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A COMMON FIXED POINT THEOREM FOR SIX WEAKLY COMPATIBLE MAPPINGS IN COMPLETE METRIC SPACE USING RATIONAL INEQUALITY

ARVIND KUMAR SHARMA[1,*], V. H. BADSHAH[2] AND V. K. GUPTA[3]

[1]Department of Applied Mathematics, Mahakal Group of Institutes, Ujjain (M.P.) India

[2]School of Studies in Mathematics, Vikram University, Ujjain (M.P.) India

[3]Department of Mathematics, Govt. Madhav Science College, Ujjain (M.P.) India

Abstract: The purpose of this paper is to present a common unique fixed-point theorem for six self mappings in complete metric space using weaker condition such as weakly compatible and associated sequence in place of compatible mappings and completeness of the metric space. More over the condition of continuity of any one of mapping is being dropped. Our result generalizes the results of Singh and Chouhan [19], Fisher [4], Lohani and Badshah [13], Sharma, Badshah and Gupta [18].

Keywords: complete metric space; compatible mappings; weakly compatible mappings; common fixed point.

Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

Jungck [6], proved a common fixed point theorem for commuting mappings in 1976, that generalizes the Banach's [1] fixed point theorem in a complete metric space. This result was generalized and extended in various ways by Iseki and Singh [5], Park [15], Das and Naik [2], Singh [20], Singh and Singh [21], Fisher [3], Park and Bae [16]. Recently, some common fixed point theorems of three and four commuting mappings were proved by Fisher [3], Khan and Imdad [12], Kang and Kim [10] and Lohani and Badshah [13].The result of Jungck [6] has so many applications but it requires the continuity of mappings. Sessa [17], introduced the concept

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*Corresponding author

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of weak commutativity and proved a common fixed point theorem for weakly commuting maps. Further Jungck [7], introduced more generalized commutativity; known as compatibility, which is weaker than weakly commuting maps. Since various authors proved a common fixed point theorem for compatible mappings satisfying contractive type conditions and continuity of one of the mapping is required.

In 1968 ,kannan [11] ,proved that there exists maps that have a discontinuity in the domain but which have fixed points, moreover the maps involved in every case were continuous at the fixed point. In 1979, Jungck and Rhodes [9] introduced the weaker condition; known as weakly compatible and showed that compatible maps are weakly compatible but the converse is not true in general.

In1998, Pant [14] introduced the notion of reciprocal continuity and used it to prove common fixed point theorems for contraction type self mappings. Also showed that if two mappings are continuous then they are obviously reciprocally continuous but the converse is not true. Moreover, in the setting of common fixed point theorems for compatible maps satisfying contractive conditions, continuity of one the two mappings implies their reciprocal continuity but not conversely.

The purpose of this paper is to generalize results of Sharma, Badshah and Gupta [18] for six self mappings in complete metric space using weaker condition such as weakly compatible and associated sequence in place of compatible mappings and completeness of the metric space. More over the condition of continuity of any one of mapping is being dropped. Our result extends results of Fisher [4], Jungck [7, 8] Singh and Chouhan [19], Lohani and Badshah [13]. To illustrate our main theorems, an example is also given.

2. Preliminaries

Definition 2.1 Two mapping S and T from a metric space (X, d) into itself, are called commuting on *X*, if $d(STx, TSx) = 0$ that is $STx = TSx$ for all x in *X*.

Definition 2.2 Two mapping S and T from a metric space (X, d) into itself, are called weakly commuting on X, if $d(STx, TSx) \leq d(Sx, Tx)$ for all x in X.

Clearly, commuting mappings are weakly commuting, but converse is not necessarily true, given by following example :

Example2.1

Let $X = [0, 1]$ with the Euclidean metric d. Define S and $T : X \rightarrow X$ by

$Sx = \frac{x}{7-x}$ and $Tx = \frac{x}{7}$ for all x in X.

Then for any x in X ,

$$\begin{aligned} d(STx, TSx) &= \left| \frac{x}{49-x} - \frac{x}{49-7x} \right| \\ &= \left| \frac{-6x^2}{(49-x)(49-7x)} \right| \\ &\leq \frac{x^2}{49-7x} \\ &= \left| \frac{x}{7-x} - \frac{x}{7} \right| \\ &= d(Sx, Tx) \end{aligned}$$

i.e. $d(STx, TSx) \leq d(Sx, Tx)$ for all x in X.

Thus S and T are weakly commuting mappings on X, but they are not commuting on X, because

$$STx = \frac{x}{49-x} < \frac{x}{49-7x} = TSx \quad \text{for any } x \neq 0 \text{ in } X.$$

i.e. $STx < TSx$ for any $x \neq 0$ in X.

Definition 2.3. If Two mapping S and T from a metric space (X, d) into itself, are called compatible mappings on X, if $\lim_{m\to\infty} d(STx_m, TSx_m) = 0$, when there is a sequence $\{x_m\}$ is in X such that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some x in X.

Clearly Two mapping S and T from a metric space (X, d) into itself, are called compatible mappings on X, then $d(STx, TSx) = 0$ when $d(Sx, Tx) = 0$ for some x in X . Note that weakly commuting mappings are compatible, but the converse is not necessarily true.

Example2.2 [18]

Let $X = [0, 1]$ with the Euclidean metric d. Define S and $T : X \to X$ by

$Sx = x$ and $Tx = \frac{x}{x+1}$ for all x in X.

Then for any x in X,

$$STx = S(Tx) = S\left(\frac{x}{x+1}\right) = \frac{x}{x+1}$$

$$TSx = T(Sx) = T(x) = \frac{x}{x+1}$$

$$d(Sx, Tx) = \left|x - \frac{x}{x+1}\right| = \left|\frac{x^2}{x+1}\right|$$

Thus we have

$$d(STx, TSx) = \left|\frac{x}{x+1} - \frac{x}{x+1}\right|$$

$$= 0 \le \frac{x^2}{x+1} \text{ for all } x \text{ in } X.$$

$$= d(Sx, Tx)$$

i.e. $d(STx, TSx) \le d(Sx, Tx)$ for all x in X.

Thus S and T are weakly commuting mappings on X, and then obviously S and T are compatible mappings on X.

Example2.3 [18]

Let $X = R$ with the Euclidean metric d. Define S and $T : X \to X$ by

$Sx = x^2$ and $Tx = 2x^2$ for all x in X.

Then for any x in X ,

$$STx = S(Tx) = S(2x^2) = 4x^4$$

$TSx = T(Sx) = T(x^2) = 2x^4$ are compatible mappings on X , because

$$d\left(Sx,Tx\right)=\left|x^2-2x^2\right|=\left|-x^2\right|\to 0 \ \ as \ \ x\to 0$$

Then

$$d\left(STx,TSx\right)=\left|4x^4-2x^4\right|=2\left|x^4\right|\to 0 \ \ as \ \ x\to 0$$

But $d\left(STx,TSx\right) \le d\left(Sx,Tx\right)$ is not true for all x in X.

Thus S and T are not weakly commuting mappings on X.

Hence all weakly commuting mappings are compatible, but converse is not true.

Definition 2.4 If Two mapping S and T from a metric space $\left(X,\, d\right)$ into itself, are called Weakly compatible mappings on X, if they commute at their coincidence point i.e. if $Su=Tu$ for some u in X ,then $STu = TSu$.

Example2.4

Let X = [0, 1] with the usual metric $d(x, y) = |x - y|$.

Define S and $T : X \to X$ by

$$Sx=\begin{cases} x & \text{when } 0\le x\le \dfrac{1}{2} \\ 1 & \text{when } \dfrac{1}{2}<x\le 1 \end{cases}$$

$$Tx= 1-x \quad \text{for all } x \text{ in } X.$$

Then clearly $\frac{1}{2}$ is coincidence point of S and T , because $S\left(\frac{1}{2}\right)=T\left(\frac{1}{2}\right)=\frac{1}{2}$.

Also $$ST\left(\frac{1}{2}\right) = S\left\{T\left(\frac{1}{2}\right)\right\}=S\left\{1-\frac{1}{2}\right\}=S\left(\frac{1}{2}\right)=\frac{1}{2}$$

$$TS\left(\frac{1}{2}\right) = T\left\{S\left(\frac{1}{2}\right)\right\}=T\left\{\frac{1}{2}\right\}=1-\frac{1}{2}=\frac{1}{2}$$

Hence $\left(S,T\right)$ is weakly compatible on X , because they commutes at their coincidence point $\frac{1}{2}$.

But $\left(S,T\right)$ is not compatible on X, for this take a sequence $x_n=\frac{1}{2}-\frac{1}{n}$, $n\ge 2$.

Then $\lim\limits_{n\to\infty} Sx_n = \frac{1}{2}$, $\lim\limits_{n\to\infty} Tx_n = \frac{1}{2}$ Also $\lim\limits_{n\to\infty} TSx_n = \lim\limits_{n\to\infty} T\left(\frac{1}{2}-\frac{1}{n}\right)=\lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{1}{n}\right)=\frac{1}{2}$.

But $\lim_{n\to\infty} STx_n = \lim_{n\to\infty} S\left(\frac{1}{2}+\frac{1}{n}\right) = \lim_{n\to\infty}(1) = 1$ so that $\lim_{n\to\infty}(TSx_n, STx_n) \neq 0$.

Hence (S,T) is not compatible on X. Note that compatible mappings are weakly compatible, but the converse is not necessarily true.

Definition 2.5 If Two mapping S and T from a metric space (X, d) into itself, are called reciprocally continuous mappings on X, if $\lim_{n\to\infty} STx_n = Su$ and $\lim_{n\to\infty} TSx_n = Tu$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$ for some u in X.

Example 2.5

Let X = [2, 15] with the Euclidean metric d.

Define S and $T : X \to X$ by

$$Sx = 7 \text{ when } x \in [2,15]$$

$$Tx = \begin{cases} 8 & \text{when } x < 4 \text{ and } > 12 \\ 3+x & \text{when } 4 \le x \le 12 \end{cases}$$

Take a sequence $x_n = 4 + \frac{1}{n}$, $n \ge 1$ in X.

Then $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} S\left(4+\frac{1}{n}\right) = 7$, $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} T\left(4+\frac{1}{n}\right) = \lim_{n\to\infty}\left(3+4+\frac{1}{n}\right) = 7$

Also $\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T\left\{S\left(4+\frac{1}{n}\right)\right\} = \lim_{n\to\infty} T(7) = 3+7 = 10 = T(7)$.

But $\lim_{n\to\infty} STx_n = \lim_{n\to\infty} S\left(4+\frac{1}{n}\right) = \lim_{n\to\infty} S\left(4+\frac{1}{n}\right) = \lim_{n\to\infty}\left(7+\frac{1}{n}\right) = 7 = S(7)$.

Hence S and T are Reciprocally continuous mappings on X , but S and T are not continuous on X. Hence if S and T both continuous then they are obviously reciprocally continuous but converse is not true converse is not necessarily true.

Singh and Chouhan [19] proved the following theorem.

Theorem 2.1 Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the following conditions:

$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X) \tag{3.1}$$

One of P, Q, S and T is continuous,

$$\begin{aligned}[d(Ax,By)]^2 \le k_1[d(Ax,Sx)d(By,Ty)+d(By,Sx)d(Ax,Ty)] \\ +k_2[d(Ax,Sx)d(Ax,Ty)+d(By,Ty)d(By,Sx)]\end{aligned} \tag{3.2}$$

for all $x, y \in X$, where $k_1, k_2 \ge 0$ and $0 \le k_1 + k_2 < 1$.

The pairs (A, S) and (B, T) are compatible on X, then A, B, S and T have a unique common fixed point in X.

Sharma, Badshah and Gupta [18] proved the following theorem.

Theorem 2.2 Let P, Q, S and T *be* mappings from a complete metric space (X, d) into itself satisfying the conditions

$$S(X) \subseteq Q(X),\ T(X) \subseteq P(X) \tag{3.3}$$

$$d(Sx,Ty) \le \left\{\alpha+\beta\frac{d(Sx,Px)}{1+d(Px,Qy)}\right\}d(Ty,Qy) \tag{3.4}$$

for all $x, y \in X$, where $\alpha, \beta \ge 0$ and $\alpha + \beta < 1$.

Suppose that

(i) One of P, Q, S and T is continuous,

(ii) Pairs (S, P) and (T, Q) are compatible on X.

Then P, Q, S and T have a unique common fixed point in X.

Now we generalize the theorem 2.2 for six self mappings by using weaker condition weakly compatible in place of compatible mapping also the condition that one of the mappings should be continuous also dropped.

Associated Sequence. Suppose A, B, P, Q, S and T be self mappings from a complete metric space (X, d) into itself satisfying the conditions:

$$S(X) \subseteq AB(X) \text{ and } T(X) \subseteq PQ(X) \tag{3.5}$$

$$d(Sx,Ty) \le \left\{\alpha+\beta\frac{d(Sx,PQx)}{1+d(PQx,ABy)}\right\}d(Ty,ABy) \tag{3.6}$$

for all $x, y \in X$, where $\alpha, \beta \ge 0$ and $\alpha + \beta < 1$.

Then for an arbitrary point x_0 in X, by (3.5) we choose a point x_1 in X such that $Sx_0 = ABx_1$ and for this point x_1, there exists a point x_2 in X such that $Tx_1 = PQx_2$ and so on. Proceeding in the similar manner, we can define a sequence $\{y_m\}$ in X such that

$y_{2m+1} = ABx_{2m+1} = Sx_{2m}$ for $m \geq 0$ and $y_{2m} = PQx_{2m} = Tx_{2m-1}$ for $m \geq 1$ (3.7)

we shall call this sequence as an "Associated sequence of x_0" relative to six self mappings *A*, *B*, *P*, *Q*, *S* and *T* .

Lemma 2.1. Let *A*, *B*, *P*, *Q*, *S* and *T* be mappings from a complete metric space (X, d) into itself satisfying the conditions (3.5) and (3.6). Then the Associated sequence $\{y_m\}$ relative to six self mappings *A*, *B*, *P*, *Q*, *S* and *T* defined in (3.7) is a Cauchy sequence in *X*.

Proof. By definition (3.6) we have

$$d(y_{2m+1}, y_{2m}) = d(Sx_{2m}, Tx_{2m-1})$$

$$\leq \left\{\alpha + \beta \frac{d(Sx_{2m}, PQx_{2m})}{1 + d(PQx_{2m}, ABx_{2m-1})}\right\} d(Tx_{2m-1}, ABx_{2m-1})$$

$$\leq \left\{\alpha + \beta \frac{d(y_{2m+1}, y_{2m})}{1 + d(y_{2m}, y_{2m-1})}\right\} d(y_{2m}, y_{2m-1})$$

$$\leq \alpha d(y_{2m}, y_{2m-1}) + \beta d(y_{2m+1}, y_{2m})$$

i.e $d(y_{2m+1}, y_{2m}) \leq \dfrac{\alpha}{1-\beta} d(y_{2m}, y_{2m-1})$

Hence $d(y_{2m+1}, y_{2m}) \leq h\, d(y_{2m}, y_{2m-1})$

Where $h = \dfrac{\alpha}{1-\beta} < 1$

Similarly we can show that

$$d(y_{2m+1}, y_{2m}) \leq h^{2m}\, d(y_1, y_0)$$

For k > m , we have

$$d(y_{m+k}, y_m) \leq \sum_{i=1}^{k} d(y_{n+i}, y_{n+i-1})$$

$$\leq \sum_{i=1}^{k} h^{n+i-1} d(y_1, y_0)$$

i.e. $d(y_{m+k}, y_m) \leq \left(\dfrac{h^n}{1-h}\right) d(y_1, y_0) \to 0$ as $n \to \infty$

Since $h < 1$, $h^n \to 0$ as $n \to \infty$, so that $d(y_m, y_{m+k}) \to 0.$ This Show that the sequence $\{y_m\}$ is a Cauchy's sequence in *X*. and since *X* is a complete metric space, it converges to a limit, say *u* in

X. The converse of the lemma is not true, that is A, B, P, Q, S and T satisfying (3.5) and (3.6) , even if for x_0 in X and the Associated sequence of x_0 converges, the metric space (X, d) need not be complete. The following example establishes this.

Example 2.6

Let X = (-1, 1) with usual metric $d(x, y) = |x - y|$

$$Sx = Tx = \begin{cases} \frac{1}{4} & \text{if } -1 < x < \frac{1}{5} \\ \frac{1}{5} & \text{if } \frac{1}{5} \le x < 1 \end{cases} \quad , \quad Ax = Px = x \text{ if } -1 < x < 1 \quad , \quad Bx = \begin{cases} \frac{1}{4} & \text{if } -1 < x < \frac{1}{5} \\ \frac{2}{5} - x & \text{if } \frac{1}{5} \le x < 1 \end{cases}$$

$$Qx = \begin{cases} \frac{1}{4} & \text{if } -1 < x < \frac{1}{5} \\ \frac{5x+4}{25} & \text{if } \frac{1}{5} \le x < 1 \end{cases}$$ Then $S(X) = T(X) = \left\{\frac{1}{4}, \frac{1}{5}\right\}$ while $A(X) = P(X) = (-1, 1)$,

$AB(X) = \left\{\frac{1}{4} \cup \left[\frac{1}{5} \quad \frac{-3}{5}\right)\right\}$ $\quad PQ(X) = \left\{\frac{1}{4} \cup \left[\frac{1}{5} \quad \frac{9}{25}\right)\right\}$ Clearly $S(X) \subset AB(X)$ and $T(X) \subset PQ(X)$ are satisfied. Also inequality (3.6) can be easily verified with appropriate values of α and β. Also the sequence $Sx_0, Tx_1, Sx_2, Tx_3, \ldots Sx_{2n}, Tx_{2n+1}, \ldots$ converges to $\frac{1}{5}$ if $\frac{1}{5} \le x < 1$. But (X, d) is not a complete metric space.

3. Main Result

Theorem 3.1 Let A, B, P, Q, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions (3.5) and (3.6). Suppose that the pair (S, PQ) is reciprocally continuous and compatible and (T, AB) is weakly compatible on X. Also $AB = BA$, $PQ = QP$, $QS = SQ$, $TA = AT$. Further the associated sequence relative to four self mappings A, B, P, Q, S and T such that Sx_0, Tx_1, Sx_2, $Tx_3, \ldots$, Sx_{2m}, Tx_{2m+1} converges to u in X as $n \to \infty$ .Then A, B, P, Q, S and T have a unique common fixed point u in X.

Proof. Let $\{y_m\}$ be the associated sequence in X defined by (3.7). By lemma, Associated $\{y_m\}$ is a Cauchy sequence in X and hence it converges to some point u in X. Consequently, the subsequences $\{Sx_{2m}\}$, $\{PQx_{2m+2}\}$, $\{ABx_{2m+1}\}$ and $\{Tx_{2m+-1}\}$ of $\{y_m\}$ also converges to u.

Since $S(X) \subseteq AB(X)$ then there exists $v \in X$ such that $u = ABv$. Also from the condition $T(X) \subseteq PQ(X)$ implies that there exists $v' \in X$ in such that $PQv' = u$. since (S, PQ) is reciprocally continuous $S(PQ)x_{2m} \to Su$ and Also (S, PQ) is compatible,

$\lim_{m\to\infty} d\left(S(PQ)x_{2m}, (PQ)Sx_{2m}\right) = 0$ then $d\left(Su, PQu\right) = 0$ implies that $Su = PQu$.

To prove $Su = u$, put $x = u$, $y = x_{2m+1}$ in (3.6), we obtain

$$d(Su, Tx_{2m+1}) \le \left\{\alpha + \beta \frac{d(Su, PQu)}{1 + d(PQu, ABx_{2m+1})}\right\} d(Tx_{2m+1}, ABx_{2m+1})$$

Letting $m \to \infty$ we get

$$d(Su, u) \le \left\{\alpha + \beta \frac{d(Su, Su)}{1 + d(Su, u)}\right\} d(u, u)$$

$$d(Su, u) \le 0$$

So that $u = Su$.Therefore $u = Su = PQu$.

To prove $Tv = u$, put $x = u$, $y = v$ in (3.6), we obtain

$$d(Su, Tv) \le \left\{\alpha + \beta \frac{d(Su, PQu)}{1 + d(PQu, ABv)}\right\} d(Tv, ABv)$$

$$d(u, Tv) \le \left\{\alpha + \beta \frac{d(u, u)}{1 + d(u, u)}\right\} d(Tv, u)$$

$$(1 - \alpha) d(u, Tv) \le 0$$

$d(u, Tv) \le 0$.So that $u = Tv$.

Since (T, AB) is weakly compatible and $ABv = Tv = u$, then $(AB)Tv = T(AB)v$ therefore $ABu = Tu$.

To prove $Tu = u$, put $x = x_{2m}$, $y = u$ in (3.6), we obtain

$$d(Sx_{2m}, Tu) \leq \left\{\alpha + \beta \frac{d(Sx_{2m}, PQx_{2m})}{1 + d(PQx_{2m}, ABu)}\right\} d(Tu, ABu)$$

Letting $m \to \infty$ we get

$$d(u, Tu) \leq \left\{\alpha + \beta \frac{d(u,u)}{1 + d(u,Tu)}\right\} d(Tu, Tu)$$

$$d(u, Tu) \leq 0$$

So that $u = Tu$.Therefore $u = Tu = ABu.$

To prove $Au = u$, put $x = u$, $y = Au$ in (3.6), we obtain

$$d(Su, TAu) \leq \left\{\alpha + \beta \frac{d(Su, PQu)}{1 + d(PQu, ABAu)}\right\} d(TAu, ABAu)$$

$$d(u, ATu) \leq \left\{\alpha + \beta \frac{d(Su, PQu)}{1 + d(PQu, ABAu)}\right\} d(ATu, ABAu)$$

$$d(u, Au) \leq \left\{\alpha + \beta \frac{d(u,u)}{1 + d(u, Au)}\right\} d(Au, Au)$$

$$d(u, Au) \leq 0$$

So that $u = Au$. Now $ABu = u \Rightarrow BAu = u \Rightarrow Bu = u$. Hence $u = Au = Bu.$

To prove $Qu = u$, put $x = Qu$, $y = u$ in (3.6), we obtain

$$d(SQu, Tu) \leq \left\{\alpha + \beta \frac{d(SQu, PQQu)}{1 + d(PQQu, ABu)}\right\} d(Tu, ABu)$$

$$d(Qu, u) \leq \left\{\alpha + \beta \frac{d(Qu, Qu)}{1 + d(Qu, Tu)}\right\} d(Tu, Tu)$$

$$d(u, Au) \leq \left\{\alpha + \beta \frac{d(u,u)}{1 + d(u, Au)}\right\} d(Au, Au)$$

$$d(Qu, u) \leq 0$$

So that $u = Qu$. Now $PQu = u$ then $Pu = u$. Thus $u = Au = Bu = Pu = Qu = Su = Tu.$ Hence u is common fixed point of *A*, *B*, *P*, *Q*, *S* and *T*.

For uniqueness of u , suppose u and z , $u \neq z$, are common fixed points of *A*, *B*, *P*, *Q*, *S* and *T*. Then by (3.6), we obtain

$$d(u, z) = d(Su, Tz)$$

$$\leq \left\{ \alpha + \beta \frac{d(Su, PQu)}{1 + d(PQu, ABz)} \right\} d(Tz, ABz)$$

$$\leq \left\{ \alpha + \beta \frac{d(Su, Pu)}{1 + d(Pu, Az)} \right\} d(Tz, Az)$$

$$\leq \left\{ \alpha + \beta \frac{d(u, u)}{1 + d(u, z)} \right\} d(z, z)$$

$$\leq 0$$

i.e. $d(u, z) \leq 0$

which is a contradiction .Hence $u = z$. Hence u is unique common fixed point of A, B, P, Q, S and T.

This completes the proof.

Remark 3.1 From the example 2.6 given above , clearly the pair (S, PQ) is reciprocally continuous and compatible and (T, AB) is weakly compatible on X as they commute at their coincidence point $\frac{1}{5}$. But the pairs (T, AB) is not compatible on X ,for this take a sequence $x_n = \frac{1}{5} + \frac{1}{n}$, $n \geq 2$.

Then $\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} AB\left(\frac{1}{5} + \frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{2}{5} - \frac{1}{5} - \frac{1}{n}\right) = \frac{1}{5}$, $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} T\left(\frac{1}{5} + \frac{1}{n}\right) = \frac{1}{5}$

Also $\lim_{n\to\infty} (AB)Tx_n = \lim_{n\to\infty} AB\left\{T\left(\frac{1}{5} + \frac{1}{n}\right)\right\} = \lim_{n\to\infty} AB\left(\frac{1}{5}\right) = \frac{2}{5} - \frac{1}{5} = \frac{1}{5}$.

But $\lim_{n\to\infty} T(AB)x_n = \lim_{n\to\infty} T\left\{AB\left(\frac{1}{5} + \frac{1}{n}\right)\right\} = \lim_{n\to\infty} T\left(\frac{1}{5} - \frac{1}{n}\right) = \frac{1}{4}$. so that $\lim_{n\to\infty}\left(T(AB)x_n, (AB)Tx_n\right) \neq 0.$

Hence (T, AB) is not compatible on X. Also note that none of the mappings are continuous and the rational inequality holds for appropriate value of α, β with α, $\beta \geq 0$ and $\alpha + \beta < 1$. Clearly $\frac{1}{5}$ is the unique common fixed point of P, Q, S and T.

Conclusion:- The conclusion of this paper that we shown a unique common fixed point theorem with generalize the result of Sharma, Badshah and Gupta [18] for six mappings, using weaker condition weakly compatible, instead of compatible mappings.

Conflict of Interests

The authors declare that there is no conflict of interests.

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