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COMMON FIXED POINT THEOREM FOR GENERALIZED T-HARDY-ROGERS CONTRACTION MAPPING IN A CONE METRIC SPACE

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Abstract: In the present paper we improve and generalize common fixed point theorem for T-Hardy-Rogers contraction mapping in the setting of cone metric space.

Keywords: Cone metric space, Banach operator pair, T-contraction, Hardy-Rogers type contraction, common fixed point.

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1. Introduction

It is well known that the classic contraction mapping principle of Banach is a fundamental result in fixed point theory. Several authors have obtained various extensions and generalizations of Banach's theorem by considering contractive mappings on many different metric spaces.

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In 1977, Rhoades [16] considered 250 types of contractive definitions and analyzed the relationship among them. In 2009, A Beiranvand et al. [1] introduced new classes of contractive functions-T-contraction and T-contractive mappings and then they established and extended the Banach contraction principle and the Edelstein's fixed point theorems.

Recently, Huang and Zhang [2] introduce the notion of cone metric space. He replaced real number system by ordered Banach space. He also gave the condition in the setting of cone metric spaces. These authors also described the convergence of sequences in the cone metric spaces and introduce the corresponding notion of completeness. Subsequently, many authors have generalized the results of Huang and Zhang and have studied fixed point theorems for different types of cones, see for instance [5], [6], [9], [10], [11], [14], [19] and [20].

In sequel, J.R. Morales and E. Rojas [12], [13] obtained sufficient condition for the existence of a unique fixed point of T-Kannan contractive, T-Zamfirescu, T-contractive mappings etc, on complete cone metric spaces. Afterwards; in [3] R. Sumitra et al. have proved common fixed point theorem for a Banach pair of mappings satisfying T-Hardy-Rogers type contraction condition in cone metric spaces. In the sequel, we need a definition which was introduced and called Banach operator of type k by Subrahmanyam [17]. Recently Chen and Li [7] extended the concept of Banach operator of type k to Banach operator pair and proved various best approximation results using common fixed point theorems for f −nonexpansive mappings.

The aim of this paper is to prove common fixed point theorem for generalized T-Hardy-Rogers contraction mappings in the setting of cone metric spaces. Our result is generalization of the result [3].

2. Preliminary Notes

First, we recall some standard notations and definitions which we need them in the sequel.

Definition 2.1([2]) .Let E be a real Banach space and P a subset of E. P is called a cone if and only if

(i) P is closed, non-empty and $P \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, a partial ordering is defined as $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. It is denoted as $x \ll y$ will stand for $y - x \in int\ P$, $int\ P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\| \qquad (2.1)$$

The least positive number K satisfying the above is called the normal constant of P.

Definition 2.2([2]) .Let X be a non-empty set. Suppose E is a real Banach space, P is a cone with $int\ P \neq \emptyset$ and $\leq$ is partial ordering with respect to P. If the mapping $d: X \times X \to E$ satisfies,

(i) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(iii) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called cone metric space .

Example 2.3([2]). Let $E=\mathbb{R}^2, P=\{(x,y)\in E: x,y\geq 0\}\subset R^2, X=\mathbb{R}$ and $d: X\times X\to E$ such that $d(x,y)=(|x-y|, \propto |x-y|)$,where $\propto\geq 0$ is a constant.Then (X,d) is a cone metric space.

Lemma 2.4([2]). Let (X,d) be a cone metric space and Pbe a normal cone with normal constant K. A sequence $\{x_n\}$ in X converges to x if and only if $d(x_n,x)\to 0$ as $n\to\infty$.

Lemma 2.5([2]). Let (X,d) be a cone metric space and P be a normal cone with normal constant K. A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $d(x_n,x_m)\to 0$ as $n,m\to\infty$.

Definition 2.6[12]. Let (X,d) be a cone metric space and $\{x_n\}$ be a sequence in X. Then,

(i) $\{x_n\}$ converges to $x\in X$, if for every $c\in E$ with $0\ll c$, there is $n_o\in N$,the set of all natural numbers such that for all $n\geq n_o$,

$d(x_n,x)\ll c$.

It is denoted by $lim_{n\to\infty} x_n = x$ or $x_n\to x, (n\to\infty)$.

(ii) If for every$c\in E$,there is a number $n_o\in N$ such that for all $m,n\geq n_o$,

$d(x_n,x_m)\ll c$,

then $\{x_n\}$ is called a Cauchy sequence in X;

(iii) (X,d) is called a complete cone metric space, if every Cauchy sequence in X is convergent.

(iv)A self mapping $T: X \to X$ is said to be continuous at a point $x \in X$, if $lim_{n\to\infty} x_n = x$ implies that $lim_{n\to\infty} Tx_n = Tx$ for every $\{x_n\}$ in X.

Definition 2.7. A self mapping T of a metric space (X, d) is said to be a contraction mapping, if there exists a real number $0 \leq k < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y) \qquad (2.2)$$

The following definition is given by Beiranvand et al. [1].

Definition 2.8([1]). Let T and f be two self-mappings of a metric space (X, d).The self mapping f of X is said to be T-contraction, if there exists a real number $0 \leq k < 1$ such that

$$d(Tfx, Tfy) \leq kd(Tx, Ty) \qquad (2.3)$$

for all $x, y \in X$.

If $T = I$, the identity mapping, then the Definition 2.8 reduces to Banach contraction mapping.

The following example shows that a T-contraction mapping need not be a contraction mapping.

Example 2.9. Let $X = [0, \infty)$ be with the usual metric. Let define two mappings $T, f: X \to X$ as

$$fx = \beta x, \beta > 1$$

$$Tx = \frac{\propto}{x^2}, \propto \in R.$$

It is clear that, f is not contraction but f is T-contraction, since,

$$d(Tfx, Tfy) = \left|\frac{\propto}{\beta^2 x^2} - \frac{\propto}{\beta^2 y^2}\right| = \frac{1}{\beta^2}|Tx - Ty|.$$

Definition 2.10([1]). Let T be a self mapping of a metric space (X, d).Then

i) A mapping T is said to be sequentially convergent, if the sequence $\{y_n\}$in X is convergent whenever $\{Ty_n\}$ is convergent.

ii) A mapping T is said to be sub-sequentially convergent, if $\{y_n\}$ has a convergent subsequence whenever $\{Ty_n\}$ is convergent.

Definition 2.11([17]). Let T be a self mapping of a normed space X. Then T is called a Banach operator of type k if

$$\|T^2x - Tx\| \leq k\|Tx - x\|$$

for some $k \geq 0$ and for all $x \in X$.

This concept was introduced by Subrahmanyam [17], then Chen and Li [7] extended this as following:

Definition 2.12([7]). Let T and f be two self mappings of a non-empty subset M of a normed linear space X. Then (T,f) is a Banach operator pair, if any one of the following conditions is satisfied:

(i) $T[F(f)] \subseteq F(f)$i.e. $F(f)$ is T-invariant.

(ii) $fTx = Tx$ for each $x \in F(f)$.

(iii)$fTx = Tfx$ for each $x \in F(f)$.

(iv)$\|Tfx - fx\| \leq k\|fx - x\|$ for some $k \geq 0$.

The following corollary of Rezapour [15] will be needed in the sequel.

Corollary 2.13([15]). Let a,b,c,u$\in E$, the real Banach space.

(i) If $a \leq b$ and b$\ll c$, then $a \ll c$.

(ii) If $a \ll b$ and b$\ll c$, then $a \ll c$.

(iii)If $0 \leq u \ll c$ for each $c \in int\ P$, then $u = 0$.

Remark 2.14([10]). If $c \in int\ P$, 0$\leq a_n$ and $a_n \to 0$, then there exists $n_0$such that for all $n > n_0$, it follows that $a_n \ll c$.

3. Main Result

Theorem 3.1. Let T, f and g be three continuous self mappings of a complete cone metric space (X, d). Assume that T is an injective mapping and P is a normal cone with normal constant. If the mappings T, f and g satisfy

$$d(Tfx,Tgy) \le a_1 d(Tx,Ty) + a_2 d(Tx,Tfx) + a_3 d(Ty,Tgy) + a_4 d(Tx,Tgy) + a_5 d(Ty,Tfx) \quad (3.1)$$

for all $x, y \in X$, where $a_i, i = 1,2,3,4,5$ are all non negative constants such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then f and g have a unique common fixed point in X. Moreover, if (T, f) and (T, g) are Banach pairs, then T, f and g have a unique common fixed point in X.

Proof. Let $x_o \in X$ as an arbitrary element and define the sequences $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ in X such that

$x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for each $n = 0,1,2,- - - - \infty$. Consider,

$$d(Tx_{2n+1}, Tx_{2n}) = d(Tfx_{2n}, Tgx_{2n-1})$$

$$\le a_1 d(Tx_{2n}, Tx_{2n-1}) + a_2 d(Tx_{2n}, Tfx_{2n}) + a_3 d(Tx_{2n-1}, Tgx_{2n-1}) + a_4 d(Tx_{2n}, Tgx_{2n-1}) + a_5 d(Tx_{2n-1}, Tfx_{2n})$$

$$\le a_1 d(Tx_{2n}, Tx_{2n-1}) + a_2 d(Tx_{2n}, Tx_{2n+1})$$

$+a_3 d(Tx_{2n-1}, Tx_{2n}) + a_4 d(Tx_{2n}, Tx_{2n})$

$+a_5 d(Tx_{2n-1}, Tx_{2n+1})$

$\leq (a_1 + a_3 + a_5) d(Tx_{2n}, Tx_{2n-1})$

$$+(a_2 + a_5) d(Tx_{2n}, Tx_{2n+1}) \qquad (3.2)$$

Next consider,

$d(Tx_{2n}, Tx_{2n+1}) = d(Tgx_{2n-1}, Tfx_{2n})$

$\leq a_1 d(Tx_{2n-1}, Tx_{2n}) + a_2 d(Tx_{2n-1}, Tgx_{2n-1})$

$+a_3 d(Tx_{2n}, Tfx_{2n}) + a_4 d(Tx_{2n-1}, Tfx_{2n})$

$+a_5 d(Tx_{2n}, Tgx_{2n-1})$

$\leq a_1 d(Tx_{2n-1}, Tx_{2n}) + a_2 d(Tx_{2n-1}, Tx_{2n})$

$+a_3 d(Tx_{2n}, Tx_{2n+1}) + a_4 d(Tx_{2n-1}, Tx_{2n+1})$

$+a_5 d(Tx_{2n}, Tx_{2n})$

$\leq (a_1 + a_2 + a_4) d(Tx_{2n-1}, Tx_{2n})$

$$+(a_3 + a_4) d(Tx_{2n}, Tx_{2n+1}) \qquad (3.3)$$

Adding inequalities (3.2) and (3.3),

$$2d(Tx_{2n},Tx_{2n+1}) \leq (2a_1 + a_2 + a_3 + a_4 + a_5)d(Tx_{2n},Tx_{2n-1})$$

$$+(a_2 + a_3 + a_4 + a_5)d(Tx_{2n},Tx_{2n+1})$$

$$d(Tx_{2n},Tx_{2n+1}) \leq \frac{(2a_1 + a_2 + a_3 + a_4 + a_5)}{(2-a_2 - a_3 - a_4 - a_5)} d(Tx_{2n},Tx_{2n-1})$$

$$= kd(Tx_{2n},Tx_{2n-1}),$$

where $k = \frac{2a_1+a_2+a_3+a_4+a_5}{2-a_2-a_3-a_4-a_5} < 1 \ \ as \ \ a_1 + a_2 + a_3 + a_4 + a_5 < 1.$

Proceeding further,

$$d(Tx_{2n},Tx_{2n+1}) \leq k^{2n} d(Tx_0,Tx_1) \tag{3.4}$$

Next, to claim that $\{Tx_{2n}\}$ is a Cauchy sequence. Consider $m, n \in N$ such that $m > n$,

$$d(Tx_{2n},Tx_{2m}) \leq d(Tx_{2n},Tx_{2n+1}) + d(Tx_{2n+1},Tx_{2n+2})$$

$$+ - - - - - + d(Tx_{2m-1},Tx_{2m})$$

$$\leq (k^{2n} + k^{2n+1} + - - - + k^{2m-1})d(Tx_1,Tx_0)$$

$$= \frac{k^{2n}}{1-k} d(Tx_0,Tx_1).$$

From (2.1), it follows that

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$$d(Tx_{2m},Tx_{2n})\| \leq \frac{k^{2n}}{1-k} \|d(Tx_0,Tx_1)\| \tag{3.5}$$

Since $k \in (0,1), k^{2n} \to 0$ as $n \to \infty$. Therefore $\|d(Tx_{2m}, Tx_{2n})\| \to 0$ as $m, n \to \infty$. Thus $\{Tx_{2n}\}$ is a Cauchy sequence in X. As X is a complete cone metric space, there exists $z \in X$ such that $lim_{n\to\infty} Tx_{2n} = z$.

Since T is sub-sequentially convergent, $\{x_{2n}\}$ has a convergent sub-sequence $\{x_{2m}\}$ such that $lim_{m\to\infty} x_{2m} = u$. As T is continuous

$$lim_{m\to\infty} Tx_{2m} = Tu. \tag{3.6}$$

By the uniqueness of the limit, $z = Tu$.

Since f is continuous $lim_{m\to\infty} fx_{2m} = fu$. Again as T is continuous,

$lim_{m\to\infty} Tfx_{2m} = Tfu$. Therefore

$$lim_{m\to\infty} Tx_{2m+1} = Tfu \tag{3.7}$$

Now consider,

$$d(Tfu, Tu) \leq d(Tfu, Tx_{2m}) + d(Tx_{2m}, Tu)$$

$$\leq d(Tfu, Tgx_{2m-1}) + d(Tx_{2m}, Tu)$$

$$\leq a_1 d(Tu, Tx_{2m-1}) + a_2 d(Tu, Tfu)$$

$$+ a_3 d(Tx_{2m-1}, Tgx_{2m-1}) + a_4 d(Tu, Tgx_{2m-1})$$

$$+ a_5 d(Tx_{2m-1}, Tfu) + d(Tx_{2m}, Tu)$$

$$= a_1 d(Tu, Tx_{2m-1}) + a_2 d(Tu, Tfu) + a_3 d(Tx_{2m-1}, Tx_{2m}) + a_4 d(Tu, Tx_{2m}) + a_5 d(Tx_{2m-1}, Tfu) + d(Tx_{2m}, Tu)$$

$$\leq \frac{a_1}{1-a_2} d(Tu, Tx_{2m-1}) + \frac{a_3}{1-a_2} d(Tx_{2m-1}, Tx_{2m})$$

$$+ \frac{1+a_4}{1-a_2} d(Tx_{2m}, Tu) + \frac{a_5}{1-a_2} d(Tx_{2m-1}, Tfu)$$

$$\leq \frac{a_1}{1-a_2} [d(Tu, Tx_{2m}) + d(Tx_{2m}, Tx_{2m-1})]$$

$$+ \frac{a_3}{1-a_2} d(Tx_{2m-1}, Tx_{2m}) + \frac{1+a_4}{1-a_2} d(Tx_{2m}, Tu)$$

$$+ \frac{a_5}{1-a_2} [d(Tx_{2m-1}, Tx_{2m}) + d(Tx_{2m}, Tu) + d(Tu, Tfu)]$$

$$\left(1 - \frac{a_5}{1-a_2}\right) d(Tu, Tfu) \leq \frac{1+a_1+a_4+a_5}{1-a_2} d(Tu, Tx_{2m})$$

$$+ \frac{a_1+a_3+a_5}{1-a_2} d(Tx_{2m-1}, Tx_{2m})$$

Therefore,

$$d(Tu, Tfu) \leq \frac{1+a_1+a_4+a_5}{1-a_2-a_5} d(Tu, Tx_{2m})$$

$$+ \frac{a_1+a_3+a_5}{1-a_2-a_5} d(Tx_{2m-1}, Tx_{2m}) \qquad (3.8)$$

Let $0 \ll c$ be arbitrary. By (3.6),

$$d(Tu, Tx_{2m}) \ll \frac{c(1-a_2-a_5)}{2(1+a_1+a_4+a_5)}$$

Similarly by (3.7), it follows that

$$d(Tx_{2m-1}, Tx_{2m}) \ll \frac{c(1-a_2-a_5)}{2(a_1+a_3+a_5)}.$$

Then, (3.8) becomes

$$d(Tu, Tfu) \ll \frac{c}{2} + \frac{c}{2} = c.$$

Thus $d(Tu, Tfu) \ll c$ for each $c \in int\ P$. Now, using Corollary 2.13 (iii), it follows that

$d(Tu, Tfu) = 0$

which implies that $Tu = Tfu$. As T is injective, $u = fu$. Thus u is the fixed point of f.

Similarly, it can be established that, u is also the fixed point of g. That means u is common fixed point of f and g.

To prove uniqueness: If w is another common fixed point of $f$and g, then $fw = w = gw$.

$d(Tu, Tw) = d(Tfu, Tgw)$

$\leq a_1 d(Tu, Tw) + a_2 d(Tu, Tfu)$

$+a_3 d(Tw, Tgw) + a_4 d(Tu, Tgw) + a_5 d(Tw, Tfu)$

$\leq a_1 d(Tu, Tw) + a_4 d(Tu, Tw) + a_5 d(Tw, Tu)$

$= (a_1 + a_4 + a_5) d(Tu, Tw)$

$\leq (\ a_1 + a_2 + a_3 + a_4 + a_5\ ) d(Tu, Tw)$

$< d(Tu, Tw)\ as\ \ a_1 + a_2 + a_3 + a_4 + a_5 < 1$

a contradiction. Hence $d(Tu,Tw) = 0$ which implies $Tu = Tw$. As T is injective, $u = w$ is the unique common fixed point of f and g.

Since we have assumed that (T,f) and $(T,g)$are Banach pairs; (T,f) and (T,g)commutes at the fixed point of f and g respectively. This implies that $Tfu = fTu$ for $u \in F(f)$. So $Tu = fTu$ which gives that Tu is another fixed point of f. It is true for g, too. By the uniqueness of fixed point of f, $Tu = u$. Hence $u = Tu = fu = gu$, u is the unique common fixed point of T, f and g in X.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, Two fixed point theorem for special mappings, arxiv:0903.1504v1[math.FA](2009).

[2] L.G. Huang and X.Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math.Anal. Appl., 332 (2007), 1468-1476.

[3] R. Sumitra, V.R. Uthariaraj and R. Hemavathy, Common fixed point theorem for T-Hardy-Rogers contraction mapping in a Cone metric space, Int. Math.Forum 5 (2010), 1495-1506.

[4] M.A. Al-Thagafi and N. Shahzad, Banach operator pairs, common fixed points, invariant approximations, and non-expansive multimaps, Nonlinear Anal.,69 (2008), 2733-2739.

[5] M. Abbas andB.E.Rhoades, Fixed and periodic point results in Cone metric spaces, Appl.Math. Lett. 22(2009), 512-516.

[6] M. Abbas and G. Jungck, Common fixed point results for non-commuting mappings without continuity in Cone metric spaces, J. Math. Anal. Appl., 341 (2008), 416-420.

[7] J.Chen and Z.Li,Common fixed points for Banach operator pairs in best approximation, J. Math Anal.Appl. 336 (2007), 1466-1475.

[8] N. Hussain, Common fixed points in best approximation for Banach operator pairs with ciric type I-contractions, J. Math. Anal. Appl.,338, (2008), 1351-1363.

[9] Z. Kadelburg, S. Radenovic and B. Rosic, Strict contractive conditions and common fixed point theorems in cone metric spaces, Fixed point theory Appl. 2009 (2009), 1-14.

[10] G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic, Common fixed point theorems for weakly compatible pairs on cone metric spaces, Fixed point theory Appl. 2009 (2009), Article ID 643840 .

[11] D. Ilic and V. Rakocevic, Quasi-contraction on a Cone metric space, Appl. Math. Lett. 22 (2009), 728-731.

[12] J.R. Morales and E. Rojas, Cone metric spaces and fixed point theorems for T-Kannan contractive mappings, arxiv:0907.3949v1[math.FA](2009).

[13] J. R.Morales and E. Rojas, T-Zamfirescu and T-Weak contraction mappings on Cone metric spaces ,arxiv:0909.1255v1[math.FA].

[14] V. Raja and S.M. Vaezpour, Some extensions of Banach’s contraction principle in complete cone metric spaces, Fixed point Theory Appl. 2008 (2008),1-11.

[15] Sh. Rezapour, A review on topological properties of cone metric spaces, Analysis, Topology and Applications (ATA’08), VrnjackaBanja, Serbia, May-June 2008.

[16] B.E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer.Math. Soc., 226 (1977),257-290.

[17] P.V. Subrahmanyam, Remarks on some fixed point theorems related to Banach's contraction principle, J. Math. Phys.Sci.,8 (1974),445-457.

[18] V. Berinde, Iterate approximation of fixed points, Lect. Notes Math. 1912(2007), Springer Verlag, Berlin, 2nd Edition.

[19] A.K.Dubey and A. Narayan, Cone metric spaces and fixed point theorems for pair of contractive maps, Math. Aeterna, 2 (2012), 839-845.

[20] A.K Dubey Rita Shukla and R.P. Dubey, An Extension of the paper “Cone metric spaces and fixed point theorems of contractive mappings”, Int. J. Appl. Math. Res. 2 (2013), 84-90.