Available online at http://scik.org
Adv. Inequal. Appl. 2014, 2014:43
ISSN: 2050-7461

FIXED POINT THEOREMS IN PROBABILISTIC CONE METRIC SPACES

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Abstract. We define Probabilistic Cone metric space and find some fixed point results for weak contraction condition. In support an example is furnished.

Keywords: probabilistic cone metric space; Cauchy sequence, fixed point.

2010 AMS Classification: 47H10, 54H25.

1. INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space introduced in 1942 by Menger [2] who used distribution functions instead of nonnegative real numbers as values of the metric, the notion of probabilistic metric space correspond to situations when we do not know exactly the distance between the two points but we know probabilities of possible values of this distance. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological threshold. It is also a fundamental importance in probabilistic functional analysis. Schweizer and Sklar [4] studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid [5].The development of fixed point theory in PM-spaces was due to Schweizer and Sklar [3, 4].

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Received February 11, 2014

In [1] Huang and Zhang generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space, hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. Subsequently, various authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cones. There exist a lot of works involving fixed points used the Banach contraction principle. This principle has been extended kind of contraction mappings by various authors [6].

2. PRELIMINARY

Definition 2.1: Let (E, τ) be a topological vector space and P a subset of E, P is called a cone if

1. P is non-empty and closed, $P \neq \{0\}$,
2. For $x, y \in P$ and $a, b \in R \Longrightarrow ax + by \in P$ where $a,b \geq 0$
3. If $x \in P$ and $-x \in P \Longrightarrow x = 0$

For a given cone $P \subseteq E$, a partial ordering $\geq$ with respect to P is defined by $x \geq y$ if and only if $x - y \in P$, $x > y$ if $x \geq y$ and $x \neq y$, while $x \gg y$ will stand for $x - y \in \operatorname{int} P$, int P denotes the interior of P.

Definition 2.2 A probabilistic metric space (FPM space) is an ordered pair (X,F) consisting of a nonempty set X and a mapping F from XxX into the collections of all distribution functions .For $x, y \in X$ we denote the distribution function $\mathit{F}(x,y)$ by $F_{x,y}$ and $F_{x,y}(u)$ is the value of $F_{x,y}$ at u in R. The functions $F_{x,y}$ assumed to satisfy the following conditions:

2.1.1 $F_{x,y}(u) = 1 \ \forall\ u > 0$ iff $x = y$,

2.1.2 $F_{x,y}(0) = 0 \ \forall\ x , y$ in X,

2.1.3 $F_{x,y} = F_{y,x} \ \forall\ x , y$ in X,

2.1.4 If $F_{x,y}(u) = 1$ and $F_{y,z}(v) = 1$ then $F_{x,z}(u+v) = 1 \ \forall\ x ,y, z$ in X and $u, v > 0$.

Definition 2.3 A commutative, associative and non-decreasing mapping $t: [0,1] \times [0,1] \to [0,1]$ is a t-norm if and only if $t(a,1)=a$ for all $a \in [0,1]$, $t(0,0)=0$ and $t(c,d) \geq t(a,b)$ for $c \geq a$, $d \geq b$.

Definition 2.4 A Menger space is a triplet (X,F,t), where (X,F) is a PM-space,t is a t-norm and the generalized triangle inequality for all x, y, z in X $u, v > 0$.

$$F_{x,y}(u+v) \geq t\,(F_{x,z}(u), F_{z,y}(v))$$

The concept of neighborhoods in Menger space is introduced as:

Definition 2.5 Let (X,F,t) be a Menger space. If $x\in X$, $\varepsilon > 0$ and $\lambda \in(0,1)$, then (ε,λ) -neighborhood of x, called $U_x(\varepsilon,\lambda)$, is defined by

$$U_x(\varepsilon,\lambda) = \{y\in X: F_{(x,y)}(\varepsilon)>(1-\lambda)\}$$

An (ε,λ)-topology in X is the topology induced by the family $\{U_x(\varepsilon,\lambda): x \in X, \varepsilon > 0, \lambda\in(0,1)\}$ of neighborhood.

Remark: If t is continuous, then Menger space (X,F,t) is a Housdroff space in (ε,λ)-topology.

Let (X,F,t) be a complete Fuzzy Menger space and $A\subset X$. Then A is called a bounded set if for $u>0$ $\lim_{u\to\infty} \inf_{x,y\in A} F_{x,y}(u) =1$.

Definition 2.6 A sequence $\{x_n\}$ in (X,F,t) is said to be convergent to a point x in X if for every $\varepsilon>0$and $\lambda>0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $x_n \in U_x(\varepsilon,\lambda)$ for all $n \geq N$ or equivalently $F(x_n, x; \varepsilon) > 1-\lambda$ for all $n \geq N$.

Definition 2.7 A sequence $\{x_n\}$in (X,F,t) is said to be cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $F(x_n,x_m; \varepsilon) > 1-\lambda\ \forall\, n, m \geq N$.

Definition 2.8A Menger space (X,F,t) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X .

Lemma 1 Let $\{x_n\}$ be a sequence in a Menger space (X,F,t), where t is continuous and $t(p,p) \geq p$ for all $p\in[0,1]$, if there exists a constant k(0,1) such that $\forall p > 0$ and $n\in N$

$$t(F(x_n,x_{n+1}; kp)) \geq t(F(x_{n-1},x_n; p)),$$

then $\{x_n\}$ is cauchy sequence.

Lemma 2 If (X,d) is a metric space, then the metric d induces, a mapping F: $XxX\to L$ defined by $F(p, q) = H(x- d(p, q))$, $p, q \in R$. Further if $t: [0,1] \times [0,1]\to [0,1]$ is defined by $t(a,b) = \min\{a,b\}$, then (X,F,t) is a Menger space. It is complete if (X,d) is complete.

Definition 2.9: Let M be a nonempty set and the mapping $d: M \to X$ and $P\subset X$ be a cone, satisfies the following conditions:

2.8.1) $F_{x,y}(u) > 1 \forall\ x, y\in X \Leftrightarrow x = y$

2.8.2) $F_{x,y}(u) = F_{y,x}(u)\ \ \forall\, x, y\in X$,

2.8.3) $F_{x,y}(u+v) \geq t(F_{x,z}(u)\ ,\ F_{z,y}(v)) \forall\, x, y\in X$.

2.8.4) $For\ any\ x, y\in X, (x, y)\quad is\ non-increa\sin g\ and\ left\ continuous$.

Definition 2.10: Implicit Relation

Let Φ be the class of all real-valued continuous functions $\varphi : (R^+)^5 \to R^+$ non -decreasing in the first argument and satisfying the following conditions:

For x, y ≥ 0,

$x \geq \varphi(y,1,y,y, y)$ or $x \geq \varphi(y,y, x, y,x)$ or $x \geq \varphi(1, 1,y,1, y)$ such that $x \geq y$.

3. MAIN RESULTS

Theorem 3.1 : Let (X,F,t) be a complete Menger cone metric space and let M be a nonempty separable closed subset of cone metric space X and let T be continuous mapping defined on M satisfying contraction.

$F_{T(x), T(y)}(p) \geq \phi (F_{x,y}(p), F_{y,T(x)}(p), F_{y,T(y)}(p), F_{x,T(x)}(p), F_{x,T(y)}(p)$ ....(3.1)

for all x, y $\in$ X. Then T has a fixed point in X.

Proof: For each $x\in X$ and $n \geq 1$, let $x_1 = T x_0$ and $x_{n+1} = T (x_n) = T^{n+1}x_0$. Then

$$F_{x_n,x_{n+1}}(p)= F_{T(x_{n-1}), T(x_n)}(p) \geq \phi(F_{x_{n-1}, x_n}(p) , F_{x_n, T(x_{n-1})}(p), F_{x_n, T(x_n)}(p),F_{x_{n-1}, T(x_{n-1})}(p), F_{x_{n-1}, T(x_n)}(p))$$
$$\geq \phi (F_{x_{n-1}, x_n}(p), F_{x_n, x_n}(p), F_{x_n, x_{n+1}}(p), F_{x_{n-1}, x_n}(p), F_{x_{n-1}, x_{n+1}}(p))$$
$$\geq \phi(F_{x_{n-1}, x_n}(p), 1, F_{x_n, x_{n+1}}(p), F_{x_{n-1}, x_n}(p), t(F_{x_{n-1}, x_n}(u), F_{x_n, x_{n+1}}(v)))$$

Hence from (2.10) we have

$$F_{x_n, x_{n+1}}(p) \geq F_{x_{n-1}, x_n}(p)$$

Similarly

$$F_{x_{n-1}, x_n}(p) \geq F_{x_{n-2}, x_{n-1}}(p)$$

Hence $F_{x_n, x_{n+1}}(p) \geq F_{x_{n-1}, x_n}(p) \geq F_{x_{n-2}, x_{n-1}}(p)$

On continuing this process

$$F_{x_n, x_{n+1}}(p) \geq F_{x_{n-2}, x_{n-1}}(p) \geq F_{x_{n-3}, x_{n-2}}(p) \geq \dots\dots\dots \geq F_{x_0, x_1}(p)$$

Therefore the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since (F, X, t) is complete, there exists z $\in$ X such that $x_n \to z$.

$$F_{z, Tz}(p) \geq t(F_{z, x_{n+1}}(p) , F_{x_{n+1}, Tz}(p))$$
$$= t(F_{z, x_{n+1}}(p) , F_{Tx_n, Tz}(p))$$
$$\geq t(F_{z, x_{n+1}}(p) , \phi (F_{x_n, z}(p), F_{z,Tx_n}(p), F_{z,Tz}(p), F_{x_n,Tx_n}(p), F_{x_n,Tz}(p)))$$
$$\geq t(F_{z, x_{n+1}}(p) , \phi (F_{x_n, z}(p), F_{z, x_{n+1}}(p), F_{z,Tz}(p), F_{x_n,x_{n+1}}(p), F_{x_n,Tz}(p)))$$

Taking n→∞ we have

$F_{z,Tz}(p) \geq t(1, \phi(1, 1, F_{z,Tz}(p), 1, F_{z,Tz}(p)))$

(I) If $F_{z,Tz}(p) \geq 1$

Thus $F_{z,Tz} \in$ extP. But $F_{z,Tz} \in P$.

Therefore $F_{z,Tz} = 1$ and so Tz= z.

(II) If $F_{z,Tz}(p) \geq F_{z,Tz}(p)$

⇒ Tz= z.

This completes the proof.

Example: Let M=R and P={x∈M:x≥0} Let X = [0,∞) and metric d is defined by

$d(x,y) = \frac{|x-y|}{1+|x-y|}$. For each p define $F(x, y, p) = \begin{cases} 1 & for.x = y \\ H(p) & for.x \neq y \end{cases}$,

where $H(p) = \begin{cases} 0 & if \quad p \leq 0 \\ p.d(x,y) & if \quad 0 < p < 1 \\ 1 & if \quad p \geq 1 \end{cases}$

Clearly, (X, *F*, p) is a complete probabilistic space where t is defined by t(p,p) ≥ p.

The sequence $\{x_n\}$ is defined as $x_n = 2 - \frac{1}{2n}$. $Tx = \begin{cases} x & 0 \leq x \leq 1 \\ \frac{4-x}{2} & x > 1 \end{cases}$,

we see the all conditions of Theorem 3.1 are satisfied and hence 1 is the common fixed point in X.

4. CONCLUSION

In the paper we have fined that in pr**o**babilistic cone the continuity is required. With the required condition we fine that fixed point lie on axis of the cone.

Conflict of Interests

The authors declare that there is no conflict of interests.

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