mathfrak{f}\mathfrak{w}) = 0, \forall \beta > 0.$$

$$\Rightarrow \qquad \mathfrak{f}\mathfrak{w} = \mathfrak{f}\mathfrak{y} = \mathfrak{w}.$$

Therefore, by (39), we get $\mathfrak{w} = \mathfrak{f}\mathfrak{w} = \mathfrak{g}\mathfrak{w}$. Hence, $\mathfrak{w}$ is a common $\mathcal{FP}$ of $\mathfrak{f}$ and $\mathfrak{g}$. The proof of the theorem is thus complete. □

Corollary 5.6. *In addition to the hypothesis of Theorem 5.5, if $\mathfrak{poc}(\mathfrak{f},\mathfrak{g})$ is unique, then the mappings $\mathfrak{f}$ and $\mathfrak{g}$ possess a unique common $\mathcal{FP}$.*

Remark 5.1. *The outcome established in Theorem 5.5 remains true when the contraction condition (31) holds for every pair $\mathfrak{q},\mathfrak{z} \in \mathcal{V}_{\wp^\omega}$, where $\mathfrak{g}\mathfrak{z} \preceq \mathfrak{g}\mathfrak{q}$ and when condition (33) is substituted with the following: If $\{\mathfrak{g}\mathfrak{y}_\mathfrak{j}\}$ is non-decreasing sequence converges to $\mathfrak{g}\mathfrak{y}$ in $\mathfrak{g}\mathcal{V}_{\wp^\omega}$, then $\mathfrak{g}\mathfrak{y}_\mathfrak{j} \preceq \mathfrak{g}\mathfrak{y}$ and $\mathfrak{g}\mathfrak{y} \preceq \mathfrak{g}(\mathfrak{g}\mathfrak{y})$ for all $\mathfrak{j} \in \mathbb{W}$.*

Corollary 5.7. *Consider the complete $\mathcal{OPMMS}$ $(\mathcal{V},\preceq,\wp^\omega)$. Suppose a non-decreasing self-map $\mathfrak{f}:\mathcal{V}_{\wp^\omega} \to \mathcal{V}_{\wp^\omega}$ with $\mathfrak{y}_0 \preceq \mathfrak{f}\mathfrak{y}_0$ for some $\mathfrak{y}_0$ in $\mathcal{V}_{\wp^\omega}$ is such that for $\mathfrak{z} \preceq \mathfrak{q}$, where $\mathfrak{q},\mathfrak{z} \in \mathcal{V}_{\wp^\omega}$,*

$$\varphi(\wp^\omega_\beta(\mathfrak{f}\mathfrak{q},\mathfrak{f}\mathfrak{z})) \le \varphi(\mathcal{W}^\mathfrak{f}(\mathfrak{q},\mathfrak{z})) - \psi(\mathcal{W}^\mathfrak{f}(\mathfrak{q},\mathfrak{z})), \tag{41}$$

where, $\varphi \in \Phi$, $\psi \in \Psi$ and

$$\mathcal{W}^\mathfrak{f}(\mathfrak{q},\mathfrak{z}) = \max\left\{\wp^\omega_\beta(\mathfrak{q},\mathfrak{z}), \wp^\omega_\beta(\mathfrak{f}\mathfrak{q},\mathfrak{q}), \wp^\omega_\beta(\mathfrak{z},\mathfrak{f}\mathfrak{z}), \frac{\wp^\omega_{2\beta}(\mathfrak{f}\mathfrak{q},\mathfrak{z}) + \wp^\omega_{2\beta}(\mathfrak{q},\mathfrak{f}\mathfrak{z})}{2}\right\},$$

then the mapping $\mathfrak{f}$ possess a $\mathcal{FP}$ $\mathfrak{y}$ in $\mathcal{V}_{\wp^\omega}$, provided that $\mathfrak{f}$ is continuous. Furthermore, $\wp^\omega_\beta(\mathfrak{y},\mathfrak{y}) = 0$, for all $\beta > 0$.

Theorem 5.8. *Suppose all the assumptions of Corollary (5.7) are satisfied, and assume further that for arbitrary elements $\mathfrak{q},\mathfrak{z} \in \mathcal{V}_{\wp^\omega}$, there exists some $\varpi \in \mathcal{V}_{\wp^\omega}$ such that $\varpi \preceq \mathfrak{q}$ and $\varpi \preceq \mathfrak{z}$. Under these condition, the mapping $\mathfrak{f}$ possess a unique $\mathcal{FP}$.*

Proof. Let $\mathfrak{q}$ and $\mathfrak{z}$ be two $\mathcal{FP}$ of the mapping $\mathfrak{f}$, that is, $\mathfrak{f}(\mathfrak{q}) = \mathfrak{q}$ and $\mathfrak{f}(\mathfrak{z}) = \mathfrak{z}$. We consider the two cases:

(i) If $\mathfrak{q}$ and $\mathfrak{z}$ are comparable, then invoking relation (41), yields

$$\varphi(\wp^\omega_\beta(\mathfrak{q},\mathfrak{z})) = \varphi(\wp^\omega_\beta(\mathfrak{f}\mathfrak{q},\mathfrak{f}\mathfrak{z})) \le \varphi(\mathcal{W}(\mathfrak{q},\mathfrak{z})) - \psi(\mathcal{W}(\mathfrak{q},\mathfrak{z})), \tag{42}$$

where $$\mathcal{W}^\mathfrak{f}(\mathfrak{q},\mathfrak{z}) = \max\left\{\wp^\omega_\beta(\mathfrak{q},\mathfrak{z}), \wp^\omega_\beta(\mathfrak{f}\mathfrak{q},\mathfrak{q}), \wp^\omega_\beta(\mathfrak{z},\mathfrak{f}\mathfrak{z}), \frac{\wp^\omega_{2\beta}(\mathfrak{f}\mathfrak{q},\mathfrak{z}) + \wp^\omega_{2\beta}(\mathfrak{q},\mathfrak{f}\mathfrak{z})}{2}\right\}$$

$$= \wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{z}), \forall \beta > 0.$$

Hence, from (42) $\quad \varphi(\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{z})) \leq \varphi(\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{z})) - \psi(\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{z}))$

$\Rightarrow \quad \psi(\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{z})) \leq 0 \Rightarrow \wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{z}) = 0, \forall \beta > 0.$

Also, from (21) $\quad \wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{q}) = 0, \forall \beta > 0.$

By $(\wp_1)$, $\mathfrak{q} = \mathfrak{z}$. Therefore, $\mathfrak{q}$ is the unique $\mathcal{FP}$ of $\mathfrak{f}$.

(ii) Now suppose that $\mathfrak{q}$ and $\mathfrak{z}$ are not comparable. Select an element $\boldsymbol{\varpi} \in \mathcal{V}_{\wp^{\omega}}$ such that $\mathfrak{q} \preceq \boldsymbol{\varpi}$ and $\mathfrak{z} \preceq \boldsymbol{\varpi}$. Since, $\mathfrak{f}$ is non-decreasing, it follows that $\mathfrak{q} = \mathfrak{f}^{\mathfrak{j}}(\mathfrak{q}) \preceq \mathfrak{f}^{\mathfrak{j}}(\boldsymbol{\varpi})$ for every $\mathfrak{j} \in \mathbb{N}$. By applying condition (41), we obtain

$$\begin{aligned}\varphi(\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi})) = \varphi(\wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{f}^{\mathfrak{j}-1}\mathfrak{q},\mathfrak{f}\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi})) &\leq \varphi(\mathcal{W}^{\mathfrak{f}}(\mathfrak{f}^{\mathfrak{j}-1}\mathfrak{q},\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi})) \\ &\quad - \psi(\mathcal{W}^{\mathfrak{f}}(\mathfrak{f}^{\mathfrak{j}-1}\mathfrak{q},\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi})),\end{aligned}$$

where,

$$\begin{aligned}\mathcal{W}^{\mathfrak{f}}(\mathfrak{f}^{\mathfrak{j}-1}\mathfrak{q},\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi}) = \max\Bigg\{&\wp^{\omega}_{\beta}(\mathfrak{f}^{\mathfrak{j}-1}\mathfrak{q},\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi}), \wp^{\omega}_{\beta}(\mathfrak{f}\mathfrak{f}^{\mathfrak{j}-1}\mathfrak{q},\mathfrak{f}^{\mathfrak{j}-1}\mathfrak{q}), \\ &\wp^{\omega}_{\beta}(\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi},\mathfrak{f}\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi}), \frac{\wp^{\omega}_{2\beta}(\mathfrak{f}\mathfrak{f}^{\mathfrak{j}-1}\mathfrak{q},\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi}) + \wp^{\omega}_{2\beta}(\mathfrak{f}^{\mathfrak{j}-1}\mathfrak{q},\mathfrak{f}\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi})}{2}\Bigg\} \\ \leq \max\{&\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi}), \wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi})\},\end{aligned}$$

as $\wp^{\omega}_{\beta}(\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi})$ approaches to zero, for all sufficiently large values of $\mathfrak{j}$. In a manner similar to the proof of Lemma 5.1, it can be demonstrated that

$$\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi}) \leq \mathcal{W}^{\mathfrak{f}}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi}) \leq \wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}-1}\boldsymbol{\varpi}), \forall \beta > 0.$$

Therefore, the sequence $\{\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi})\}$ is non-increasing and converges to some limit $\mathfrak{l} \geq 0$. Suppose, $\mathfrak{l} > 0$ and if we take the limit in the relation

$$\varphi(\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi})) \leq \varphi(\mathcal{W}^{\mathfrak{f}}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi})) - \psi(\mathcal{W}^{\mathfrak{f}}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi})),$$

we end up with a contradiction. Therefore, $\mathfrak{l} = 0$ that is, $\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi}) \to 0$, for all $\beta > 0$. Similarly, we can show that $\wp^{\omega}_{\beta}(\mathfrak{z},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi}) \to 0$, for all $\beta > 0$ as $\mathfrak{j} \to +\infty$.

Then by $(\wp_4)$ $\quad \wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{z}) \leq \wp^{\omega}_{\frac{\beta}{2}}(\mathfrak{q},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi}) + \wp^{\omega}_{\frac{\beta}{2}}(\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi},\mathfrak{z}) - \wp^{\omega}_{\frac{\beta}{2}}(\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi},\mathfrak{f}^{\mathfrak{j}}\boldsymbol{\varpi}).$

Taking $\mathfrak{j} \to +\infty$, we get $\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{z}) = 0, \forall \beta > 0.$

Also, from (21) $\wp^{\omega}_{\beta}(\mathfrak{q},\mathfrak{q}) = 0, \forall \beta > 0.$

By $(\wp_1)$ $\mathfrak{q} = \mathfrak{z}.$

This confirms the $\mathcal{FP}$ is unique. □

Example 5.1. *Suppose that* $\mathcal{V} = [0,1]$ *and* $\wp^{\omega}_{\beta}(\xi,\zeta) = \frac{1}{\beta}|\xi-\zeta| + \max\{\xi,\zeta\}, \forall \xi,\zeta \in \mathcal{V}$ *and* $\beta > 0$*, then* $(\mathcal{V},\preceq,\wp^{\omega})$ *is an* $\mathcal{OPMMS}$ *with usual order* $\preceq$ *that is* $\zeta \preceq \xi$*, if and only if* $\zeta \leq \xi$*. Assume that* $\mathfrak{f} : \mathcal{V}_{\wp^{\omega}} \to \mathcal{V}_{\wp^{\omega}}$ *is such that* $\mathfrak{f}\rho = \frac{\exp(-\ell)\rho^2}{(1+\rho)}$*, for* $\rho \in \mathcal{V}_{\wp^{\omega}}$ *and* $\ell > 0$*. Then,* $\mathfrak{f}$ *is non-decreasing and continuous in* $\mathcal{V}_{\wp^{\omega}}$ *and also* $\mathfrak{y}_0 \preceq \mathfrak{f}\mathfrak{y}_0$ *for* $\mathfrak{y}_0 = 0 \in \mathcal{V}$*. Let* $\varphi \in \Phi$*,* $\psi \in \Psi$ *such that* $\varphi(\varpi) = \varpi$ *and* $\psi(\varpi) = (1-\exp(-\ell))\varpi$*, for* $\varpi \in [0,+\infty)$*. Let* $\zeta \preceq \xi$ *then* $\zeta \leq \xi$*. As* $\mathfrak{f}$ *is non-decreasing. Therefore,* $\mathfrak{f}(\zeta) \leq \mathfrak{f}(\xi)$*.*

$$
\begin{aligned}
\textit{Now,}\qquad \wp^{\omega}_{\beta}(\mathfrak{f}\xi,\mathfrak{f}\zeta) &= \frac{1}{\beta}|\mathfrak{f}\xi-\mathfrak{f}\zeta| + \max\{\mathfrak{f}\xi,\mathfrak{f}\zeta\}\\
&= \frac{1}{\beta}\left|\frac{\exp(-\ell)\xi^2}{(1+\xi)} - \frac{\exp(-\ell)\zeta^2}{(1+\zeta)}\right| + \mathfrak{f}(\xi)\\
&\leq \frac{1}{\beta}\left|\frac{\exp(-\ell)(\xi-\zeta)(\zeta+\xi+\zeta\xi)}{(1+\zeta+\xi+\zeta\xi)}\right| + \exp(-\ell)\frac{\xi^2}{(1+\xi)},\\
&\leq \frac{\exp(-\ell)}{\beta}|\xi-\zeta| + \exp(-\ell)\xi\\
&\qquad\qquad \textit{since, } \frac{\zeta+\xi+\zeta\xi}{1+\zeta+\xi+\zeta\xi} < 1; \textit{ and } \frac{\xi^2}{(1+\xi)} \leq \xi \textit{ in } \mathcal{V}\\
&= \exp(-\ell)\left(\frac{1}{\beta}|\xi-\zeta| + \max\{\xi,\zeta\}\right)\\
&= \exp(-\ell)\wp^{\omega}_{\beta}(\xi,\zeta)\\
&\leq \exp(-\ell)\mathcal{W}^{\mathfrak{f}}(\xi,\zeta)\\
&= \mathcal{W}^{\mathfrak{f}}(\xi,\zeta) - (\mathcal{W}^{\mathfrak{f}}(\xi,\zeta) - \exp(-\ell)\mathcal{W}^{\mathfrak{f}}(\xi,\zeta))
\end{aligned}
$$

$$
\implies \quad \varphi(\wp^{\omega}_{\beta}(\mathfrak{f}\xi,\mathfrak{f}\zeta)) \leq \varphi(\mathcal{W}^{\mathfrak{f}}(\xi,\zeta)) - \psi(\mathcal{W}^{\mathfrak{f}}(\xi,\zeta)),
$$

where
$$
\mathcal{W}^{\mathfrak{f}}(\xi,\zeta) = \max\left\{\wp^{\omega}_{\beta}(\xi,\zeta), \wp^{\omega}_{\beta}(\mathfrak{f}\xi,\xi), \wp^{\omega}_{\beta}(\zeta,\mathfrak{f}\zeta), \frac{\wp^{\omega}_{2\beta}(\mathfrak{f}\xi,\zeta) + \wp^{\omega}_{2\beta}(\xi,\mathfrak{f}\zeta)}{2}\right\}.
$$

As there exists elements which are comparable to both ξ *and* ζ *in* $\mathcal{V}_{\wp^{\omega}}$*. So, by Theorem 5.8,* $\mathfrak{f}$ *has a unique* $\mathcal{FP}$*, 0.*

Remark 5.2. *The outcome established in Theorems 5.7 and 5.8 remains true when the continuity condition is substituted by the following condition;*
whenever a non-decreasing sequence $\{\mathfrak{y}_{\mathfrak{j}}\}$ *converges to* $\mathfrak{y}$ *in* $\mathcal{V}_{\wp^{\omega}}$, *then* $\mathfrak{y}_{\mathfrak{j}} \preceq \mathfrak{y}$ *for all* $\mathfrak{j} \in \mathbb{N}$.

6. Application to Volterra Integro-Differential Equations

This section aims to demonstrate, by applying Theorem 5.8, the existence and uniqueness of a solution for the nonlinear $\mathcal{VIDE}$ of the form,

$$\mathfrak{I}'(\varpi) = \mathfrak{l}(\varpi) + \int_0^{\varpi} \mho(\varpi, \tau, \mathfrak{I}(\tau))d\tau,\ \mathfrak{I}(0) = b. \tag{43}$$

The solution of equation (43) is equivalent to the solution of the equation,

$$\mathfrak{I}(\varpi) = b + \int_0^{\varpi} \mathfrak{l}(s)ds + \int_0^{\varpi}\int_0^{s} \mho(s, \tau, \mathfrak{I}(\tau))d\tau ds, \tag{44}$$

where $\mathfrak{I} \in C^1[0,h]$, which is the set of all continuously differentiable function on $[0,h]$ and $b, h > 0$, $\varpi, \tau, s \in [0,h]$, the functions $\mathfrak{l} : [0,h] \to \mathbb{R}$, and $\mho : [0,h] \times [0,h] \times \mathbb{R} \to \mathbb{R}$ are continuous. Let $\mathcal{V} = C^1[0,h]$ and define the mapping $\wp^{\omega} : (0,+\infty) \times \mathcal{V} \times \mathcal{V} \to [0,+\infty)$, for all $\mathfrak{I}_1, \mathfrak{I}_2 \in \mathcal{V}$ and $\beta > 0$

$$\wp^{\omega}_{\beta}(\mathfrak{I}_1, \mathfrak{I}_2) = \max_{\varpi \in [0,h]} (\exp(-\beta)|\mathfrak{I}_1(\varpi) - \mathfrak{I}_2(\varpi)| + \max\{|\mathfrak{I}_1(\varpi)|, |\mathfrak{I}_2(\varpi)|\}).$$

It is readily seen that $(\mathcal{V}, \wp^{\omega})$ constitutes a complete $\mathcal{PMMS}$. Then, $(\mathcal{V}, \preceq, \wp^{\omega})$ is a complete $\mathcal{OPMMS}$ with usual order $\preceq$ that is $\mathfrak{I}_1 \preceq \mathfrak{I}_2$, if and only if $\mathfrak{I}_1 \leq \mathfrak{I}_2$ or $\mathfrak{I}_2 \leq \mathfrak{I}_1$. Define a continuous self map $\mathcal{P}$ on $C^1[0,h]$ by,

$$(\mathcal{P}\mathfrak{I})(\varpi) = b + \int_0^{\varpi} \mathfrak{l}(s)ds + \int_0^{\varpi}\int_0^{s} \mho(s, \tau, \mathfrak{I}(\tau))d\tau ds. \tag{45}$$

Finding the solution of (44) is equivalent in finding the $\mathcal{FP}$ of (45).

Theorem 6.1. *Consider the nonlinear* $\mathcal{VIDE}$ *(43). Furthermore, suppose that the following conditions are satisfied:*

- (H1) *The functions* $\mathfrak{l}$ *and* $\mho$ *are continuous;*
- (H2) *for all* $\beta > 0$, $\varpi, s, \tau \in [0,h]$, $\mathfrak{I}_1, \mathfrak{I}_2 \in C^1[0,h]$, $h > 0$ *such that* $\mathfrak{I}_1$ *and* $\mathfrak{I}_2$ *are comparable that is* $\mathfrak{I}_1 \leq \mathfrak{I}_2$ *or* $\mathfrak{I}_2 \leq \mathfrak{I}_1$ *and* $\phi_{\mho} : [0,h] \times [0,h] \to [0,+\infty)$ *is a non-negative function,*

$$|\mho(s, \tau, \mathfrak{I}_1(\tau)) - \mho(s, \tau, \mathfrak{I}_2(\tau))| \leq |\mathfrak{I}_1(\varpi) - \mathfrak{I}_2(\varpi)|\phi_{\mho}(s, \tau);$$

(H3) $\max\{|(\mathcal{P}\mathfrak{I}_1)(\varpi)|, |(\mathcal{P}\mathfrak{I}_2)(\varpi)|\} \leq \exp(-\ell)\max\{|\mathfrak{I}_1(\varpi)|, |\mathfrak{I}_2(\varpi)|\}$, *for some* $\ell > 0$*;*

(H4) $\int_0^{\varpi}\int_0^{s}|\phi_{\mho}(s,\tau)|dsd\tau \leq \exp(-\ell)$, *for some* $\ell > 0$.

Under these conditions, the equation (43) has a unique solution.

Proof. For $\mathfrak{I}_1, \mathfrak{I}_2 \in \mathcal{V}$ and every $\varpi \in [0,h]$, it follows that

$$\begin{aligned}
|(\mathcal{P}\mathfrak{I}_1)(\varpi) - (\mathcal{P}\mathfrak{I}_2)(\varpi)| &= |b + \int_0^{\varpi} \mathfrak{l}(s)ds + \int_0^{\varpi}\int_0^{s} \mho(s,\tau,\mathfrak{I}_1(\tau))d\tau ds \\
&\quad - (b + \int_0^{\varpi} \mathfrak{l}(s)ds + \int_0^{\varpi}\int_0^{s} \mho(s,\tau,\mathfrak{I}_2(\tau))d\tau ds)| \\
&\leq \int_0^{\varpi}\int_0^{s} |\mho(s,\tau,\mathfrak{I}_1(\tau)) - \mho(s,\tau,\mathfrak{I}_2(\tau))|d\tau ds \\
&\leq |\mathfrak{I}_1(\varpi) - \mathfrak{I}_2(\varpi)| \int_0^{\varpi}\int_0^{s} \phi_{\mho}(s,\tau)d\tau ds \\
&\leq \exp(-\ell)\exp(\beta)\max\{|\mathfrak{I}_1(\varpi)|, |\mathfrak{I}_2(\varpi)|\} \\
&\quad - \exp(\beta)\max\{|(\mathcal{P}\mathfrak{I}_1)(\varpi)|, |(\mathcal{P}\mathfrak{I}_2)(\varpi)|\} \\
&+ \exp(-\ell)|\mathfrak{I}_1(\varpi) - \mathfrak{I}_2(\varpi)|; \qquad \forall\, \beta > 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & \exp(-\beta)|(\mathcal{P}\mathfrak{I}_1)(\varpi) - (\mathcal{P}\mathfrak{I}_2)(\varpi)| + \max\{|(\mathcal{P}\mathfrak{I}_1)(\varpi)|, |(\mathcal{P}\mathfrak{I}_2)(\varpi)|\} \\
& \leq \exp(-\ell)(\exp(-\beta)|\mathfrak{I}_1(\varpi) - \mathfrak{I}_2(\varpi)| + \max\{|\mathfrak{I}_1(\varpi)|, |\mathfrak{I}_2(\varpi)|\}).
\end{aligned}$$

Taking the maximum with respect to $\varpi \in [0,h]$ on both sides,

$$\begin{aligned}
\wp_{\beta}^{\omega}(\mathcal{P}\mathfrak{I}_1, \mathcal{P}\mathfrak{I}_2) &\leq \exp(-\ell)\wp_{\beta}^{\omega}(\mathfrak{I}_1, \mathfrak{I}_2) \\
&\leq \exp(-\ell)\mathcal{W}^{\mathfrak{f}}(\mathfrak{I}_1, \mathfrak{I}_2) \\
&= \mathcal{W}^{\mathfrak{f}}(\mathfrak{I}_1, \mathfrak{I}_2) - ((1 - \exp(-\ell))\mathcal{W}^{\mathfrak{f}}(\mathfrak{I}_1, \mathfrak{I}_2))
\end{aligned} \tag{46}$$

$$\Longrightarrow \qquad \varphi(\wp_{\beta}^{\omega}(\mathcal{P}\mathfrak{I}_1, \mathcal{P}\mathfrak{I}_2) \leq \varphi(\mathcal{W}^{\mathfrak{f}}(\mathfrak{I}_1, \mathfrak{I}_2)) - \psi(\mathcal{W}^{\mathfrak{f}}(\mathfrak{I}_1, \mathfrak{I}_2)),$$

where,
$$\mathcal{W}(\mathfrak{I}_1, \mathfrak{I}_2) = \max\left\{\wp_{\beta}^{\omega}(\mathfrak{I}_1, \mathfrak{I}_2), \wp_{\beta}^{\omega}(\mathcal{P}\mathfrak{I}_1, \mathfrak{I}_1), \wp_{\beta}^{\omega}(\mathfrak{I}_2, \mathcal{P}\mathfrak{I}_2), \frac{\wp_{2\beta}^{\omega}(\mathcal{P}\mathfrak{I}_1, \mathfrak{I}_2) + \wp_{2\beta}^{\omega}(\mathfrak{I}_1, \mathcal{P}\mathfrak{I}_2)}{2}\right\}. \tag{47}$$

Where $\varphi(\flat) = \flat$ and $\psi(\flat) = (1 - \exp(-\ell))\flat$, for $\flat \in [0, +\infty)$ and $\ell > 0$. Therefore, by Theorem 5.8, $\mathcal{P}$ has a unique $\mathcal{FP}$. Hence, the nonlinear $\mathcal{VIDE}$ (43) has a unique solution. □

Example 6.1. *Consider the nonlinear $\mathcal{VIDE}$*

$$\mathfrak{I}'(\varpi) = \frac{1}{16} - \frac{\varpi}{8} - \exp(-\varpi) - \frac{1}{16}\exp(-2\varpi) + \int_0^{\varpi} \frac{(\varpi-\tau)\mathfrak{I}^2(\tau)}{4} d\tau,\ \mathfrak{I}(0) = 1. \tag{48}$$

Define a self-map $\mathcal{P}$, as in (45)*, and suppose*

$$|(\mathcal{P}\mathfrak{I})(\varpi)| \le \|\mathcal{P}\|\|\mathfrak{I}(\varpi)\|_\infty,\ \textit{with } \|\mathcal{P}\| < 1 \textit{ and } \|\mathfrak{I}(\varpi)\|_\infty < 1. \tag{49}$$

Here, $\mathfrak{l}(\varpi) = \frac{1}{16} - \frac{\varpi}{8} - \exp(-\varpi) - \frac{1}{16}\exp(-2\varpi)$ *and* $\mho(s,\tau,\mathfrak{I}(\tau)) = \frac{(s-\tau)\mathfrak{I}^2(\tau)}{4}$ *are continuous. The solution of equation (48) is equivalent to finding the $\mathcal{FP}$ of (45). Now, for all* $\mathfrak{I}_1, \mathfrak{I}_2 \in \mathcal{V}$ *and* $s, \tau \in [0,1]$*,*

$$\begin{aligned} |\mho(s,\tau,\mathfrak{I}_1(\tau)) - \mho(s,\tau,\mathfrak{I}_2(\tau))| &= \left| \frac{(s-\tau)\mathfrak{I}_1^2(\tau)}{4} - \frac{(s-\tau)\mathfrak{I}_2^2(\tau)}{4} \right| \\ &= \frac{(s-\tau)}{4}|(\mathfrak{I}_1(\tau)-\mathfrak{I}_2(\tau))(\mathfrak{I}_1(\tau)+\mathfrak{I}_2(\tau))| \\ &= |\mathfrak{I}_1(\tau)-\mathfrak{I}_2(\tau)||\phi_{\mho}(s,\tau)|, \end{aligned}$$

where, $\phi_{\mho}(s,\tau) = \frac{(s-\tau)}{4}(\mathfrak{I}_1(\tau)+\mathfrak{I}_2(\tau))$

Now, $\quad |\phi_{\mho}(s,\tau)| = \left|\frac{(s-\tau)}{4}\right| |\mathfrak{I}_1(\tau)+\mathfrak{I}_2(\tau)| \le \frac{1}{2} \le \exp(-\ell)$*, for some* $l \in (0, \ln 2]$.

Also, $\quad \max\{|(\mathcal{P}\mathfrak{I}_1)(\varpi)|, |(\mathcal{P}\mathfrak{I}_2)(\varpi)|\} \le \exp(-\ell)\max\{|\mathfrak{I}_1(\varpi)|, |\mathfrak{I}_2(\varpi)|\}.$

Since, all of the hypotheses of Theorem 6.1 are fulfilled, we conclude that the nonlinear $\mathcal{VIDE}$ (48) admits a unique solution that is the mapping $\mathcal{P}$ has a unique $\mathcal{FP}$. The exact solution of equation (48) is $\mathfrak{I}(\varpi) = \exp(-\varpi)$*. Applying the Picard iteration method, we obtain:*

$$\mathfrak{I}_{i+1}(\varpi) = (\mathcal{P}\mathfrak{I}_i)(\varpi) = 1 + \int_0^{\varpi} \mathfrak{l}(s)ds + \int_0^{\varpi}\int_0^s \mho(s,\tau,\mathfrak{I}_i(\tau))d\tau ds,$$

$i = 0,1,2,3\ldots$*. The initial approximation is chosen as* $\mathfrak{I}_0(\varpi) = 1$.

Table *(1) presents the computed values for* $\mathfrak{I}_i$, $i = 1,2,3,\ldots$ *obtained using MATLAB for various values of* $\varpi \in [0,1]$ *and also displays the corresponding absolute errors* $\mathcal{E}_i = |\mathfrak{I}_i(\varpi) - \mathfrak{I}(\varpi)|$; $i = 1,2,3,\ldots$. ***Figure*** *(4) illustrates the comparison between the exact and numerical solutions, whereas* ***Figure*** *(5) illustrates the graph of the absolute error. Finally, the* ***Figures*** *(6) and (7) depicts that* $\mathfrak{I}_i(0.7)$ *and* $\mathfrak{I}_i(0.9)$ *approaches to* $\mathfrak{I}(0.7)$ *and* $\mathfrak{I}(0.9)$ *respectively. Using the Theorem 6.1, we have established the existence and uniqueness of the solution to nonlinear*

$\varpi \rightarrow$	0.1	0.2	0.3	0.4	0.5
Exact	0.904837	0.818731	0.740818	0.670320	0.606531
$\Im_1(\varpi)$	0.904839	0.818762	0.740969	0.670778	0.607610
$\Im_2(\varpi)$	0.904837	0.818731	0.740818	0.670320	0.606531
$\Im_3(\varpi)$	0.904837	0.818731	0.740818	0.670320	0.606531
$\varpi \rightarrow$	0.6	0.7	0.8	0.9	1.0
Exact	0.548812	0.496585	0.449329	0.406570	0.367879
$\Im_1(\varpi)$	0.550974	0.500458	0.455722	0.416485	0.382525
$\Im_2(\varpi)$	0.548812	0.496588	0.449334	0.406581	0.367901
$\Im_3(\varpi)$	0.548812	0.496585	0.449329	0.406570	0.367879
$\varpi \rightarrow$	0.1	0.2	0.3	0.4	0.5
$\mathcal{E}_1$	0.000002	0.000031	0.000150	0.000458	0.001080
$\mathcal{E}_2$	0.000000	0.000000	0.000000	0.000000	0.000000
$\mathcal{E}_3$	0.000000	0.000000	0.000000	0.000000	0.000000
$\varpi \rightarrow$	0.6	0.7	0.8	0.9	1.0
$\mathcal{E}_1$	0.002162	0.003873	0.006393	0.009916	0.014646
$\mathcal{E}_2$	0.000001	0.000002	0.000005	0.000011	0.000022
$\mathcal{E}_3$	0.000000	0.000000	0.000000	0.000000	0.000000

TABLE 1. Iterated values and the absolute errors.

$\mathcal{VIDE}$ (48). The theoretical result is further supported by a numerical investigation employing Picard iteration.

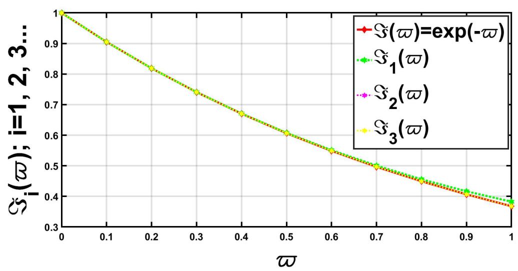

FIGURE 4. The graph depicts that the iterations $\mathfrak{I}_i(\varpi) = (\mathcal{P}\mathfrak{I}_i)(\varpi)$ converges to the exact solution $\exp(-\varpi)$.

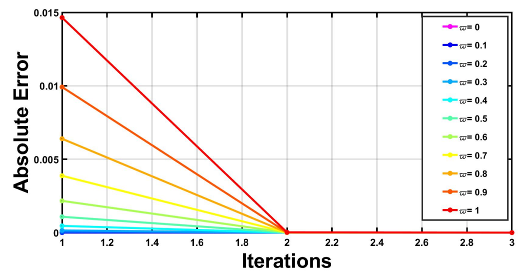

FIGURE 5. The graph depicts absolute errors $\mathcal{E}_i = |\mathfrak{I}_i(\varpi) - \mathfrak{I}(\varpi)|$; $i = 1,2,3,\dots$ for different values of $\varpi \in [0,1]$.

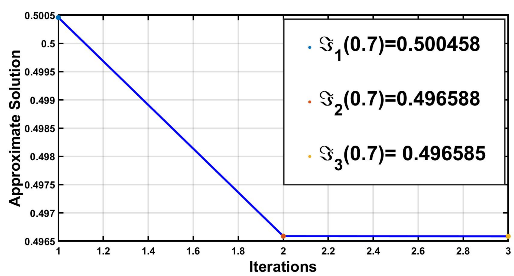

FIGURE 6. $\mathfrak{I}_i(0.7), i = 1,2,3\dots$ approaches to $\mathfrak{I}(0.7)$.

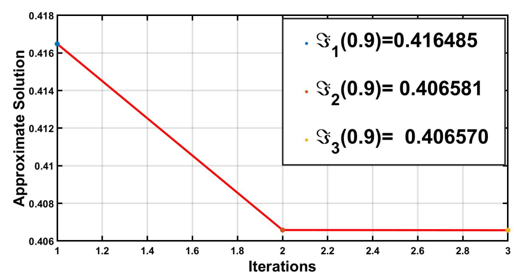

FIGURE 7. $\mathfrak{I}_i(0.9), i = 1,2,3\dots$ approaches to $\mathfrak{I}(0.9)$.

The convergence behaviour is depicted in the ***Figure*** *(4) and the error* ***Figure*** *(5) illustrates that with the increase in number of iteration for different values of ϖ, the error decreases, which confirms that the equation (48) admits a unique solution $\mathfrak{I}(\varpi) = \exp(-\varpi)$. Hence,* $\exp(-\varpi)$ *is the $\mathcal{FP}$ of the equation* (45)*, that is* $\mathcal{P}(\mathfrak{I}(\varpi)) = \mathfrak{I}(\varpi) = \exp(-\varpi)$.

7. Hyers-Ulam-Rassias Stability

This section employs the $\mathcal{FP}$ technique to demonstrate the $\mathcal{HUR}$ stability of the nonlinear $\mathcal{VIDE}$ in the framework of $\mathcal{OPMMS}$.

Definition 7.1. *The nonlinear $\mathcal{VIDE}$ (43) is said to exhibit $\mathcal{HUR}$ stability if, whenever a function satisfies the inequality*

$$|\mathfrak{I}'(\varpi) - \mathfrak{l}(\varpi) - \int_0^{\varpi} \mho(\varpi,\tau,\mathfrak{I}(\tau))d\tau| \leq \mathcal{Q}(\varpi), \tag{50}$$

for a non-negative function $\mathcal{Q}$, there exists an exact solution $\mathfrak{I}_0$ of equation (43) and a constant $\mathcal{S} > 0$ such that

$$|\mathfrak{I}(\varpi) - \mathfrak{I}_0(\varpi)| \leq \mathcal{S}\mathcal{Q}(\varpi), \tag{51}$$

where $\mathcal{S}$ is independent of $\mathfrak{I}$ and $\mathfrak{I}_0$.

Remark 7.1. *If we replace $\mathcal{Q}(\varpi)$ by a constant $\alpha \geq 0$ in (50) and (51), then the nonlinear $\mathcal{VIDE}$ (43) is said to exhibit $\mathcal{HU}$ stability. The subsequent Lemma is essential for establishing the main result of this section.*

Motivated by the works of [10] and [25], we construct the following Lemma in the framework of $\mathcal{OPMMS}$.

Lemma 7.1. *Suppose $(\mathcal{V}, \preceq, \wp^{\omega})$ is complete $\mathcal{OPMMS}$. Further, let $\mathcal{P} : \mathcal{V}_{\wp^\omega} \to \mathcal{V}_{\wp^\omega}$ be the contraction mapping satisfying $\wp^{\omega}_{\beta}(\mathcal{P}\xi, \mathcal{P}\mathfrak{I}) \leq \mathcal{G}\wp^{\omega}_{\beta}(\xi, \mathfrak{I})$, $\xi, \mathfrak{I} \in \mathcal{V}_{\wp^\omega}$ with $\mathcal{G} < 1$ and $\xi \preceq \mathfrak{I}$. Assume that there is a non-negative integer m for which $\wp^{\omega}_{\beta}(\mathcal{P}^{m+1}\xi, \mathcal{P}^m\xi) < +\infty$ for some $\xi \in \mathcal{V}_{\wp^\omega}$, then the following conditions holds:*

- *(i) $\{\mathcal{P}^m\xi\}_{m\in\mathbb{N}}$ converges towards ξ^*, a $\mathcal{FP}$ of $\mathcal{P}$;*
- *(ii) ξ^* represents the unique $\mathcal{FP}$ of $\mathcal{P}$ within*

$$\mathcal{V}^*_{\wp^\omega} = \{\mathfrak{I} \in \mathcal{V}_{\wp^\omega} : \wp^{\omega}_{\beta}(\mathcal{P}^m\xi, \mathfrak{I}) < +\infty\};$$

- *(iii) if $\mathfrak{I} \in \mathcal{V}^*_{\wp^\omega}$, then $\wp^{\omega}_{\beta}(\mathfrak{I}, \xi^*) \leq \frac{1}{1-\mathcal{G}}\wp^{\omega}_{\beta}(\mathcal{P}\mathfrak{I}, \mathfrak{I})$.*

Theorem 7.2. *Suppose that the conditions (H1), (H2), (H3) and (H4) holds. Additionally, if a continuously differentiable function $\mathfrak{I}$ satisfies*

$$|\mathfrak{I}'(\varpi) - \mathfrak{l}(\varpi) - \int_0^{\varpi} \mho(\varpi, \tau, \mathfrak{I}(\tau))d\tau| \leq \mathcal{Q}(\varpi),\ \forall \varpi \in [0, h], \tag{52}$$

where $\varpi, \tau \in [0, h]$ and $\mathfrak{l} : [0, h] \to \mathbb{R}$, and $\mho : [0, h] \times [0, h] \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{Q} : [0, h] \to [0, +\infty)$ are continuous mapping with

$$\int_0^{\varpi} \mathcal{Q}(\tau)d\tau + \exp(\beta)\max\{|\mathcal{P}\mathfrak{I}|, |\mathfrak{I}|\} \leq \exp(\beta)\mathcal{Z}\mathcal{Q}(\varpi),$$

for each $\beta > 0$*, some* $\mathcal{Z} > 0$ *and define a self-map* $\mathcal{P}$*, as in* (45)*, then there exists a unique solution* $\Im_0 \in C^1[0,h]$ *of* (43)*, satisfying* (44)*, with*

$$|\Im(\varpi) - \Im_0(\varpi)| \le \frac{\mathcal{Z}\exp(\beta)\mathcal{Q}(\varpi)}{1-\exp(-\ell)},\ \forall \varpi \in [0,h] \text{ and for some } \ell > 0.$$

Proof. Let $\mathcal{V} = C^1[0,h]$, and define the mapping $\wp^\omega : (0,+\infty)\times\mathcal{V}\times\mathcal{V} \to [0,+\infty)$ by

$$\wp^\omega_\beta(\Im,\Re) = \max_{\varpi\in[0,h]}(\exp(-\beta)|\Im(\varpi)-\Re(\varpi)| + \max\{|\Im(\varpi)|,|\Re(\varpi)|\}),\ \forall \Im,\Re \in \mathcal{V},$$

then $(\mathcal{V},\wp^\omega)$ constitutes a complete $\mathcal{PMMS}$. Also, $(\mathcal{V},\preceq,\wp^\omega)$ is a complete $\mathcal{OPMMS}$ with usual order $\preceq$ that is $\Im \preceq \Re$, if and only if $\Im \le \Re$ or $\Re \le \Im$. Now, from Theorem 6.1, we find that the equation (43) has a unique solution. By equation (46), we have

$$\wp^\omega_\beta(\mathcal{P}\Im,\mathcal{P}\Re) \le \exp(-\ell)\wp^\omega_\beta(\Im,\Re).$$

From Theorem 5.8, we infer that the operator $\mathcal{P}$ is a generalized $(\varphi,\psi)-$contraction. By fundamental theorem of calculus $\mathcal{P}(\Re) \in \mathcal{V}_{\wp^\omega}$, $\Re \in \mathcal{V}_{\wp^\omega}$ and thus we have

$$\exp(-\beta)|\mathcal{P}\Re(\varpi) - \Re(\varpi)| + \max\{|\mathcal{P}\Re(\varpi)|,|\Re(\varpi)|\} < +\infty,$$

for any $\Re \in \mathcal{V}_{\wp^\omega}$ and $\varpi \in [0,h]$. That is, $\wp^\omega_\beta(\mathcal{P}\Re,\Re) < +\infty$, for all $\Re \in \mathcal{V}_{\wp^\omega}$. Similarly, $\exp(-\beta)|\mathcal{P}\Im(\varpi) - \Re(\varpi)| + \max\{|\mathcal{P}\Im(\varpi)|,|\Re(\varpi)|\} < +\infty$, for all $\Im \in \mathcal{V}_{\wp^\omega}$ and $\varpi \in [0,h]$, then $\wp^\omega_\beta(\mathcal{P}\Im,\Re) < +\infty$, for all $\Re \in \mathcal{V}_{\wp^\omega}$. That is

$$\{\Im \in \mathcal{V}_{\wp^\omega} : \wp^\omega_\beta(\mathcal{P}\Im,\Re) < +\infty\} = \mathcal{V}_{\wp^\omega},\ \forall \Re \in \mathcal{V}_{\wp^\omega}.$$

Now let $\Im \in \mathcal{V}_{\wp^\omega}$. Again, by (52), for all $\varpi \in [0,h]$,

$$-\mathcal{Q}(\varpi) \le \Im'(\varpi) - \mathfrak{l}(\varpi) - \int_0^\varpi \mho(\varpi,\tau,\Im(\tau))d\tau \le \mathcal{Q}(\varpi). \tag{53}$$

Integrating each term of inequality of (53) from 0 to ϖ, we get

$$\begin{aligned}
&|\Im(\varpi) - \Im(0) - \int_0^\varpi \mathfrak{l}(\tau)d\tau - \int_0^\varpi\int_0^s \mho(s,\tau,\Im(\tau))d\tau ds| \le \int_0^\varpi \mathcal{Q}(\tau)d\tau \\
\Rightarrow\quad &|\Im(\varpi) - \mathcal{P}\Im(\varpi)| \le \exp(\beta)(\mathcal{Z}\mathcal{Q}(\varpi) - \max\{|\mathcal{P}\Im|,|\Im|\}) \\
\Rightarrow\quad &\exp(-\beta)|\mathcal{P}\Im(\varpi) - \Im(\varpi)| + \max\{|\mathcal{P}\Im|,|\Im|\} \le \mathcal{Z}\mathcal{Q}(\varpi) \\
\Rightarrow\quad &\wp^\omega_\beta(\mathcal{P}\Im,\Im) \le \mathcal{Z}\mathcal{Q}(\varpi).
\end{aligned}$$

As the conditions of Lemma 7.1 are satisfied, a unique solution $\mathfrak{I}_0$ exists satisfying

$$\wp_\beta^\omega(\mathfrak{I},\mathfrak{I}_0) \leq \frac{1}{1-\exp(-\ell)}\wp_\beta^\omega(\mathcal{P}\mathfrak{I},\mathfrak{I}) \leq \frac{\mathcal{Z}\mathcal{Q}(\varpi)}{1-\exp(-\ell)}.$$

Further, by the definition of $\mathcal{PMMS}$, we have, for all $\mathfrak{I},\mathfrak{I}_0 \in \mathcal{V}$,

$$\wp_\beta^\omega(\mathfrak{I},\mathfrak{I}_0) = \max_{\varpi\in[0,h]}(\exp(-\beta)|\mathfrak{I}(\varpi)-\mathfrak{I}_0(\varpi)| + \max\{|\mathfrak{I}(\varpi)|,|\mathfrak{I}_0(\varpi)|\})$$

$$\Rightarrow \qquad \exp(-\beta)|\mathfrak{I}(\varpi)-\mathfrak{I}_0(\varpi)| \leq \wp_\beta^\omega(\mathfrak{I},\mathfrak{I}_0), \forall\varpi\in[0,h]$$

$$\Rightarrow \qquad |\mathfrak{I}(\varpi)-\mathfrak{I}_0(\varpi)| \leq \frac{\mathcal{Z}\exp(\beta)\mathcal{Q}(\varpi)}{1-\exp(-\ell)}, \forall\varpi\in[0,h].$$

Hence, under the given conditions, the nonlinear $\mathcal{VIDE}$ (43) possesses $\mathcal{HUR}$ stability. □

The subsequent corollary establishes the $\mathcal{HU}$ stability result of the nonlinear $\mathcal{VIDE}$ (43).

Corollary 7.3. *Suppose that the conditions (H1), (H2), (H3) and (H4) holds. Additionally, if a continuously differentiable function $\mathfrak{I}$ satisfies*

$$|\mathfrak{I}'(\varpi)-\mathfrak{l}(\varpi)-\int_0^\varpi \mho(\varpi,\tau,\mathfrak{I}(\tau))d\tau| \leq \alpha, \ \forall\varpi\in[0,h], \tag{54}$$

where $\varpi,\tau\in[0,h]$, $\mathfrak{l}:[0,h]\to\mathbb{R}$, $\mho:[0,h]\times[0,h]\times\mathbb{R}\to\mathbb{R}$, $\alpha\geq 0$, and define a self-map $\mathcal{P}$, as in (45), then there exists a unique solution $\mathfrak{I}_0\in C^1[0,h]$ of (43), satisfying (44), with $\max\{|\mathcal{P}\mathfrak{I}|,|\mathfrak{I}|\} \leq \alpha(\mathcal{Z}-\varpi\exp(-\beta))$, for each $\beta>0$, some $\mathcal{Z}>0$, then

$$|\mathfrak{I}(\varpi)-\mathfrak{I}_0(\varpi)| \leq \frac{\mathcal{Z}\exp(\beta)\alpha}{1-\exp(-\ell)},$$

for all $\varpi\in[0,h]$ and for some $\ell>0$.

Example 7.1. *Consider the nonlinear $\mathcal{VIDE}$*

$$\mathfrak{I}'(\varpi) = \frac{1}{16}-\frac{\varpi}{8}-\exp(-\varpi)-\frac{\exp(-2\varpi)}{16}+\int_0^\varpi \frac{(\varpi-\tau)\mathfrak{I}^2(\tau)}{4}d\tau, \ \mathfrak{I}(0)=1. \tag{55}$$

*The exact solution of equation (55) is $\mathfrak{I}_0(\varpi)=\exp(-\varpi)$ and the numerical solution are computed in **Table** (1). To see the $\mathcal{HUR}$ stability, we consider $\mathcal{Q}(\varpi)=\exp(\varpi)$ and $\mathfrak{I}(\varpi) = \frac{\varpi}{16}-\frac{\varpi^2}{16}+\frac{\varpi^3}{24}-\frac{1}{32}+\exp(-\varpi)+\frac{\exp(-2\varpi)}{32}$ be any other numerical solution. Now,*

$$|\mathfrak{I}'(\varpi)-\mathfrak{l}(\varpi)-\int_0^\varpi \mho(\varpi,\tau,\mathfrak{I}(\tau))d\tau|$$
$$= \Bigg|\frac{\varpi^2}{8}-\Bigg(\frac{10243\varpi}{49152}+\frac{23\exp(-\varpi)}{64}+\frac{257\exp(-2\varpi)}{4096}+\frac{\exp(-3\varpi)}{576}$$

$$
\begin{aligned}
&+\frac{\exp(-4\varpi)}{65536}+\frac{9\varpi\exp(-\varpi)}{32}+\frac{\varpi\exp(-2\varpi)}{2048}+\frac{3\varpi^2\exp(-\varpi)}{32}\\
&+\frac{\varpi^2\exp(-2\varpi)}{4096}+\frac{\varpi^3\exp(-\varpi)}{48}+\frac{\varpi^3\exp(-2\varpi)}{6144}+\frac{\varpi^2}{8192}-\frac{\varpi^3}{6144}\\
&+\frac{\varpi^4}{6144}-\frac{\varpi^5}{7680}+\frac{7\varpi^6}{92160}-\frac{\varpi^7}{32256}+\frac{\varpi^8}{129024}-\frac{25009}{589824}\Big)\Big|\\
&=|\mathcal{L}(\varpi)|\leq\exp(\varpi)=\mathcal{Q}(\varpi),
\end{aligned}
$$

*for all $\varpi\in[0,1]$ (verified using MATLAB). The values of $\mathcal{L}(\varpi)$ and $\mathcal{D}(\varpi)=|\mathfrak{I}(\varpi)-\mathfrak{I}_0(\varpi)|$ are computed using MATLAB. Here, $\mathfrak{I}(0)=\mathfrak{I}_0(0)=1$ and $\mathcal{D}(0)=0$. The corresponding values of $\mathcal{D}(\varpi)$ for different values of ϖ are presented in the **Table** (2), and the graphs of $\mathcal{L}(\varpi)$ and $\mathcal{D}(\varpi)$ are illustrated in **Figures** (8) and (9) respectively. Choose $\beta=0.2$, $\ell=\ln 2$, and $\mathcal{Z}=2$ then,*

$$
\begin{aligned}
\int_0^{\varpi}\mathcal{Q}(\tau)d\tau+\exp(\beta)\max\{|\mathcal{P}\mathfrak{I}|,|\mathfrak{I}|\}&=\exp(\varpi)+\exp(\beta)\max\{|\mathcal{P}\mathfrak{I}|,|\mathfrak{I}|\}\\
&\leq\exp(\beta)\mathcal{Z}\exp(\varpi)=\exp(\beta)\mathcal{Z}\mathcal{Q}(\varpi),\ \forall\varpi\in[0,1].
\end{aligned}
$$

$\varpi\to$	0.1	0.2	0.3	0.4	0.5
$\mathfrak{I}_0$	0.904837	0.818731	0.740818	0.670320	0.606531
$\mathfrak{I}(\varpi)$	0.904839	0.818762	0.740969	0.670778	0.607610
$\mathcal{D}(\varpi)$	0.000002	0.000031	0.000150	0.000458	0.001080
$\varpi\to$	0.6	0.7	0.8	0.9	1.0
$\mathfrak{I}_0$	0.548812	0.496585	0.449329	0.406570	0.367879
$\mathfrak{I}(\varpi)$	0.550974	0.500458	0.455722	0.416485	0.382525
$\mathcal{D}(\varpi)$	0.002162	0.003873	0.006393	0.009916	0.014646

TABLE 2. Values of $\mathfrak{I}_0(\varpi)$, $\mathfrak{I}(\varpi)$ and $\mathcal{D}(\varpi)$.

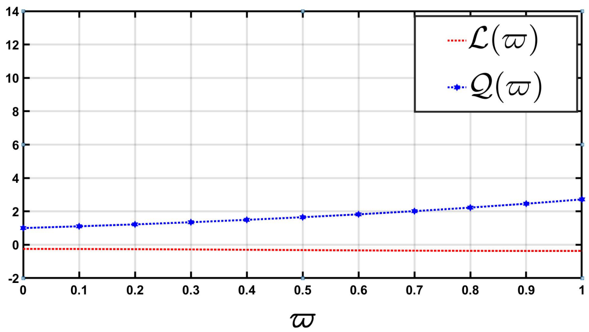

FIGURE 8. Comparison between $\mathcal{L}(\varpi)$ and $\mathcal{Q}(\varpi)$.

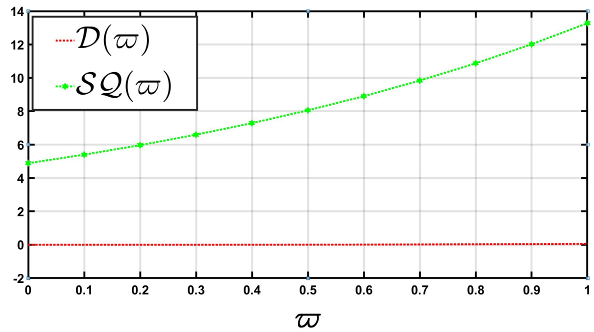

FIGURE 9. Comparison between $\mathcal{D}(\varpi)$ and $\mathcal{S}\mathcal{Q}(\varpi)$.

Since, all of the hypotheses of Theorem 7.2 are fulfilled, there exists a unique solution $\Im_0(\varpi)$ such that,

$$\begin{aligned}\mathcal{D}(\varpi) = |\Im(\varpi) - \Im_0(\varpi)| &= \left|\frac{\varpi}{16} - \frac{\varpi^2}{16} + \frac{\varpi^3}{24} - \frac{1}{32} + \frac{\exp(-2\varpi)}{32}\right| \\ &\leq \frac{\mathcal{Z}\exp(\beta)\exp(\varpi)}{1-\exp(-\ln 2)} = 2\mathcal{Z}\exp(\beta)\mathcal{Q}(\varpi) \\ &= \mathcal{S}\mathcal{Q}(\varpi), \forall \varpi \in [0,h], \textit{ where } \mathcal{S} = 2\mathcal{Z}\exp(\beta).\end{aligned}$$

Hence, if an approximate solution $\Im(\varpi)$ satisfies (50) for some non-negative function $\mathcal{Q}(\varpi)$ then there a unique solution $\Im_0(\varpi)$ which satisfies (51) for some constant $\mathcal{S} > 0$.

8. CONCLUSION

In this article, we have established the existence and uniqueness results for $\mathcal{CP}$ and $\mathcal{FP}$ of generalized (φ, ψ)-weak contraction within the framework of $\mathcal{OPMMS}$, considering cases both with and without continuity assumptions on the involved mappings. The theoretical results were validated through illustrative examples. Our primary contribution is the demonstration of the applicability of our theoretical result through detailed analysis of nonlinear $\mathcal{VIDE}$ that guarantee both the existence and uniqueness of solution. Additionally, these results are supported by numerical example that validate our result. Furthermore, we have established the $\mathcal{HUR}$ stability for nonlinear $\mathcal{VIDE}$ and is validated through a numerical example. The numerical computations and graphical illustrations were performed using MATLAB to validate the theoretical $\mathcal{FP}$ results.

Our contribution generalize and unify, several well-known existing results in the literature, thereby significantly extending several results in nonlinear analysis to a broader class of nonlinear contraction within $\mathcal{PMMS}$. This generalization facilitates the use of $\mathcal{FP}$ method for solving and finding the $\mathcal{HUR}$ stability of nonlinear $\mathcal{VIDE}$, thus deepening insights into $\mathcal{FP}$ and stability theory within nonlinear analysis.

The established results can be extended to the framework of generalized metric structures, like $\mathcal{PM}\mathsf{b}-\mathcal{MS}$ [13] and C^*-algebra valued $\mathcal{PMMS}$ [18]. Additionally, the application to nonlinear $\mathcal{VIDE}$ and $\mathcal{HUR}$ stability of such functional equations can be studied in these generalized spaces.

Author contributions

All the authors contributed equally to the writing of the manuscript and reviewed it.

Conflict of Interests

The authors declare that there is no conflict of interests.

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