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SOME GENERALIZATION OF POWER INEQUALITIES FOR THE NUMERICAL RADIUS OF SUM OF PRODUCTS OF TWO OPERATORS IN A HILBERT SPACE

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Abstract. In this paper, we generalize some power inequalities numerical radius for the finite number sum of product of two operators in a Hilbert space. Also we generalized some inequalities for the sum of two products using the generalized Lagrange's identity.

Keywords: numerical range, numerical radius.

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1. Introduction

Let H be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. The *numerical range* of T is the subset

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\} \tag{1}$$

of the complex plane, where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the standard inner product and norm on H.

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It is known that the numerical range is always convex (see [4]). The spectrum of an operator is contained in the closure of its numerical range.

On a complex Hilbert space the numerical radius $w(T)$ of an operator T is given by

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\}. \tag{2}$$

Hence, we have

$$w(\alpha T) = |\alpha| w(T) \tag{3}$$

for any real α. It is well known that $w(T)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$. The following inequality is well known

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\| \tag{4}$$

For other properties of numerical range and numerical radius, the reader may consult [4] and for other inequalities. See for example [3], [6] and [8].

2. Some Numerical Radius Inequalities

Theorem 2.1. *For $A_i, B_i \in B(H), i = 1,2,...,n$ and $r \geq 1$ we have*

$$w\left(\sum_{i=1}^{n} B_i^* A_i\right) \leq 2^{-\frac{1}{r}} \cdot \sum_{i=1}^{n} \|(A_i^* A_i)^r + (B_i^* B_i)^r\|^{\frac{1}{r}}. \tag{5}$$

Proof. Using the Schwarz inequality in the Hilbert space $(H; \langle .,. \rangle)$ we have

$$\begin{aligned}
|\langle (B_1^* A_1 + B_2^* A_2 + \cdots B_n^* A_n)x, x\rangle| &= |\langle (B_1^* A_1)x, x\rangle + \cdots + \langle (B_n^* A_n x, x)\rangle| \\
&= |\langle A_1 x, B_1 x\rangle + \langle A_2 x, B_2 x\rangle + \cdots \langle A_n x, B_n x\rangle| \\
&\leq |\langle A_1 x, B_1 x\rangle| + |\langle A_2 x, B_2 x\rangle| + \cdots |\langle A_n x, B_n x\rangle| \\
&\leq \|A_1 x\| \cdot \|B_1 x\| + \cdots + \|A_n x\| \cdot \|B_n x\| \\
&= \sum_{i=1}^{n} \langle (A_i^* A_i)x, x\rangle^{\frac{1}{2}} \cdot \langle (B_i^* B_i)x, x\rangle^{\frac{1}{2}}, \qquad (6)
\end{aligned}$$

for any $x \in H$.

Utilizing the arithmetic-geometric mean inequality and byu the power mean inequality for $r \geq 1$ we have,

$$\begin{aligned}\sum_{i=1}^{n} \langle (A_i^* A_i)x, x\rangle^{\frac{1}{2}} \cdot \langle (B_i^* B_i)x, x\rangle^{\frac{1}{2}} &\leq \sum_{i=1}^{n} \frac{\langle (A_i^* A_i)x, x\rangle + \langle (B_i^* B_i)x, x\rangle}{2} \\ &\leq \sum_{i=1}^{n} \left(\frac{\langle A_i^* A_i x, x\rangle^r + \langle B_i^* B_i x, x\rangle^r}{2} \right)^{\frac{1}{r}} \quad (7)\end{aligned}$$

for any $x \in H$.

It is known that if P is a positive operator then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see [6])

$$\langle Px, x\rangle^r \leq \langle P^r x, x\rangle. \quad (8)$$

Applying this property to the positive operators $A_i^* A$ and $B_i^* B_i$, for $i = 1, \ldots n$, we deduce that

$$\begin{aligned}\sum_{i=1}^{n} \left(\frac{\langle A_i^* A_i x, x\rangle^r + \langle B_i^* B_i x, x\rangle^r}{2} \right)^{\frac{1}{r}} &\leq \sum_{i=1}^{n} \left(\frac{\langle (A_i^* A_i)^r x, x\rangle + \langle (B_i^* B_i)^r x, x\rangle}{2} \right)^{\frac{1}{r}} \\ &= \sum_{i=1}^{n} \left(\frac{\langle [(A_i^* A_i)^r + (B_i^* B_i)^r] x, x\rangle}{2} \right)^{\frac{1}{r}} \quad (9)\end{aligned}$$

for any $x \in H$. Now combining the inequalities (6), (7) and (9) then the result follows by taking supremum over all unit vectors in H.

This completes the proof.

Corollary 2.2. *For $A, B \in B(H)$ we have,*

$$w^r(B^* A) \leq \frac{1}{2} \|(A^* A)^r + (B^* B)^r\| \quad (10)$$

for any $r \geq 1$.

Proof.

In particular $n = 1$, and if we put $A_1 = A$, $B_1 = B$ in (5) then we get (10).

Corollary 2.3. *For $A_i \in B(H), i = 1, \ldots, n$ we have,*

$$w(\sum_{i=1}^{n} A_i) \leq 2^{-\frac{1}{r}} \sum_{i=1}^{n} \|(A_i^* A_i)^r + I\|^{\frac{1}{r}} \quad (11)$$

for any $r \geq 1$.

Proof.

If we put $B_i = I$ for all $i = 1,2,\dots,n$ in (5) then we get (11).

Corollary 2.4. *For $A_i \in B(H), i = 1,\dots,n$ we have,*

$$w(\sum_{i=1}^{n} A_i^2) \leq 2^{-\frac{1}{r}} \sum_{i=1}^{n} \|(A_i^* A_i)^r + (A_i A_i^*)^r\|^{\frac{1}{r}} \tag{12}$$

for any $r \geq 1$.

Proof. If we put $B_i^* = A_i$ for all $i = 1,2,\dots,n$ in (5) then we get (12).

Corollary 2.5. *For $A \in B(H)$ we have,*

$$w^r(A) \leq \frac{1}{2}\|(A^*A)^r + I\| \tag{13}$$

for any $r \geq 1$.

Proof. If we put $B_i = I$ and $A_i = A$ for all $i = 1,2,\dots,n$ in (5) then we get (13).

Theorem 2.6. *For $A_i, B_i \in B(H), i = 1,\dots,n$ we have,*

$$w\left(\sum_{i=1}^{n} B_i^* A_i\right) \leq \frac{n^{1-\frac{1}{r}}}{2}\left[\left(\sum_{i=1}^{n}\|A_i^*A_i\|^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^{n}\|B_i^*B_i\|^r\right)^{\frac{1}{r}}\right], \tag{14}$$

for $r \geq 1$.

Proof. Using the Schwarz's inequality in the Hilbert space $(H;\langle .,.\rangle)$ we have

$$\begin{aligned}
|\langle (B_1^*A_1 + B_2^*A_2 + \cdots B_n^*A_n)x, x\rangle| &= |\langle (B_1^*A_1)x, x\rangle + \cdots + \langle (B_n^*A_n x, x)\rangle| \\
&= |\langle A_1x, B_1x\rangle + \langle A_2x, B_2x\rangle + \cdots \langle A_nx, B_nx\rangle| \\
&\leq |\langle A_1x, B_1x\rangle| + |\langle A_2x, B_2x\rangle| + \cdots |\langle A_nx, B_nx\rangle| \\
&\leq \|A_1x\|\cdot\|B_1x\| + \cdots + \|A_nx\|\cdot\|B_nx\| \\
&= \sum_{i=1}^{n} \langle (A_i^*A_i)x, x\rangle^{\frac{1}{2}} \cdot \langle (B_i^*B_i)x, x\rangle^{\frac{1}{2}}, \qquad (15)
\end{aligned}$$

for any $x \in H$.

Utilizing the arithmetic-geometric mean inequality and power mean inequality $r \geq 1$ we have successively,

$$\begin{aligned}
\sum_{i=1}^{n} \langle (A_i^*A_i)x,x\rangle^{\frac{1}{2}} \cdot \langle (B_i^*B_i)x,x\rangle^{\frac{1}{2}} &\leq \sum_{i=1}^{n} \frac{\langle (A_i^*A_i)x,x\rangle + \langle (B_i^*B_i)x,x\rangle}{2} \\
&= \frac{1}{2}\left[\sum_{i=1}^{n} \langle (A_i^*A_i)x,x\rangle + \sum_{i=1}^{n} \langle (B_i^*B_i)x,x\rangle\right] \\
&\leq \frac{1}{2}\left[n \cdot \left(\frac{\sum_{i=1}^{n} \langle (A_i^*A_i)x,x\rangle^r}{n}\right)^{\frac{1}{r}} + n \cdot \left(\frac{\sum_{i=1}^{n} \langle (B_i^*B_i)x,x\rangle^r}{n}\right)^{\frac{1}{r}}\right] \\
&= \frac{n^{1-\frac{1}{r}}}{2}\left[\left(\sum_{i=1}^{n} \langle (A_i^*A_i)x,x\rangle^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^{n} \langle (B_i^*B_i)x,x\rangle^r\right)^{\frac{1}{r}}\right]. \qquad (16)
\end{aligned}$$

It is known that if P is a positive operator, then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for example [6])

$$\langle Px,x\rangle^r \leq \langle P^r x,x\rangle. \qquad (17)$$

Applying this property to the positive operators A_i^*A and $B_i^*B_i$, for $i = 1, \ldots n$, we deduce that

$$\begin{aligned}
&\frac{n^{1-\frac{1}{r}}}{2}\left[\left(\sum_{i=1}^{n} \langle (A_i^*A_i)x,x\rangle^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^{n} \langle (B_i^*B_i)x,x\rangle^r\right)^{\frac{1}{r}}\right] \\
\leq\ &\frac{n^{1-\frac{1}{r}}}{2}\left[\left(\sum_{i=1}^{n} \langle (A_i^*A_i)^r x,x\rangle\right)^{\frac{1}{r}} + \left(\sum_{i=1}^{n} \langle (B_i^*B_i)^r x,x\rangle\right)^{\frac{1}{r}}\right] \qquad (18)
\end{aligned}$$

for any $x \in H$.

Now combining the inequalities (15), (16) and (18) then the result follows by taking supremum over all unit vectors in H.

Theorem 2.7. *For $A_i, B_i \in B(H), i = 1,2,...,n$ and $r \geq 1, \alpha \in (0,1)$ we have,*

$$w^2\left(\sum_{i=1}^{n} B_i^*A_i\right) \leq n \cdot \sum_{i=1}^{n} \|\alpha \cdot (A_i^*A_i)^{\frac{r}{\alpha}} + (1-\alpha)(B_i^*B_i)^{\frac{r}{1-\alpha}}\|^{1/r} \qquad (19)$$

Proof. By the Schwarz's inequality and arithmetic-geometric mean inequality,we have

$$
\begin{aligned}
|\langle (B_1^*A_1+B_2^*A_2+\cdots B_n^*A_n)x,x\rangle|^2 &= |\langle (B_1^*A_1)x,x\rangle+\cdots+\langle B_n^*A_n\rangle|^2 \\
&\leq \left(\sum_{i=1}^{n}|\langle (B_i^*A_i)x,x\rangle|\right)^2 \\
&\leq n\cdot\sum_{i=1}^{n}|\langle (B_i^*A_i)x,x\rangle|^2 \\
&\leq n\cdot\sum_{i=1}^{n}\langle (A_i^*A_i)x,x\rangle\cdot\langle (B_i^*B_i)x,x\rangle
\end{aligned}
\tag{20}
$$

for any $x\in H$. It is well known that (see for example [6]) if P is positive operator and $q\in(0,1]$, then for any $u\in H,\|u\|=1$, we have

$$
\langle P^q u,u\rangle \leq \langle Pu,u\rangle^q \tag{21}
$$

Applying this property to the positive operators $(A_i^*A_i)^{\frac{1}{\alpha}}$ and $(B_i^*B_i)^{\frac{1}{1-\alpha}}$, for $i=1,2,\ldots,n,(\alpha\in(0,1])$, we have

$$
\begin{aligned}
&n\cdot\sum_{i=1}^{n}\langle [(A_i^*A_i)^{\frac{1}{\alpha}}]^{\alpha}x,x\rangle\cdot\langle [(B_i^*B_i)^{\frac{1}{1-\alpha}}]^{1-\alpha}x,x\rangle \\
&\qquad\leq n\cdot\sum_{i=1}^{n}\langle (A_i^*A_i)^{\frac{1}{\alpha}}x,x\rangle^{\alpha}\cdot\langle (B_i^*B_i)^{\frac{1}{1-\alpha}}x,x\rangle^{1-\alpha}
\end{aligned}
\tag{22}
$$

for any $x\in H,\|x\|=1$. Now, utilizing the weighted arithmetic mean-geometric mean inequality, i.e., $a^{\alpha}b^{1-\alpha}\leq\alpha a+(1-\alpha)b,\alpha\in(0,1),a,b\geq 0$, we get

$$
\begin{aligned}
&n\cdot\sum_{i=1}^{n}\langle (A_i^*A_i)^{\frac{1}{\alpha}}x,x\rangle^{\alpha}\cdot\langle (B_i^*B_i)^{\frac{1}{1-\alpha}}x,x\rangle^{1-\alpha} \\
&\qquad\leq n\cdot\sum_{i=1}^{n}[\alpha\langle (A_i^*A_i)^{\frac{1}{\alpha}}x,x\rangle+(1-\alpha)\langle (B_i^*B_i)^{\frac{1}{1-\alpha}}x,x\rangle]
\end{aligned}
\tag{23}
$$

for any $x\in H,\|x\|=1$. Moreover, by the elementary inequality following from the convexity of the function $f(t)=t^r,r\geq,1$ namely

$$
\alpha a+(1-\alpha)b\leq(\alpha a^r+(1-\alpha)b^r)^{\frac{1}{r}},\alpha\in(0,1),a,b\geq 0,
$$

we deduce that

$$\begin{aligned} & n\cdot\sum_{i=1}^{n}[\alpha\langle(A_i^*A_i)^{\frac{1}{\alpha}}x,x\rangle+(1-\alpha)\langle(B_i^*B_i)^{\frac{1}{1-\alpha}}x,x\rangle] \\ \leq\ & n\cdot\sum_{i=1}^{n}[\alpha\langle(A_i^*A_i)^{\frac{1}{\alpha}}x,x\rangle^r+(1-\alpha)\langle(B_i^*B_i)^{\frac{1}{1-\alpha}}x,x\rangle^r]^{\frac{1}{r}} \\ \leq\ & n\cdot\sum_{i=1}^{n}[\alpha\langle(A_i^*A_i)^{\frac{r}{\alpha}}x,x\rangle+(1-\alpha)\langle(B_i^*B_i)^{\frac{r}{1-\alpha}}x,x\rangle]^{\frac{1}{r}} \\ =\ & n\cdot\sum_{i=1}^{n}[\langle(\alpha(A_i^*A_i)^{\frac{r}{\alpha}}+(1-\alpha)(B_i^*B_i)^{\frac{r}{1-\alpha}})x,x\rangle]^{\frac{1}{r}} \end{aligned} \tag{24}$$

for any $x\in H,\|x\|=1$, where, for last inequality we used the inequality (8) for the positive operators $(A_i^*A_i)^{\frac{1}{\alpha}}$ and $(B_i^*B_i)^{\frac{1}{1-\alpha}}, i=1,2,\ldots,n$.

Now combining the inequalities (20), (22),(23) and (24) then result follows by taking supremum over all unit vectors in H.

Corollary 2.8. *For $A,B\in B(H)$ we have,*

$$w^{2r}(B^*A)\leq\|\alpha(A^*A)^{\frac{r}{\alpha}}+(1-\alpha)(B^*B)^{\frac{r}{1-\alpha}}\| \tag{25}$$

for any $r\geq 1$ and $\alpha\in(0,1)$.

Proof. If we put $B_i=I$ and $A_i=A$ for all $i=1,2,\ldots,n$ in (5) then we get (25).

Corollary 2.9. *If we put in (19) $B_i=I$ for all $i=1,2,\ldots,n$ then we have,*

$$w^2(\sum_{i=1}^{n}A_i)\leq n\cdot\sum_{i=1}^{n}\|\alpha(A_i^*A_i)^{\frac{r}{\alpha}}+(1-\alpha)I\| \tag{26}$$

for any $r\geq 1$ and $\alpha\in(0,1)$. In particular for $\alpha=\frac{1}{2}$ we have,

$$w^2(\sum_{i=1}^{n}A_i)\leq\frac{n}{2}\cdot\sum_{i=1}^{n}\|(A_i^*A_i)^{2r}+I\| \tag{27}$$

3. Inequalities for the sum of two products

Theorem 3.1. *For $A,B\in B(H)$ and $r\geq 1$ we have*

$$\|B^*A\|^r\leq\frac{1}{2}\left(\|(A^*A)^r\|+\|(B^*B)^r\|\right). \tag{28}$$

Proof. Using the Schwarz inequality in the Hilbert space $(H;\langle .,.\rangle)$ we have

$$\begin{aligned} |\langle (B^*A)x,y\rangle| &= |\langle Ax,By\rangle| \leq \|Ax\|\cdot\|By\| \\ &= \langle (A^*A)x,x\rangle^{\frac{1}{2}}\cdot\langle (B^*B)y,y\rangle^{\frac{1}{2}}, \end{aligned} \tag{29}$$

for any $x,y\in H$.

Using the arithmetic-geometric mean inequality and power means inequality $r\geq 1$ we have,

$$\begin{aligned} \langle (A^*A)x,x\rangle^{\frac{1}{2}}\cdot\langle (B^*B)y,y\rangle^{\frac{1}{2}} &\leq \frac{\langle (A^*A)x,x\rangle+\langle (B^*B)y,y\rangle}{2} \\ &\leq \left(\frac{\langle A^*Ax,x\rangle^r+\langle B^*By,y\rangle^r}{2}\right)^{\frac{1}{r}} \end{aligned} \tag{30}$$

for any $x\in H$.

It is known that if P is a positive operator then for any $r\geq 1$ and $x\in H$ with $\|x\|=1$ we have the inequality (see [6])

$$\langle Px,x\rangle^r \leq \langle P^r x,x\rangle. \tag{31}$$

Applying this property to the positive operators A^*A and B^*B, for $i=1,\dots n$, we deduce that

$$\left(\frac{\langle A^*Ax,x\rangle^r+\langle B^*By,y\rangle^r}{2}\right)^{\frac{1}{r}} \leq \left(\frac{\langle (A^*A)^r x,x\rangle+\langle (B^*B)^r y,y\rangle}{2}\right)^{\frac{1}{r}} \tag{32}$$

for any $x,y\in H$. Now combining the inequalities (30) and (32) then the result follows by taking supremum over all unit vectors in H.

The Theorem 3.1 may be generalized as follows, since the proof is similar , we will omit the proof.

Theorem 3.2. *For $A_i,B_i\in B(H), i=1,2,...,n$ and $r\geq 1$ we have*

$$\|\sum_{i=1}^{n} B_i^*A_i\| \leq 2^{-\frac{1}{r}}\cdot\left(\sum_{i=1}^{n}\|(A_i^*A_i)^r\|+\sum_{i=1}^{n}\|(B_i^*B_i)^r\|\right)^{\frac{1}{r}}. \tag{33}$$

Theorem 3.3. *For $A,B\in B(H)$ and $r\geq 1, \alpha\in(0,1)$ we have,*

$$\|B^*A\|^{2r} \leq \alpha\cdot\|(A^*A)^{\frac{r}{\alpha}}\|+(1-\alpha)\|(B^*B)^{\frac{r}{1-\alpha}}\| \tag{34}$$

Proof. By the Schwarz's inequality and arithmetic-geometric mean inequality,we have

$$
\begin{aligned}
|\langle (B^*Ax, y\rangle|^2 &= |\langle (Ax, By\rangle|^2 \leq \|Ax\|^2\|By\|^2 \\
&= \langle (A^*A)x, x\rangle \cdot \langle (B^*B)y, y\rangle \qquad (35)
\end{aligned}
$$

for any $x \in H$. It is well known that (see for example [6]) if P is positive operator and $q \in (0,1]$, then for any $u \in H, \|u\| = 1$, we have

$$
\langle P^q u, u\rangle \leq \langle Pu, u\rangle^q \qquad (36)
$$

Applying this property to the positive operators $(A_i^*A_i)^{\frac{1}{\alpha}}$ and $(B_i^*B_i)^{\frac{1}{1-\alpha}}$, for $i = 1, 2, \ldots, n, (\alpha \in (0,1])$, we have

$$
\langle [(A^*A)^{\frac{1}{\alpha}}]^{\alpha} x, x\rangle \cdot \langle [(B^*B)^{\frac{1}{1-\alpha}}]^{1-\alpha} y, y\rangle \leq \langle (A^*A)^{\frac{1}{\alpha}} x, x\rangle^{\alpha} \cdot \langle (B^*B)^{\frac{1}{1-\alpha}} y, y\rangle^{1-\alpha}
$$

for any $x, y \in H, \|x\| = \|y\| = 1$. Now, utilizing the weighted arithmetic-geometric mean inequality, i.e., $a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b, \alpha \in (0,1), a, b \geq 0$, we get

$$
\langle (A^*A)^{\frac{1}{\alpha}} x, x\rangle^{\alpha} \cdot \langle (B^*B_i)^{\frac{1}{1-\alpha}} y, y\rangle^{1-\alpha} \leq \alpha \langle (A^*A)^{\frac{1}{\alpha}} x, x\rangle + (1-\alpha)\langle (B^*B)^{\frac{1}{1-\alpha}} y, y\rangle
$$

for any $x, y \in H, \|x\| = \|y\| = 1$.

Moreover, by the elementary inequality following from the convexity of the function $f(t) = t^r, r \geq 1$, namely

$$
\alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}, \alpha \in (0,1), a, b \geq 0,
$$

we deduce that

$$
\begin{aligned}
\alpha \langle (A^*A)^{\frac{1}{\alpha}} x, x\rangle + (1-\alpha)\langle (B^*B)^{\frac{1}{1-\alpha}} y, y\rangle &\leq \left[\alpha \langle (A^*A)^{\frac{1}{\alpha}} x, x\rangle^r + (1-\alpha)\langle (B^*B)^{\frac{1}{1-\alpha}} y, y\rangle^r\right]^{\frac{1}{r}} \\
&\leq [\alpha \langle (A^*A)^{\frac{r}{\alpha}} x, x\rangle + (1-\alpha)\langle (B^*B)^{\frac{r}{1-\alpha}} y, y\rangle]^{\frac{1}{r}} \qquad (37)
\end{aligned}
$$

for any $x, y \in H, \|x\| = \|y\| = 1$, where, for last inequality we used the inequality (36) for the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$.

Now combining the inequalities (35) and (37) then result follows by taking supremum over all unit vectors in H.

The Theorem 3.3 may be generalized as follows, since the proof is similar , so we will omit the proof.

Theorem 3.4. *For* $A_i, B_i \in B(H), i = 1,2,...,n$ *and* $r \geq 1, \alpha \in (0,1)$ *we have,*

$$\|\sum_{i=1}^{n} B_i^* A_i\|^2 \leq n \cdot \sum_{i=1}^{n} \left(\alpha \cdot \|(A_i^* A_i)^{\frac{r}{\alpha}}\| + (1-\alpha)\|(B_i^* B_i)^{\frac{r}{1-\alpha}}\|\right)^{1/r} \tag{38}$$

Theorem 3.5. *For* $A,B,C,D \in B(H)$ *,* $k \in \mathbb{R}$ *and* $r,s \geq 1$ *we have,*

$$\begin{aligned} \left\|\frac{B^*A + kD^*C}{2}\right\|^2 &\leq \left\|\frac{(A^*A)^r + |k|^r (C^*C)^r}{2}\right\|^{\frac{1}{r}} \left\|\frac{(B^*B)^s + |k|^s (D^*D)^s}{2}\right\|^{\frac{1}{s}} \\ &- |k| \left(\sqrt{\|A^*A\| \, \|D^*D\|} - \sqrt{\|B^*B\| \, \|C^*C\|}\right)^2 \end{aligned} \tag{39}$$

Proof.

By the Schwarz's inequality in the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ we have:

$$\begin{aligned} |\langle (B^*A + kD^*C)x, y \rangle|^2 &= |\langle B^*Ax, y \rangle + k \langle D^*Cx, y \rangle|^2 \\ &\leq [|\langle B^*Ax, y \rangle + |k| \langle D^*Cx, y \rangle|]^2 \\ &\leq \left[\langle A^*Ax, x \rangle^{\frac{1}{2}} \langle B^*By, y \rangle^{\frac{1}{2}} + |k| \langle C^*Cx, x \rangle^{\frac{1}{2}} \langle D^*Dy, y \rangle^{\frac{1}{2}}\right]^2, \end{aligned} \tag{40}$$

for any $x, y \in H$.

Now, using the Lagrange's identity

$$(ab + kcd)^2 = (a^2 + kc^2)(b^2 + kd^2) - k(ad - bc)^2, \; a,b,c,d,k \in \mathbb{R}$$

we then conclude that:

$$\begin{aligned} &\left[\langle A^*Ax, x \rangle^{\frac{1}{2}} \langle B^*By, y \rangle^{\frac{1}{2}} + |k| \langle C^*Cx, x \rangle^{\frac{1}{2}} \langle D^*Dy, y \rangle^{\frac{1}{2}}\right]^2 \\ = \; &(\langle A^*Ax, x \rangle + |k| \langle C^*Cx, x \rangle)(\langle B^*By, y \rangle + |k| \langle D^*Dy, y \rangle) \\ - \; &|k| \left(\langle A^*Ax, x \rangle^{\frac{1}{2}} \langle D^*Dy, y \rangle^{\frac{1}{2}} - \langle B^*By, y \rangle^{\frac{1}{2}} \langle C^*Cx, x \rangle^{\frac{1}{2}}\right)^2 \end{aligned} \tag{41}$$

for any $x, y \in H$.

Now, on making use of a similar argument to the one in the proof of Theorem 2.1, we have for

$r,s \geq 1$ that

$$\begin{aligned}
&(\langle A^*Ax,x\rangle + |k|\langle C^*Cx,x\rangle)\,(\langle B^*By,y\rangle + |k|\langle D^*Dy,y\rangle) \\
(42)\qquad &\leq\ 4\left\langle \left[\frac{(A^*A)^r + |k|^r(C^*C)^r}{2}\right]x,x\right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^*B)^s + |k|^s(D^*D)^s}{2}\right]x,x\right\rangle^{\frac{1}{s}}
\end{aligned}$$

for any $x,y \in H$, $\|x\| = \|y\| = 1$.

Consequently, by (40), (41) and (42) we have

$$\begin{aligned}
|\langle (B^*A + kD^*C)x,y\rangle|^2 \ &\leq\ 4\left(\frac{\langle [(A^*A)^r + |k|^r(C^*C)^r]x,x\rangle}{2}\right)^{\frac{1}{r}}\left(\frac{\langle [(B^*B)^s + |k|^s(D^*D)^s]y,y\rangle}{2}\right)^{\frac{1}{s}} \\
&-\ |k|\left(\sqrt{\langle A^*Ax,x\rangle\langle D^*Dy,y\rangle} - \sqrt{\langle B^*By,y\rangle\langle C^*Cx,x\rangle}\right)^2
\end{aligned}$$

for any $x,y \in H$, $\|x\| = \|y\| = 1$.

Taking the supremum over $x,y \in H$, $\|x\| = \|y\| = 1$ we deduce the desired inequality (39).

Corollary 3.6. *For $A,B,C,D \in B(H)$ we have,*

$$\begin{aligned}
\left\|\frac{B^*A + kD^*C}{2}\right\|^2 \ &\leq\ \left\|\frac{(A^*A)^r + |k|^r(C^*C)^r}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(B^*B)^r + |k|^r(D^*D)^r}{2}\right\|^{\frac{1}{r}} \\
(43)\qquad &-\ |k|\left(\sqrt{\|A^*A\|\,\|D^*D\|} - \sqrt{\|B^*B\|\,\|C^*C\|}\right)^2
\end{aligned}$$

for any $r \geq 1$.

Proof. If we choose in (39) such that $r = s$ then we get (43).

Corollary 3.7. *For $A,B,C,D \in B(H)$ we have,*

$$\begin{aligned}
\left\|\frac{B^*A + D^*C}{2}\right\|^2 \ &\leq\ \left\|\frac{(A^*A)^r + (C^*C)^r}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(B^*B)^s + (D^*D)^s}{2}\right\|^{\frac{1}{s}} \\
(44)\qquad &-\ \left(\sqrt{\langle A^*Ax,x\rangle\langle D^*Dy,y\rangle} - \sqrt{\langle B^*By,y\rangle\langle C^*Cx,x\rangle}\right)^2.
\end{aligned}$$

for any $r,s \geq 1$.

Proof. Putting $k = 1$ in the proof Theorem 3.5 , then we get (44).

Theorem 3.8. *For $A_1,\ldots,A_n \in B(H)$;$B_1,\ldots,B_n \in B(H)$ and $r,s \geq 1$ we have,*

$$\left\|\frac{B_1^*A_1+B_2^*A_2\cdots++B_n^*A_n}{n}\right\|^2$$

$$\leq \left\|\frac{(A_1^*A_1)^r+\cdots+(A_n^*A_n)^r}{n}\right\|^{\frac{1}{r}}\cdot\left\|\frac{(B_1^*B_1)^s+\cdots+(B_n^*B_n)^s}{n}\right\|^{\frac{1}{s}} \tag{45}$$

Proof. By the Schwartz inequality in the Hilbert space $(H;\langle\cdot,\cdot\rangle)$ we have,

$$\begin{aligned} & |\langle(B_1^*A_1+B_2^*A_2\cdots++B_n^*A_n)x,y\rangle|^2 \\ = & |\langle(B_1^*A_1)x,y\rangle+\langle(B_2^*A_2)x,y\rangle+\cdots+\langle(B_n^*A_n)x,y\rangle|^2 \\ \leq & [|\langle(B_1^*A_1)x,y\rangle|+|\langle(B_2^*A_2)x,y\rangle|+\cdots+|\langle(B_n^*A_n)x,y\rangle|]^2 \\ \leq & \left[\langle A_1^*A_1x,x\rangle^{\frac{1}{2}}\langle B_1^*B_1y,y\rangle^{\frac{1}{2}}+\cdots+\langle A_n^*A_nx,x\rangle^{\frac{1}{2}}\langle B_n^*B_ny,y\rangle^{\frac{1}{2}}\right]^2 \end{aligned} \tag{46}$$

for any $x,y\in H$.

Now we are goning to use the following classical Cauchy-Bunyakowski- Schwartz inequality:

$$(a_1b_1+a_2b_2+\cdots+a_nb_n)^2\leq(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) \tag{47}$$

for $a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n\in\mathbb{R}$.

Then we have,

$$\begin{aligned} & \left[\langle A_1^*A_1x,x\rangle^{\frac{1}{2}}\langle B_1^*B_1y,y\rangle^{\frac{1}{2}}+\cdots+\langle A_n^*A_nx,x\rangle^{\frac{1}{2}}\langle B_n^*B_ny,y\rangle^{\frac{1}{2}}\right]^2 \\ & \leq\left(\langle A_1^*A_1x,x\rangle+\cdots+\langle A_n^*A_nx,x\rangle\right)\left(\langle B_1^*B_1y,y\rangle+\cdots+\langle B_n^*B_ny,y\rangle\right), \end{aligned} \tag{48}$$

for any $x,y\in H$. Since the function $f(t)=t^r$ is convex for $r\geq 1$, we have:

$$\begin{aligned} & \left(\langle A_1^*A_1x,x\rangle+\cdots+\langle A_n^*A_nx,x\rangle\right)\left(\langle B_1^*B_1y,y\rangle+\cdots+\langle B_n^*B_ny,y\rangle\right) \\ \leq\ & n^2\left(\left[\frac{(A_1^*A_1)^r+\cdots+(A_n^*A_n)^r}{n}\right]x,x\right)^{\frac{1}{r}}\left(\left[\frac{(B_1^*B_1)^s+\cdots+(B_n^*B_n)^s}{n}\right]y,y\right)^{\frac{1}{s}} \end{aligned} \tag{49}$$

for any $x,y\in H,\|x\|=\|y\|=1$.

Consequently, by (46), (49)we have,

$$
\begin{aligned}
&|\langle\left[\frac{B_1^*A_1+B_2^*A_2\cdots++B_n^*A_n}{n}\right]x,y\rangle|^2\\
\leq\ &\left(\left[\frac{(A_1^*A_1)^r+\cdots+(A_n^*A_n)^r}{n}\right]x,x\right)^{\frac{1}{r}}\left(\left[\frac{(B_1^*B_1)^s+\cdots+(B_n^*B_n)^s}{n}\right]y,y\right)^{\frac{1}{s}}
\end{aligned}
$$

for any $x,y\in H,\|x\|=\|y\|=1$. Taking supremum over $x,y\in H,\|x\|=\|y\|=1$ we deduce the desired inequality (45).

Corollary 3.9. *For $A_1,\ldots,A_n\in B(H);B_1,\ldots,B_n\in B(H)$ and $r,s\geq 1$ we have,*

$$
\left\|\frac{B_1^*A_1+\cdots+B_n^*A_n}{n}\right\|^2\leq\left\|\frac{(A_1^*A_1)^r+\cdots+(A_n^*A_n)^r}{n}\right\|^{\frac{1}{r}}\cdot\left\|\frac{(B_1^*B_1)^s+\cdots+(B_n^*B_n)^s}{n}\right\|^{\frac{1}{s}}. \tag{50}
$$

Proof. From the proof of the Theorem 3.8 we get the inequality (50).

Corollary 3.10. *For $A_1,\ldots,A_n\in B(H);B_1,\ldots,B_n\in B(H)$ and $r,s\geq 1$ we have,*

$$
\left\|\frac{A_1+A_2+\cdots+A_n}{n}\right\|^{2r}\leq\left\|\frac{(A_1^*A_1)^r+\cdots+(A_n^*A_n)^r}{n}\right\|. \tag{51}
$$

Proof. If we choose in the Theorem 3.8 such that $B_i=I,i=1,\ldots,n$, where I is the identity operator then we get the inequality (51).

Conflict of Interests

The authors declare that there is no conflict of interests.

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