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SOME NONLINEAR INTEGRAL INEQUALITIES FOR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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Abstract. In this paper, we establish some nonlinear Volterra-Fredholm integral inequalities which provide an explicit bound on unknown function, and can be used as a tool in the study of certain nonlinear mixed integral equations.

Keywords: integral inequality; Volterra-Fredholm integral equation; explicit bound.

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1. Introduction

The differential and integral inequalities occupy a very privileged position in the theory of differential and integral equations. On the basis of various motivations, in the recent years nonlinear integral inequalities have received considerable attention because of the important applications to a variety of problems in diverse fields of nonlinear differential and integral equations. Some integral inequalities for differential and integral equations are established by Gronwall [6], Bellman [2] and Pachpatte [8, 9] which provide explicit bounds on solutions

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of a class of differential and integral equations, which are further studied and generalized by succeeding mathematicians; see [1, 3, 4, 5, 7, 10, 11, 12].

In this paper, we establish some nonlinear Volterra-Fredholm integral inequalities, which can be used as handy tools to study the qualitative as well as the quantitative properties of solutions of some Volterra-Fredholm nonlinear integral equations. Some applications are also given to convey the importance of our results.

The following Lemma is useful in our main results.

Lemma 1.1. [13] *Assume that $a \geq 0, p \geq 1$, then $a^{\frac{1}{p}} \leq \frac{1}{p}k^{\frac{1-p}{p}}a + \frac{p-1}{p}k^{\frac{1}{p}}$, for any $k > 0$.*

Throughout this paper, we denote $R_+ = [0,\infty), I = [\alpha,\beta]$ and $D = \left\{(t,s) \in I^2 : \alpha \leq s \leq t \leq \beta\right\}$.

2. Main Results

In this section, we state and prove some Volterra-Fredholm nonlinear integral inequalities which can be useful to study certain properties of solutions of some Volterra-Fredholm nonlinear integral equations.

Theorem 2.1. *Let $u(t), f(t), g(t), c'(t) \in C(I,R_+)$ and*

$$u^p(t) \leq c(t) + \int_\alpha^t f(s)u(s)ds + \int_\alpha^\beta g(s)u^p(s)ds. \tag{2.1}$$

If $R_1 = \int_\alpha^\beta g(s)\exp\left(\int_\alpha^s m_1 f(\sigma)d\sigma\right)ds < 1$, then

$$\begin{aligned} u^p(t) \leq & \frac{c(\alpha) + \int_\alpha^\beta g(s)\left(\int_\alpha^s [c'(\tau) + m_2 f(\tau)]\exp\left(\int_\tau^s m_1 f(\sigma)d\sigma\right)d\tau\right)ds}{1 - R_1}\exp\left(\int_\alpha^t m_1 f(\sigma)d\sigma\right) \\ & + \int_\alpha^t \left[c'(s) + m_2 f(s)\right]\exp\left(\int_s^t m_1 f(\sigma)d\sigma\right)ds, \end{aligned} \tag{2.2}$$

where $p \geq 1, k > 0, m_1 = \frac{1}{p}k^{\frac{1-p}{p}}$ and $m_2 = \frac{p-1}{p}k^{\frac{1}{p}}$.

Proof. Define a function $z(t)$ by

$$z(t) = c(t) + \int_\alpha^t f(s)u(s)ds + \int_\alpha^\beta g(s)u^p(s)ds.$$

Then $u(t) \leq z^{\frac{1}{p}}(t)$,

$$z(\alpha) = c(\alpha) + \int_{\alpha}^{\beta} g(s)u^p(s)ds \tag{2.3}$$

and

$$z'(t) = c'(t) + f(t)u(t) \leq c'(t) + f(t)z^{\frac{1}{p}}(t). \tag{2.4}$$

From Lemma 1.1 and equation (2.4), we have

$$z'(t) \leq c'(t) + m_1 f(t)z(t) + m_2 f(t)$$

$$z'(t) - m_1 f(t)z(t) \leq c'(t) + m_2 f(t)$$

$$\left[\frac{z(t)}{\exp\left(\int_{\alpha}^{t} m_1 f(s)ds\right)}\right]' \leq [c'(t) + m_2 f(t)] \exp\left(-\int_{\alpha}^{t} m_1 f(s)ds\right) ds. \tag{2.5}$$

Set $t = s$; in equation (2.5) and integrate with respect to s from α to t, we get

$$\frac{z(t)}{\exp\left(\int_{\alpha}^{t} m_1 f(s)ds\right)} \leq z(\alpha) + \int_{\alpha}^{t} \left[c'(s) + m_2 f(s)\right] \exp\left(-\int_{\alpha}^{s} m_1 f(\sigma)d\sigma\right) ds,$$

and hence

$$z(t) \leq z(\alpha)\exp\left(\int_{\alpha}^{t} m_1 f(s)ds\right) + \exp\left(\int_{\alpha}^{t} m_1 f(s)ds\right)\left(\int_{\alpha}^{t} \left[c'(s) + m_2 f(s)\right] \exp\left(-\int_{\alpha}^{s} m_1 f(\sigma)d\sigma\right) ds\right). \tag{2.6}$$

As $u^p(t) \leq z(t)$ from equation (2.6), we have

$$u^p(t) \leq z(\alpha)\exp\left(\int_{\alpha}^{t} m_1 f(s)ds\right) + \int_{\alpha}^{t} \left[c'(s) + m_2 f(s)\right] \exp\left(\int_{s}^{t} m_1 f(\sigma)d\sigma\right) ds. \tag{2.7}$$

Now from equation (2.3) and (2.7), we have

$$\begin{aligned} z(\alpha) \leq c(\alpha) + \int_{\alpha}^{\beta} g(s) &\left(z(\alpha)\exp\left(\int_{\alpha}^{s} m_1 f(\sigma)d\sigma\right)\right. \\ &\left. + \int_{\alpha}^{s} \left[c'(\tau) + m_2 f(\tau)\right] \exp\left(\int_{\tau}^{s} m_1 f(\sigma)d\sigma\right) d\tau\right) ds \\ z(\alpha) \leq c(\alpha) + z(\alpha) &\int_{\alpha}^{\beta} g(s)\exp\left(\int_{\alpha}^{s} m_1 f(\sigma)d\sigma\right) ds \\ &+ \int_{\alpha}^{\beta} g(s)\left(\int_{\alpha}^{s} \left[c'(\tau) + m_2 f(\tau)\right] \exp\left(\int_{\tau}^{s} m_1 f(\sigma)d\sigma\right) d\tau\right) ds \end{aligned}$$

$$z(\alpha) \leq \frac{c(\alpha) + \int_{\alpha}^{\beta} g(s)\left(\int_{\alpha}^{s} [c'(\tau) + m_2 f(\tau)] \exp\left(\int_{\tau}^{s} m_1 f(\sigma)d\sigma\right) d\tau\right) ds}{\left(1 - \int_{\alpha}^{\beta} g(s)\exp\left(\int_{\alpha}^{s} m_1 f(\sigma)d\sigma\right) ds\right)}. \tag{2.8}$$

From equation (2.7) and (2.8), we obtain (2.2). This completes the proof.

Remark 2.2. If we take $p = 1$, then the Theorem 2.1 reduces to inequality given by Pachpatte in [10] and if $p = 1$ and $g = 0$ it reduces to one of the well known Gronwall's inequality.

Theorem 2.3. *Let* $u(t), g(t), c'(t) \in C(I, R_+), f_t(t,s) \in C(D, R_+)$ *and*

$$u^p(t) \leq c(t) + \int_\alpha^t f(t,s)u(s)ds + \int_\alpha^\beta g(s)u^p(s)ds. \tag{2.9}$$

If $R_2 = \int_\alpha^\beta g(s)\exp\left(\int_\alpha^s m_1 A(\sigma)d\sigma\right)ds < 1$, *then*

$$u^p(t) \leq \frac{c(\alpha) + \int_\alpha^\beta g(s)\left(\int_\alpha^s [c'(\tau) + m_2 A(\tau)]\exp\left(\int_\tau^s m_1 A(\sigma)d\sigma\right)d\tau\right)ds}{1 - R_2}\exp\left(\int_\alpha^t m_1 A(s)ds\right)$$

$$+ \int_\alpha^t \left[c'(s) + m_2 A(s)\right]\exp\left(\int_s^t m_1 A(\sigma)d\sigma\right)ds, \tag{2.10}$$

where p, k, m_1, m_2 *are as same defined in Theorem 2.1 and* $A(t) = f(t,t) + \int_\alpha^t f_t(t,s)ds$.

Proof. Define a function $z(t)$ by right hand side of (2.9). Then we observe that

$$z(\alpha) = c(\alpha) + \int_\alpha^\beta g(s)u^p(s)ds \tag{2.11}$$

and

$$z'(t) \leq c'(t) + A(t)z^{\frac{1}{p}}(t). \tag{2.12}$$

From Lemma 1.1 and equation (2.12), we have

$$z'(t) - m_1 A(t)z(t) \leq c'(t) + m_2 A(t),$$

$$\left[\frac{z(t)}{\exp\left(\int_\alpha^t n_1 A(s)ds\right)}\right]' \leq c'(t) + m_2 A(t). \tag{2.13}$$

Set $t = s$; in equation (2.13) and integrate with respect to s from α to t, we get

$$\frac{z(t)}{\exp\left(\int_\alpha^t m_1 A(s)ds\right)} \leq z(\alpha) + \int_\alpha^t \left[c'(s) + m_2 A(s)\right]\exp\left(-\int_\alpha^s m_1 A(s)ds\right)ds$$

$$z(t) \leq z(\alpha)\exp\left(\int_\alpha^t m_1 A(s)ds\right) + \left(\int_\alpha^t \left[c'(s) + m_2 A(s)\right]\exp\left(\int_s^t m_1 A(\sigma)d\sigma\right)ds\right). \tag{2.14}$$

As $u^p(t) \leq z(t)$ from equation (2.14), we have

$$u^p(t) \leq z(\alpha)\exp\left(\int_\alpha^t m_1 A(s)ds\right) + \int_\alpha^t \left[c'(s) + m_2 A(s)\right]\exp\left(\int_s^t m_1 A(\sigma)d\sigma\right)ds. \tag{2.15}$$

Now from equation (2.11) and (2.15), we have

$$\begin{aligned} z(\alpha) &\le c(\alpha) + z(\alpha)\int_\alpha^\beta g(s)\exp\left(\int_\alpha^s m_1A(\sigma)d\sigma\right)ds \\ &\quad + \int_\alpha^\beta g(s)\left(\int_\alpha^s \left[c'(\tau)+m_2A(\tau)\right]\exp\left(\int_\tau^s m_1A(\sigma)d\sigma\right)d\tau\right)ds, \\ &\le \frac{c(\alpha)+\int_\alpha^\beta g(s)\left(\int_\alpha^s \left[c'(\tau)+m_2A(\tau)\right]\exp\left(\int_\tau^s m_1A(\sigma)d\sigma\right)d\tau\right)ds}{\left(1-\int_\alpha^\beta g(s)\exp\left(\int_\alpha^s m_1A(\sigma)d\sigma\right)ds\right)}. \end{aligned} \tag{2.16}$$

From equation (2.15) and (2.16), we get (2.10). This completes the proof.

Theorem 2.4. *Let $u(t), c'(t) \in C(I,R_+)$, $f_t(t,s), g(t,s) \in C(D,R_+)$ and*

$$u^p(t) \le c(t) + \int_\alpha^t f(t,s)u(s)ds + \int_\alpha^\beta g(t,s)u^p(s)ds. \tag{2.17}$$

If $R_3 = \int_\alpha^\beta g(\alpha,s)\exp\left(\int_\alpha^s [m_1A(s)+B_1(\sigma)]d\sigma\right)ds < 1$, *then*

$$\begin{aligned} u^p(t) &\le \frac{c(\alpha)+\int_\alpha^\beta g(\alpha,s)\left(\int_\alpha^s \left[c'(\tau)+m_2A(\tau)\right]\exp\left(\int_\tau^s [m_1A(\sigma)+B_1(\sigma)]d\sigma\right)d\tau\right)ds}{\left(1-\int_\alpha^\beta g(\alpha,s)\exp\left(\int_\alpha^s [m_1A(\sigma)+B_1(\sigma)]d\sigma\right)ds\right)}\exp\left(\int_\alpha^t [m_1A(s)+B_1(s)]ds\right) \\ &\quad + \int_\alpha^t \left[c'(s)+m_2A(s)\right]\exp\left(\int_s^t [m_1A(\sigma)+B_1(\sigma)]d\sigma\right)ds, \end{aligned} \tag{2.18}$$

where p, k, m_1, m_2 are as same defined in Theorem 2.1, $A(t)$ is same as defined in Theorem 2.3 and $B_1(t) = \int_\alpha^\beta g_t(t,s)ds$.

Proof. Define a function $z(t)$ by

$$z(t) = c(t) + \int_\alpha^t f(t,s)u(s)ds + \int_\alpha^\beta g(t,s)u^p(s)ds.$$

Then $u(t) \le z^{\frac{1}{p}}(t)$,

$$z(\alpha) = c(\alpha) + \int_\alpha^\beta g(\alpha,s)u^p(s)ds \tag{2.19}$$

and

$$z'(t) = c'(t) + \int_\alpha^t f_t(t,s)u(s)ds + f(t,t)u(t) + \int_\alpha^\beta g_t(t,s)u^p(s)ds,$$

$$z'(t) \le c'(t) + A(t)z^{\frac{1}{p}}(t) + B_1(t)z(t), \tag{2.20}$$

From Lemma 1.1 and equation (2.20), we have

$$z'(t)-[m_1A(t)+B_1(t)]z(t)\le c'(t)+m_2A(t)$$

$$\left[\frac{z(t)}{\exp\left(\int_\alpha^t[m_1A(s)+B_1(s)]ds\right)}\right]'\le c'(t)+m_2A(t). \tag{2.21}$$

Integrating (2.21) with respect to s from α to t, we get

$$\frac{z(t)}{\exp\left(\int_\alpha^t[m_1A(s)+B_1(s)]ds\right)}\le z(\alpha)+\int_\alpha^t\left[c'(s)+m_2A(s)\right]\exp\left(-\int_\alpha^s[m_1A(\sigma)+B_1(\sigma)]d\sigma\right)ds.$$

(2.22)

$$z(t)\le z(\alpha)\exp\left(\int_\alpha^t[m_1A(s)+B_1(s)]ds\right)+\left(\int_\alpha^t\left[c'(s)+m_2A(s)\right]\exp\left(\int_s^t[m_1A(\sigma)+B_1(\sigma)]d\sigma\right)ds\right).$$

As $u^p(t)\le z(t)$ from equation (2.22), we have

(2.23)

$$u^p(t)\le z(\alpha)\exp\left(\int_\alpha^t[m_1A(s)+B_1(s)]ds\right)+\int_\alpha^t\left[c'(s)+m_2A(s)\right]\exp\left(\int_s^t[m_1A(\sigma)+B_1(\sigma)]d\sigma\right)ds.$$

Now from equation (2.19) and (2.23), we have

$$z(\alpha)\le\frac{c(\alpha)+\int_\alpha^\beta g(\alpha,s)\left(\int_\alpha^s[c'(\tau)+m_2A(\tau)]\exp\left(\int_\tau^s[m_1A(\sigma)+B_1(\sigma)]d\sigma\right)d\tau\right)ds}{\left(1-\int_\alpha^\beta g(\alpha,s)\exp\left(\int_\alpha^s[m_1A(\sigma)+B_1(\sigma)]d\sigma\right)ds\right)}. \tag{2.24}$$

From equation (2.23) and (2.24), we obtain (2.18). This completes the proof.

Theorem 2.5. *Let* $u(t)\in C(I,R_+), f_t(t,s), g_t(t,s), a_t(t,s)\in C(D,R_+)$ *and* $c\ge 0$ *be a real constant. If*

$$u^p(t)\le c+\int_\alpha^t a(t,s)\left[u(s)+\int_\alpha^s f(s,\sigma)u(\sigma)d\sigma\right]+\int_\alpha^\beta g(t,s)u^p(s)ds,\ \text{for } t\in I \tag{2.25}$$

and $R_4=\int_\alpha^\beta g(\alpha,s)\exp\left(\int_\alpha^s[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)ds<1$, *then*

$$u^p(t)\le\frac{c+\int_\alpha^\beta g(\alpha,s)\left(\int_\alpha^s m_2A_1(\tau)\exp\left(\int_\tau^s[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)d\tau\right)ds}{1-\int_\alpha^\beta g(\alpha,s)\exp\left(\int_\alpha^s[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)ds}\exp\left(\int_\alpha^t[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)$$

(2.26)

$$+\int_\alpha^t m_2A_1(s)\exp\left(\int_s^t[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right),$$

where p,k,m_1,m_2 are as same defined in Theorem 2.1,

$$A_1(t)=\int_\alpha^t a_t(t,s)\left[1+\int_\alpha^s f(s,\sigma)d\sigma\right]+a(t,t)\left[1+\int_\alpha^t f(t,\sigma)d\sigma\right]ds \text{ and } B_1(t)=\int_\alpha^\beta g_t(t,s)ds.$$

Proof. Define a function $z(t)$ by

$$z(t)=c+\int_\alpha^t a(t,s)\left[u(s)+\int_\alpha^s f(s,\sigma)u(\sigma)d\sigma\right]ds+\int_\alpha^\beta g(t,s)u^p(s)ds.$$

Then $u(t)\le z^{\frac{1}{p}}(t)$,

$$z(\alpha)=c+\int_\alpha^\beta g(\alpha,s)u^p(s)ds \tag{2.27}$$

and

$$z'(t)\le A_1(t)z^{\frac{1}{p}}(t)+z(t)B_1(t). \tag{2.28}$$

From Lemma 1.1 and equation (2.28), we have

$$z'(t)-[m_1A_1(t)+B_1(t)]\,z(t)\le m_2A_1(t),$$

$$\left[\frac{z(t)}{\exp\left(\int_\alpha^t[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)}\right]'\le m_2A_1(t)\exp\left(-\int_\alpha^t[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right). \tag{2.29}$$

Set $t=s$; in equation (2.29) and integrate with respect to s from α to t, we get

$$z(t)\le z(\alpha)\exp\left(\int_\alpha^t[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)+\int_\alpha^t m_2A_1(s)\exp\left(\int_s^t[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)ds. \tag{2.30}$$

As $u^p(t)\le z(t)$ from equation (2.30), we have

$$u^p(t)\le z(\alpha)\exp\left(\int_\alpha^t[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)+\int_\alpha^t m_2A_1(s)\exp\left(\int_s^t[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)ds. \tag{2.31}$$

Now from equation (2.27) and (2.31), we have

$$\begin{aligned} z(\alpha)&\le c+\int_\alpha^\beta g(\alpha,s)\Big\{z(\alpha)\exp\left(\int_\alpha^s[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)\\ &\quad+\int_\alpha^s m_2A_1(\tau)\exp\left(\int_\tau^s[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)d\tau\Big\}ds\\ &\le\frac{c+\int_\alpha^\beta g(\alpha,s)\left(\int_\alpha^s m_2A_1(\tau)\exp\left(\int_\tau^s[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)d\tau\right)ds}{1-\int_\alpha^\beta g(\alpha,s)\exp\left(\int_\alpha^s[m_1A_1(\sigma)+B_1(\sigma)]\,d\sigma\right)ds}. \end{aligned} \tag{2.32}$$

From equation (2.31) and (2.32), we obtain desired result given by (2.26). This completes the proof.

Theorem 2.6. *Let $u(t), f(t), a(t), g(t) \in C(I, R_+)$ and $c \geq 0$ be a real constant. If*

$$u^p(t) \leq c + \int_\alpha^t a(s)\left[u(s) + \int_\alpha^s f(\sigma)u(\sigma)d\sigma + \int_\alpha^\beta g(\sigma)u^p(\sigma)d\sigma\right]ds, \tag{2.33}$$

then

$$u^p(t) \leq c\exp\left(\int_\alpha^t [m_1A_2(\sigma) + B_2(\sigma)]\,d\sigma\right) + \int_\alpha^t m_2A_2(s)\exp\left(\int_s^t [m_1A_2(\sigma) + B_2(\sigma)]\,d\sigma\right)ds, \tag{2.34}$$

where p, k, m_1, m_2 are as same defined in Theorem 2.1, $A_2(t) = a(t)\left[1 + \int_\alpha^t f(\sigma)d\sigma\right]$ and $B_2(t) = a(t)\int_\alpha^\beta g(\sigma)d\sigma$.

Proof. Define a function $z(t)$ by right hand side of (2.33). Then we have $u(t) \leq z^{\frac{1}{p}}(t)$, $z(\alpha) = c$ and

$$z'(t) \leq A_2(t)z^{\frac{1}{p}}(t) + z(t)B_2(t). \tag{2.35}$$

Applying Lemma 1.1 to equation (2.35), we obtain

$$z'(t) - [m_1A_2(t) + B_2(t)]\,z(t) \leq m_2A_2(t),$$

$$\left[\frac{z(t)}{\exp\left(\int_\alpha^t [m_1A_2(\sigma) + B_2(\sigma)]\,d\sigma\right)}\right]' \leq m_2A_2(t)\exp\left(-\int_\alpha^t [m_1A_2(\sigma) + B_2(\sigma)]\,d\sigma\right). \tag{2.36}$$

Integrating (2.36) with respect to s and using the fact that $u^p(t) \leq z(t)$, we get

$$u^p(t) \leq z(\alpha)\exp\left(\int_\alpha^t [m_1A_2(\sigma) + B_2(\sigma)]\,d\sigma\right) + \int_\alpha^t m_2A_2(s)\exp\left(\int_s^t [m_1A_2(\sigma) + B_2(\sigma)]\,d\sigma\right)ds. \tag{2.37}$$

Now from equation (2.37) and $z(\alpha) = c$, we get (2.34). This completes the proof.

3. Applications

One of the main motivations for the study of different type inequalities given in the previous section is to apply them as tools in the study of various classes of integral equations. In the following section we give application of some theorems of previous section. In fact we discuss the boundedness behavior of solutions of a general nonlinear Volterra-Fredholm integral equations.

Example 3.1. We calculate the explicate bound on the solution of the nonlinear integral equation of the form:

$$u^3(t) = 4 + \int_0^t \frac{1}{1-s} u(s)ds + \int_0^{\frac{1}{2}} \frac{1}{(1-s)^{\frac{1}{3}}} u^3(s)ds \tag{3.38}$$

where $u(t)$ is defined as in Theorem 2.1 and we assume that every solution $u(t)$ of (3.38) exists on R_+. Also, here

$$Q_1 = \int_0^{1/2} \frac{1}{(1-s)^{\frac{1}{3}}} \exp\left(\int_\alpha^s m_1 \frac{1}{1-\sigma} d\sigma\right) ds \quad = \quad -\frac{3\left(2^{2/3} - 2^{m_1}\right)}{2^{2/3}(3m_1 - 2)} < 1, \text{ for } 0 \le m_1 < \frac{2}{3}.$$

Hence, by Theorem 2.1 and equation (3.38), we get

$$u^p(t) \le \exp\left(\frac{1}{3}\int_0^t \frac{1}{1-x} dx\right) \left(\int_0^{\frac{1}{2}} \frac{\int_0^s \frac{\exp\left(\int_x^s \frac{1}{3(1-y)} dy\right)}{1-x} dx}{3\sqrt[3]{1-s}} ds + 4\right) + \frac{1}{3}\int_0^t \frac{\exp\left(\int_x^t \frac{1}{3(1-y)} dy\right)}{1-x} dx$$

$$= \frac{\frac{3}{4}\left(2 + \sqrt[3]{2} - 2\, 2^{2/3}\right) + 4}{\sqrt[3]{1-t}} + \frac{3 - 3\sqrt[3]{1-t}}{3\sqrt[3]{1-t}}.$$

In Figure 1 we plot the graph of estimated bound of $u(t)$ for $0 \le t \le \frac{1}{2}$.

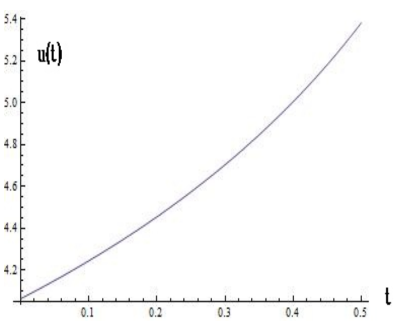

FIGURE 1

Example 3.2. We calculate the explicate bound on the solution of the nonlinear integral equation of the form:

$$u^3(t) = 5 + \int_0^t (t-s)u(s)ds + \int_0^1 \frac{t-1}{t+1} u^3(s)ds \tag{3.39}$$

where $u(t)$ are defined as in Theorem 2.3 and we assume that every solution $u(t)$ of (3.39) exists on R_+. By Theorem 2.4, we have

$$p = 3,\ m_1 = \frac{1}{p} k^{\frac{1-p}{p}} = \frac{1}{3} k^{\frac{-2}{3}}, m_2 = \frac{p-1}{p} k^{\frac{1}{p}} = \frac{2}{3} k^{\frac{1}{3}}, \alpha = 0, \beta = \frac{1}{2}, k > 0, f(s) = (t-s), g(s) = \frac{t-1}{t+1}$$

and

$$R_3 = \int_{\alpha}^{\beta} g(\alpha,s)\exp\left(\int_{\alpha}^{s} m_1 A(\sigma)d\sigma\right)ds = \sqrt{\frac{3\pi}{2}}\left(-\sqrt[3]{k}\right)\operatorname{erfi}\left(\frac{1}{\sqrt{6}\sqrt[3]{k}}\right) < 1, \text{ for any } \quad k > 0$$

Thus all the conditions of the Theorem 2.4 are satisfied, hence we obtain, for all $k > 0$

$$u(t) \leq \left(\frac{\left(5 - k\left(\sqrt{6\pi}\sqrt[3]{k}\operatorname{erfi}\left(\frac{1}{\sqrt{6}\sqrt[3]{k}}\right) - 2\right)\right)e^{\frac{t^2}{6k^{2/3}}}}{\sqrt{\frac{3\pi}{2}}\sqrt[3]{k}\operatorname{erfi}\left(\frac{1}{\sqrt{6}\sqrt[3]{k}}\right) + 1} + 2k\left(e^{\frac{t^2}{6k^{2/3}}} - 1\right)\right)^{\frac{1}{3}}.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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