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NEW INEQUALITIES OF HADAMARD TYPE FOR (h_1, h_2) – CONVEX FUNCTIONS ON THE CO-ORDINATES VIA FRACTIONAL INTEGRALS

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Abstract: In this paper we establish some Hadamard type inequalities involving Riemann – Liouville fractional integrals for (h_1, h_2) - convex functions on the co-ordinates.

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1. Introduction

If $f: I \to R$ is a convex function on the interval I, then for any $a, b \in I$ with $a < b$ we have the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \qquad (1.1)$$

This remarkable results is well known in the literature as the Hermite-Hadamard inequality. Several generalizations of the Hermite-Hadamard integral inequality are considered by many authors. For recent results and genaralizations see [1, 2, 3, 4, 6, 7, 9, 11, 12] and references therein.

Let us consider now a bidimensional interval $\Delta := [a, b] \times [c, d]$ in R^2 with $a < b$ and $c < d$. In [3] Dragomir introduced co-ordinated convex function in the following way: a mapping $f: \Delta \to R$ is said to be convex on the co-ordinates on Δ if the inequality

$$f(tx + (1-t)y),\ ru\,(1-r)w)$$

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$$\leq \boldsymbol{trf\,(x,u) + t(1-r)f(x,w) + r(1-t)f(y,u) + (1-t)(1-r)f(y,w)}$$

holds for all $t, r \in [0,1]$ and $(x,u), (y,w) \in \Delta$.

For such a mapping Dragomir proved the following Hermite-Hadamard type inequalities:

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right]$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$

$$\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx\right.$$

$$\left.+\frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right]$$

$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \tag{1.2}$$

In 2008 Alomari and Darus (see [1]) defined the co-ordinated *s*-convexity in the second sense as follows: a mapping $f\colon \Delta \to R$ is said to be *s*-convex in the second sense on the co-ordinates on Δ if the inequality

$$\boldsymbol{f(tx + (1-t)y,\ ru\,(1-r)w)}$$

$$\leq \boldsymbol{t^s r^s f\,(x,u) + t^s(1-r)^s f(x,w) + r^s(1-t)^s f(y,u) + (1-t)^s(1-r)^s f(y,w)},$$

holds for all $t, r \in [0,1]$, $(x,u), (y,w) \in \Delta$ and for some fixed $s \in [0,1]$.

For such a mapping they proved the following Hermite-Hadamard type inequalities:

$$4^{s-1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$\leq 2^{s-2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right]$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy$$

$$\leq \frac{1}{2(s+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx\right.$$

$$\left.+\frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right]$$

$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{(s+1)^2}. \tag{1.3}$$

In the paper [6], Matłoka introduced the co-ordinated (h_1, h_2) – convexity as follows: the non-negative function f on the invex set $X_1 \times X_2$ is said to be co-ordinated (h_1, h_2) – preinvex with respect to η_1 and η_2 if the inequality

$$\boldsymbol{f(x+t_1\eta_1(b,x), y+t_2\eta_2(d,y))}$$
$$\leq \boldsymbol{h_1(1-t_1)h_2(1-t_2)f(x,y) + h_1(1-t_1)h_2(t_2)f(x,d)}$$
$$+h_1(t_1)h_2(1-t_2)f(b,y)+h_1(t_1)h_2(t_2)f(b,d)$$

holds for all $t_1, t_2 \in [0,1]$, $(x,y),(b,d) \in X_1 \times X_2$, where h_1 and h_2 are the non-negative functions on $[0,1]$, $h_1 \not\equiv 0$, $h_2 \not\equiv 0$.

For such a function he proved the following Hermite-Hadamard type inequalities:

$$\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(a+\frac{1}{2}\eta_1(b,a), c+\frac{1}{2}\eta_2(d,c)\right)$$

$$\leq \frac{1}{4h_1\left(\frac{1}{2}\right)\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f\left(a+\frac{1}{2}\eta_1(b,a), y\right)dy$$

$$+\frac{1}{4h_2\left(\frac{1}{2}\right)\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f\left(x, c+\frac{1}{2}\eta_2(d,c)\right)dx$$

$$\leq \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy$$

$$\leq \frac{1}{2\eta_1(b,a)}\int_0^1 h_2(t_2)dt_2\left[\int_a^{a+\eta_1(b,a)} f(x,c)dx + \int_a^{a+\eta_1(b,a)} f(x,d)dx\right]$$

$$+\frac{1}{2\eta_2(d,c)}\int_0^1 h_1(t_1)dt_1\left[\int_c^{c+\eta_2(d,c)} f(a,y)dy+\int_c^{c+\eta_2(d,c)} f(b,y)dy\right]$$

$$\leq [f(a,c)+f(b,c)+f(a,d)+f(b,d)]\int_0^1 h_1(t_1)dt_1\cdot\int_0^1 h_2(t_2)dt_2. \tag{1.4}$$

If $\eta_1(b,a)=b-a$, $\eta_2(d,c)=d-c$, $h_1(t_1)=h_2(t_2)=t$ then inequalities (1.4) become inequalities (1.2).

If $\eta_1(b,a)=b-a$, $\eta_2(d,c)=d-c$ and $h_1(t_1)=h_2(t_2)=t^s$ then inequalities (1.4) become inequalities (1.3).

Moreover, if $\eta_1(b,a)=b-a$ and $\eta_2(d,c)=d-c$ then the function is called (h_1,h_2) – convex on the co-ordinates.

In this paper, we establish new Hermite-Hadamard type inequalities for co-ordinated (h_1,h_2) – convex functions but via Riemann – Liouville fractional integral.

Throughout this paper, we assume that considered integrals exist.

2. Main results

We give first some necessary definitions and mathematical preliminaries of fractional calculus theory which are used in this sections. For more details, one can consult [5, 8, 10].

Let $f\in L_1[a,b]$. The Riemann – Liouville integrals $I_{a^+}^{\alpha} f$ and $I_{b^-}^{\alpha} f$ of order $\alpha>0$ with $a\geq 0$ are defined by

$$I_{a^+}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad x>a$$

$$I_{b^-}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x<b,$$

where $\Gamma(\alpha)$ is the Gamma function and $I_{a^+}^{0}f(x)=I_{b^-}^{0} f(x)=f(x)$.

For the sake of convenience, we will use the following notation throughout this section:

$$A=2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)\left[I^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^-,\left(\frac{c+d}{2}\right)^-} f(a,c)+I^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^+,\left(\frac{c+d}{2}\right)^-} f(b,c)\right.$$

$$+I^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^-,\left(\frac{c+d}{2}\right)^+} f(a,d) + I^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^+,\left(\frac{c+d}{2}\right)^+} f(b.d)\Bigg]$$

$$-2^{\beta}\Gamma(\beta+1)(b-a)^{\alpha}\left[I^{\beta}_{\left(\frac{c+d}{2}\right)^-} f\left(\frac{a+b}{2},c\right) + I^{\beta}_{\left(\frac{c+d}{2}\right)^-} f\left(\frac{a+b}{2},c\right)\right.$$

$$\left.+I^{\beta}_{\left(\frac{c+d}{2}\right)^+} f\left(\frac{a+b}{2},d\right) + I^{\beta}_{\left(\frac{c+d}{2}\right)^+} f\left(\frac{a+b}{2},d\right)\right]$$

$$-2^{\alpha}\Gamma(\alpha+1)(d-c)^{\beta}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f\left(a,\frac{c+d}{2}\right) + I^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f\left(b,\frac{c+d}{2}\right)\right.$$

$$\left.+I^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f\left(a,\frac{c+d}{2}\right) + I^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f\left(b,\frac{c+d}{2}\right)\right],$$

where

$$I^{\alpha,\beta}_{a^+,c^+} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-u)^{\alpha-1}(y-v)^{\beta-1} f(u,v)\,du\,dv, \quad x>a, y>c,$$

$$I^{\alpha,\beta}_{b^-,d^-} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (u-x)^{\alpha-1}(v-y)^{\beta-1} f(u,v)\,du\,dv, \quad x<b, y<d,$$

$$I^{\alpha,\beta}_{a^+,d^-} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-u)^{\alpha-1}(v-y)^{\beta-1} f(u,v)\,du\,dv, \quad x>a, y<d,$$

$$I^{\alpha,\beta}_{b^-,c^+} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (u-x)^{\alpha-1}(y-v)^{\beta-1} f(u,v)\,du\,dv, \quad x<b, y>c.$$

To establish our main results we need the following identity:

Lemma 1. Let $f\colon \Delta \coloneqq [a,b]\times[c,d] \to R^2$ be a twice partial differentiable mapping on Δ° with $a<b$ and $c<d$. If $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$ and $\alpha,\beta>0$, $a,c \geq 0$, then the following identity holds:

$$4(b-a)^{\alpha}(d-c)^{\beta} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) + A$$

$$= \frac{(b-a)^{\alpha+1}(d-c)^{\beta+1}}{4}\left[\int_0^1\int_0^1 t^{\alpha} r^{\beta} \frac{\partial^2}{\partial r \partial t} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)c\right) dt\,dr\right.$$

$$-\int_0^1\int_0^1 t^\alpha r^\beta \frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)b, r\frac{c+d}{2}+(1-r)c\right)dtdr$$

$$-\int_0^1\int_0^1 t^\alpha r^\beta \frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)d\right)dtdr$$

$$\left.+\int_0^1\int_0^1 t^\alpha r^\beta \frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)b, r\frac{c+d}{2}+(1-r)d\right)dtdr\right]. \qquad (2.1)$$

Proof. By integration by parts and by change of the variables $u = t\frac{a+b}{2}+(1-t)a$, $v = r\frac{c+d}{2}+(1-t)c$, we have

$$\int_0^1\int_0^1 t^\alpha r^\beta \frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)c\right)drdt$$

$$=\frac{2}{b-a}\int_0^1 r^\beta \frac{\partial}{\partial r} f\left(\frac{a+b}{2}, r\frac{c+d}{2}+(1-r)c\right)dr$$

$$-\frac{2}{b-a}\int_0^1 t^{\alpha-1}\left\{\int_0^1 r^\beta \frac{\partial}{\partial r} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)c\right)dr\right\}dt$$

$$=\frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{4\beta}{(b-a)(d-c)}\int_0^1 r^{\beta-1} f\left(\frac{a+b}{2}, r\frac{c+d}{2}+(1-r)c\right)dr$$

$$-\frac{4\alpha}{(b-a)(d-c)}\int_0^1 t^{\alpha-1} f\left(t\frac{a+b}{2}+(1-t)a, \frac{c+d}{2}\right)dt$$

$$+\frac{4\alpha\beta}{(b-a)(d-c)}\int_0^1\int_0^1 t^{\alpha-1} r^{\beta-1} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)c\right)drdt$$

$$=\frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{\Gamma(\beta+1)2^{\beta+2}}{(b-a)(d-c)^{\beta+1}} I^\beta_{\left(\frac{c+d}{2}\right)^-} f\left(\frac{a+b}{2}, c\right)$$

$$-\frac{\Gamma(\alpha+1)2^{\alpha+2}}{(d-c)(b-a)^{\alpha+1}} I^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f\left(a, \frac{c+d}{2}\right)$$

$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+2}}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} I^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^-,\left(\frac{c+d}{2}\right)^-} f(a,c) \tag{2.2}$$

Similarly, by integration by parts, we also have

$$\int_0^1 \int_0^1 t^\alpha r^\beta \frac{\partial^2}{\partial r \partial t} f\left(t\frac{a+b}{2} + (1-t)a, r\frac{c+d}{2} + (1-r)d\right) drdt$$

$$= \frac{4}{(b-a)(c-d)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{\Gamma(\beta+1)2^{\beta+2}}{(b-a)(d-c)^{\beta+1}} I^{\beta}_{\left(\frac{c+d}{2}\right)^+} f\left(\frac{a+b}{2}, d\right)$$

$$-\frac{\Gamma(\alpha+1)2^{\alpha+2}}{(c-d)(b-a)^{\alpha+1}} I^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f\left(a, \frac{c+d}{2}\right)$$

$$-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+2}}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} I^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^-,\left(\frac{c+d}{2}\right)^+} f(a,d), \tag{2.3}$$

$$\int_0^1 \int_0^1 t^\alpha r^\beta \frac{\partial^2}{\partial r \partial t} f\left(t\frac{a+b}{2} + (1-t)b, r\frac{c+d}{2} + (1-r)c\right) drdt$$

$$= \frac{4}{(a-b)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{\Gamma(\beta+1)2^{\beta+2}}{(a-b)(d-c)^{\beta+1}} I^{\beta}_{\left(\frac{c+d}{2}\right)^-} f\left(\frac{a+b}{2}, c\right)$$

$$-\frac{\Gamma(\alpha+1)2^{\alpha+2}}{(d-c)(b-a)^{\alpha+1}} I^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f\left(b, \frac{c+d}{2}\right)$$

$$-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+2}}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} I^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^+,\left(\frac{c+d}{2}\right)^-} f(b,c) \tag{2.4}$$

and

$$\int_0^1 \int_0^1 t^\alpha r^\beta \frac{\partial^2}{\partial r \partial t} f\left(t\frac{a+b}{2} + (1-t)b, r\frac{c+d}{2} + (1-r)d\right) drdt$$

$$= \frac{4}{(a-b)(c-d)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{\Gamma(\beta+1)2^{\beta+2}}{(a-b)(d-c)^{\beta+1}} I^{\beta}_{\left(\frac{c+d}{2}\right)^+} f\left(\frac{a+b}{2}, d\right)$$

$$+\frac{\Gamma(\alpha+1)2^{\alpha+2}}{(c-d)(b-a)^{\alpha+1}} I^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f\left(b, \frac{c+d}{2}\right)$$

$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+2}}{(b-a)^{\alpha+1}(d-c)^{\beta+1}}\ I^{\alpha,\beta}_{\left(\frac{a+b}{2}\right)^+,\left(\frac{c+d}{2}\right)^+} f(b,d) \tag{2.5}$$

From (2.2) - (2.5), we get (2.1). This completes the proof.

Theorem 1. Let $f:\Delta\coloneqq[a,b]\times[c,d]\to R$ be a twice partial differentiable mapping on Δ° with $a<b,\ c<d,\ a,c\geq 0$ such that $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r\partial t}\right|$ is $(h_1,h_2)-$convex on the co-ordinates on Δ, then the following inequality holds:

$$\left|4(b-a)^\alpha(d-c)^\beta f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|$$

$$\leq\frac{(b-a)^{\alpha+1}(d-c)^{\beta+1}}{4}\left\{4\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\int_0^1\int_0^1 t^\alpha r^\beta h_1(t)\,h_2(r)dtdr\right.$$

$$+2\left[\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{a+b}{2},c\right)\right|\right]\int_0^1\int_0^1 t^\alpha r^\beta h_1(t)\,h_2(1-r)dtdr$$

$$+2\left[\left|\frac{\partial^2}{\partial r\partial t}f\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2}{\partial r\partial t}f\left(b,\frac{c+d}{2}\right)\right|\right]\int_0^1\int_0^1 t^\alpha r^\beta h_1(1-t)\,h_2(r)dtdr$$

$$+\left[\left|\frac{\partial^2}{\partial r\partial t}f(a,c)\right|+\left|\frac{\partial^2}{\partial r\partial t}f(b,c)\right|+\left|\frac{\partial^2}{\partial r\partial t}f(a,d)\right|+\left|\frac{\partial^2}{\partial r\partial t}f(b,d)\right|\right]$$

$$\left.\int_0^1\int_0^1 t^\alpha r^\beta\,h_1(1-t)\,h_2(1-r)dtdr\right\} \tag{2.6}$$

Proof. From Lemma 1, we have that the following inequality holds:

$$\left|4(b-a)^\alpha(d-c)^\beta f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|$$

$$\leq\frac{(b-a)^{\alpha+1}(d-c)^{\beta+1}}{4}\left[\int_0^1\int_0^1 t^\alpha r^\beta\left|\frac{\partial^2}{\partial r\partial t}f\left(t\frac{a+b}{2}+(1-t)a,r\frac{c+d}{2}+(1-r)c\right)\right|dtdr\right.$$

$$+\int_0^1\int_0^1 t^\alpha r^\beta\ \left|\frac{\partial^2}{\partial r\partial t}f\left(t\frac{a+b}{2}+(1-t)b,r\frac{c+d}{2}+(1-r)c\right)\right|dtdr$$

$$+\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)d\right)\right| dtdr$$

$$+\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)b, r\frac{c+d}{2}+(1-r)d\right)\right| dtdr\Bigg] \tag{2.7}$$

By the (h_1, h_2) −convexity of $\left|\frac{\partial^2 f}{\partial r\partial t}\right|$ on the co-ordinates on Δ, we get the following inequalities:

$$\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)c\right)\right| dtdr$$

$$\le \int_0^1\int_0^1 t^\alpha r^\beta \Bigg[h_1(1-t)h_2(1-r)\left|\frac{\partial^2}{\partial r\partial t} f(a,c)\right| + h_1(1-t)h_2(r)\left|\frac{\partial^2}{\partial r\partial t} f\left(a, \frac{c+d}{2}\right)\right|$$

$$+h_1(t)h_2(1-r)\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2}, c\right)\right| + h_1(t)h_2(r)\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|\Bigg] dtdr, \tag{2.8}$$

$$\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)b, r\frac{c+d}{2}+(1-r)c\right)\right| dtdr$$

$$\le \int_0^1\int_0^1 t^\alpha r^\beta \Bigg[h_1(1-t)h_2(1-r)\left|\frac{\partial^2}{\partial r\partial t} f(b,c)\right| + h_1(1-t)h_2(r)\left|\frac{\partial^2}{\partial r\partial t} f\left(b, \frac{c+d}{2}\right)\right|$$

$$+h_1(t)h_2(1-r)\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2}, c\right)\right| + h_1(t)h_2(r)\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|\Bigg] dtdr, \tag{2.9}$$

$$\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)d\right)\right| dtdr$$

$$\le \int_0^1\int_0^1 t^\alpha r^\beta \Bigg[h_1(1-t)h_2(1-r)\left|\frac{\partial^2}{\partial r\partial t} f(a,d)\right| + h_1(1-t)h_2(r)\left|\frac{\partial^2}{\partial r\partial t} f\left(a, \frac{c+d}{2}\right)\right|$$

$$+h_1(t)h_2(1-r)\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2}, d\right)\right| + h_1(t)h_2(r)\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|\Bigg] dtdr \tag{2.10}$$

and

$$\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)b, r\frac{c+d}{2}+(1-r)d\right)\right| dtdr$$

$$\leq \int_0^1\int_0^1 t^\alpha r^\beta \Big[h_1(1-t)h_2(1-r)\left|\frac{\partial^2}{\partial r\partial t} f(b,d)\right| + h_1(1-t)h_2(r)\left|\frac{\partial^2}{\partial r\partial t} f\left(b,\frac{c+d}{2}\right)\right|$$

$$+h_1(t)h_2(1-r)\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2},d\right)\right| + h_1(t)h_2(r)\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\Big] dtdr \qquad (2.11)$$

By using (2.8) - (2.11) in (2.7), we get the inequality (2.6). This completes the proof of the theorem.

Corollary 1. If in Theorem 1 we assume that $\left|\frac{\partial^2 f}{\partial r\partial t} f(t,r)\right| \leq M$ for all $(t,r) \in \Delta$, then the inequality (2.6) reduces to the following inequality:

$$\left|4(b-a)^\alpha(d-c)^\beta f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|$$

$\leq M\cdot K\cdot(b-a)^{\alpha+1}(d-c)^{\beta+1}$, where

$$K=\int_0^1\int_0^1 t^\alpha r^\beta h_1(t)h_2(r)dtdr+\int_0^1\int_0^1 t^\alpha r^\beta h_1(t)h_2(1-r)dtdr$$

$$+\int_0^1\int_0^1 t^\alpha r^\beta h_1(1-t)h_2(r)dtdr+\int_0^1\int_0^1 t^\alpha r^\beta h_1(1-t)h_2(1-r)dtdr.$$

Corollary 2. If in Theorem 1 we take $h_1(t)=t$ and $h_2(r)=r$ then the inequality (2.6) reduces to the following inequality for the function convex on the co-ordinates:

$$\left|4(b-a)^\alpha(d-c)^\beta f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|$$

$$\leq \frac{(b-a)^{\alpha+1}(d-c)^{\beta+1}}{4}\Big\{\frac{4}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$

$$+\frac{2}{(\beta+1)(\beta+2)(\alpha+2)}\left[\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2}{\partial r\partial t} f\left(\frac{a+b}{2},d\right)\right|\right]$$

$$+\frac{2}{(\alpha+1)(\beta+2)(\alpha+2)}\left[\left|\frac{\partial^2}{\partial r\partial t} f\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2}{\partial r\partial t} f\left(b,\frac{c+d}{2}\right)\right|\right]$$

$$+\frac{2}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left[\left|\frac{\partial^2}{\partial r\partial t}\,f(a,c)\right|+\left|\frac{\partial^2}{\partial r\partial t}\,f(b,c)\right|\right.$$
$$\left.+\left|\frac{\partial^2}{\partial r\partial t}\,f(a,d)\right| +\left|\frac{\partial^2}{\partial r\partial t}\,f(b,d)\right|\right]\Bigg\}.$$

If, additionally, we assume that $\left|\frac{\partial^2 f}{\partial r\partial t}f(t,r)\right| \le M$ for all $(t,r)\in\Delta$, then we obtain the following inequality

$$\left|4(b-a)^{\alpha}(d-c)^{\beta}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|$$
$$\le M(b-a)^{\alpha+1}(d-c)^{\beta+1}\cdot\frac{2\alpha+2\beta+\alpha\beta+3}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}.$$

Corollary 3. If in Theorem 1 we take $h_1(t)=t^s$ and $h_2(r)=r^s$ then the inequality (2.6) reduces to the following inequality for the s-convex function on the co-ordinates:

$$\left|4(b-a)^{\alpha}(d-c)^{\beta}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|$$
$$\le\frac{(b-a)^{\alpha+1}(d-c)^{\beta+1}}{4}\left\{\frac{4}{(\alpha+s+1)(\beta+s+1)}\left|\frac{\partial^2}{\partial r\partial t}\,f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right.$$
$$+\frac{2}{(\alpha+s+1)}\cdot\frac{\Gamma(s+1)\Gamma(\beta+1)}{\Gamma(\beta+s+2)}\left[\left|\frac{\partial^2}{\partial r\partial t}\,f\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2}{\partial r\partial t}\,f\left(\frac{a+b}{2},d\right)\right|\right]$$
$$+\frac{2}{(\beta+s+1)}\cdot\frac{\Gamma(s+1)\Gamma(\beta+1)}{\Gamma(\alpha+s+2)}\left[\left|\frac{\partial^2}{\partial r\partial t}\,f\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2}{\partial r\partial t}\,f\left(b,\frac{c+d}{2}\right)\right|\right]$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\Gamma(s+1))^2}{\Gamma(s+\alpha+2)\Gamma(s+\beta+2)}\left[\left|\frac{\partial^2}{\partial r\partial t}\,f(a,c)\right|+\left|\frac{\partial^2}{\partial r\partial t}\,f(b,c)\right|\right.$$
$$\left.\left.+\left|\frac{\partial^2}{\partial r\partial t}\,f(a,d)\right|+\left|\frac{\partial^2}{\partial r\partial t}\,f(b,d)\right|\right]\right\},$$

where we used the fact that $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

If, additionally, we assume that $\left|\frac{\partial^2 f}{\partial r\partial t}f(r,t)\right|\le M$ for all $(r,t)\in\Delta$, then we obtain the following inequality:

$$\left|4(b-a)^{\alpha}(d-c)^{\beta}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|$$

$$\leq M(b-a)^{\alpha+1}(d-c)^{\beta+1}\left[\frac{1}{(s+\alpha+1)(s+\beta+1)}+\frac{\Gamma(\beta+1)\Gamma(s+1)}{(\alpha+s+1)\Gamma(\beta+s+2)}\right.$$
$$\left.+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{(\beta+s+1)\Gamma(\alpha+s+2)}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\Gamma(s+1))^2}{(\alpha+s+2)\Gamma(\beta+s+2)}\right].$$

Theorem 2. Let $f:\Delta\coloneqq[a,b]\times[c,d]\to R$ be a twice partial differentiable mapping on Δ° with $a<b, c<d, a,c\geq 0$ such that $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is (h_1,h_2) - convex on the co-ordinates on Δ, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\left|\frac{\partial^2}{\partial r\partial t}f(t,r)\right|\leq M$ for all $(t,r)\in\Delta$, then the following inequality holds:

$$\left|4(b-a)^\alpha(d-c)^\beta f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|$$
$$\leq M(b-a)^{\alpha+1}(d-c)^{\beta+1}\cdot\left(\frac{1}{(\alpha p+1)(\beta p+1)}\right)^{\frac{1}{p}}\left(4\int_0^1\int_0^1 h_1(t)\,h_2(r)dtdr\right)^{\frac{1}{q}} \tag{2.12}$$

Proof. From Lemma 1, and the Hőlder inequality we have

$$\left|4(b-a)^\alpha(d-c)^\beta f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|\leq\frac{(b-a)^{\alpha+1}(d-c)^{\beta+1}}{4}\left(\int_0^1\int_0^1 t^{\alpha p}r^{\beta p}dtdr\right)^{\frac{1}{p}}$$
$$\left[\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(t\frac{a+b}{2}+(1-t)a,r\frac{c+d}{2}+(1-r)c\right)\right|^q dtdr\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(t\frac{a+b}{2}+(1-t)b,r\frac{c+d}{2}+(1-r)c\right)\right|^q dtdr\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(t\frac{a+b}{2}+(1-t)a,r\frac{c+d}{2}+(1-r)d\right)\right|^q dtdr\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(t\frac{a+b}{2}+(1-t)b,r\frac{c+d}{2}+(1-r)d\right)\right|^q dtdr\right)^{\frac{1}{q}}\right] \tag{2.13}$$

By the co-ordinated (h_1, h_2) −convexity and $\left|\frac{\partial^2}{\partial r \partial t} f(t,r)\right| \leq M$, for all $(t,r) \in \Delta$, we have:

$$\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)c\right)\right|^q dtdr$$

$$\leq M^q\left[\int_0^1\int_0^1 h_1(1-t)h_2(1-r)dtdr+\int_0^1\int_0^1 h_1(1-t)h_2(r)dtdr\right.$$

$$\left.+\int_0^1\int_0^1 h_1(t)h_2(1-r)dtdr+\int_0^1\int_0^1 h_1(t)h_2(r)dtdr\right]$$

$$= M^q\cdot 4\int_0^1\int_0^1 h_1(t)h_2(r)dtdr\,.$$

Similarly, we also have the following inequalities

$$\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)b, r\frac{c+d}{2}+(1-r)c\right)\right|^q dtdr \leq M^q\cdot 4\int_0^1\int_0^1 h_1(t)h_2(r)dtdr$$

$$\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)d\right)\right|^q dtdr \leq M^q\cdot 4\int_0^1\int_0^1 h_1(t)h_2(r)dtdr$$

and

$$\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)b, r\frac{c+d}{2}+(1-r)d\right)\right|^q dtdr \leq M^q\cdot 4\int_0^1\int_0^1 h_1(t)h_2(r)dtdr\,.$$

Using the last four inequalities in (2.13), we obtain (2,12). This completes the proof of the theorem.

Corollary 4. If in Theorem 2 we take $h_1(t) = t$ and $h_2(r) = r$ then the inequality (2.12) reduces to the following inequality for the convex function on the co-ordinates:

$$\left|4(b-a)^\alpha(d-c)^\beta f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|$$

$$\leq M(b-a)^{\alpha+1}(d-c)^{\beta+1}\cdot\left(\frac{1}{(\alpha\, p+1)(\beta p+1)}\right)^{\frac{1}{p}}.$$

Corollary 5. If in Theorem 2 we take $h_1(t) = t^s$ and $h_2(r) = r^s$ then the inequality (2.12) reduces to the following inequality for the *s*-convex function on the co-ordinates:

$$\left|4(b-a)^{\alpha}(d-c)^{\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + A\right|$$

$$\leq M(b-a)^{\alpha+1}(d-c)^{\beta+1} \cdot \left(\frac{1}{(\alpha p+1)(\beta p+1)}\right)^{\frac{1}{q}} \cdot \left(\frac{2}{s+1}\right)^{\frac{2}{q}}.$$

Theorem 3. Let $f\colon \Delta \coloneqq [a,b] \times [c,d] \to R$ be a twice differentiable mapping on Δ° with $a < b$, $c < d$, $a, c \geq 0$ such that $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r \partial t}\right|^q$ is (h_1, h_2)-convex on the co-ordinates on Δ, $q \geq 1$ and $\left|\frac{\partial^2 f}{\partial r \partial t} f(t,r)\right| \leq M$, $(t,r) \in \Delta$, then the following inequality holds:

$$\left|4(b-a)^{\alpha}(d-c)^{\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + A\right|$$

$$\leq M \cdot K^{\frac{1}{q}}(b-a)^{\alpha+1}(d-c)^{\beta+1} \cdot \left(\frac{1}{(\alpha+1)(\beta+1)}\right)^{1-\frac{1}{q}} \tag{2.14}$$

where K is defined in Corollary 1.

Proof. From Lemma 1, and the power mean inequality we have

$$\left|4(b-a)^{\alpha}(d-c)^{\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + A\right| \leq \frac{(b-a)^{\alpha+1}(d-c)^{\beta+1}}{4}\left(\int_0^1\int_0^1 t^{\alpha} r^{\beta} dtdr\right)^{1-\frac{1}{q}}$$

$$\left[\left(\int_0^1\int_0^1 t^{\alpha} r^{\beta}\left|\frac{\partial^2}{\partial r \partial t} f\left(t\frac{a+b}{2} + (1-t)a, r\frac{c+d}{2} + (1-r)c\right)\right|^q dtdr\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\int_0^1 t^{\alpha} r^{\beta}\left|\frac{\partial^2}{\partial r \partial t} f\left(t\frac{a+b}{2} + (1-t)b, r\frac{c+d}{2} + (1-r)c\right)\right|^q dtdr\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1 t^{\alpha} r^{\beta}\left|\frac{\partial^2}{\partial r \partial t} f\left(t\frac{a+b}{2} + (1-t)a, r\frac{c+d}{2} + (1-r)d\right)\right|^q dtdr\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)b, r\frac{c+d}{2}+(1-r)d\right)\right|^q dtdr\right)^{\frac{1}{q}}\Bigg] \quad (2.15)$$

By the co-ordinated (h_1, h_2) −convexity and $\left|\frac{\partial^2}{\partial r\partial t} f(t,r)\right| \leq M$, for all $(t,r) \in \Delta$, we have:

$$\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)c\right)\right|^q dtdr$$

$$\leq M^q \left[\int_0^1\int_0^1 t^\alpha r^\beta h_1(t)h_2(r)dtdr + \int_0^1\int_0^1 t^\alpha r^\beta h_1(t)h_2(1-r)dtdr\right.$$

$$\left. + \int_0^1\int_0^1 t^\alpha r^\beta h_1(1-t)h_2(r)dtdr + \int_0^1\int_0^1 t^\alpha r^\beta h_1(1-t)h_2(1-r)dtdr\right]$$

$$= M^q \cdot K$$

In a similarly way, we also have the following inequalities

$$\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)b, r\frac{c+d}{2}+(1-r)c\right)\right|^q dtdr \leq M^q \cdot K$$

$$\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)a, r\frac{c+d}{2}+(1-r)d\right)\right|^q dtdr \leq M^q \cdot K$$

and

$$\int_0^1\int_0^1 t^\alpha r^\beta \left|\frac{\partial^2}{\partial r\partial t} f\left(t\frac{a+b}{2}+(1-t)b, r\frac{c+d}{2}+(1-r)d\right)\right|^q dtdr \leq M^q \cdot K.$$

Using the last four inequalities we obtain from (2.15) the inequality (2.14)

Corollary 6. If in Theorem 3 we take $h_1(t) = t$ and $h_2(r) = r$ then the inequality (2.14) reduces to the following inequality for the co-ordinated convex function:

$$\left|4(b-a)^\alpha(d-c)^\beta f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + A\right|$$

$$\leq M(b-a)^{\alpha+1}(d-c)^{\beta+1}\cdot\frac{1}{(\alpha\ +1)(\beta+1)}\left(\frac{2\alpha+2\beta+\alpha\beta+3}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}}.$$

Corollary 7. If in Theorem 3 we take $h_1(t)=t^s$ and $h_2(r)=r^s$ then the inequality (2.14) reduces to the following inequality for the co-ordinated *s*-convex function:

$$\left|4(b-a)^{\alpha}(d-c)^{\beta}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+A\right|$$

$$\leq M(b-a)^{\alpha+1}(d-c)^{\beta+1}\cdot\left(\frac{1}{(\alpha\ +1)(\beta+1)}\right)^{\frac{1}{q}}\cdot\left[\frac{1}{(s+\alpha\ +1)(s+\beta+1)}\right.$$

$$\left.+\frac{\Gamma(\beta+1)\Gamma(s+1)}{(\alpha+s+1)\Gamma(\beta+s+2)}+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{(\beta+s+1)\Gamma(\alpha+s+2)}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\Gamma(s+1))^2}{\Gamma(\alpha+s+2)\Gamma(\beta+s+2)}\right]^{\frac{1}{q}}.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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