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NOTE ON STIRLING'S FORMULA AND WALLIS' INEQUALITY OF ORDER m

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Abstract. In this paper, we obtain Stirling's formula and improvement of it. Also we get the Wallis' inequality and Wallis' formula of order m.

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1. Introduction

The Gamma function $\Gamma : (0,\infty) \longrightarrow \mathbb{R}$ is defined by the relation

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt \quad (x > 0).$$

The Gamma function has the following well know properties [1]:

$$\Gamma(1) = 1. \tag{1.1}$$

$$\Gamma(x+1) = x\Gamma(x) \quad (\Gamma(n+1) = n!,\ n \in \mathbb{N}). \tag{1.2}$$

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(1.3) Γ is log-convex, that is $\ln\Gamma(x)$ is convex on $(0,\infty)$,

(1.4) $\Gamma(\frac{x}{2})\Gamma(\frac{x+1}{2}) = \frac{\sqrt{\pi}}{2^{x-1}}\Gamma(x)$(Gauss-Legendre duplication formula),

(1.5) $\prod_{k=0}^{m-1}\Gamma(x+\frac{k}{m}) = (2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mx},\Gamma(mx)$(Gauss multiplication formula).

Closely related to Gamma function is the Beta function B, which is the real function of two variables defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (x,y>0).$$

The following connection between the Beta and Gamma function hold [1]

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (x,y>0). \tag{1.6}$$

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a convex function, then the inequality

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.7}$$

is known as the Hermite-Hadamard inequality. I previously obtained a new refinement of Hermite-Hadamard inequality [6]

$$\begin{aligned} f(\frac{a+b}{2}) &\leq \frac{1}{n+1}\sum_{k=0}^{n} f(a+\frac{2k+1}{2n+2}(b-a)) \leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{n+1}\sum_{k=0}^{n} f(a+\frac{k}{n}(b-a)) \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.8}$$

Wallis' inequality is a well known and important inequality

$$\frac{1}{\sqrt{\pi n+\frac{\pi}{2}}} < \frac{1.3.5.\ldots.(2n-1)}{2.4.6.\ldots.(2n)} < \frac{1}{\sqrt{\pi n}} \tag{1.9}$$

or

$$\frac{1}{\sqrt{\pi n+\frac{\pi}{2}}} < \frac{(2n)!}{(n!)^2 2^{2n}} < \frac{1}{\sqrt{\pi n}} \tag{1.10}$$

and Wallis' formula is

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty}\frac{(2n)!\sqrt{2n+1}}{(n!)^2 2^{2n}}. \tag{1.11}$$

In this paper by the log-convexity of Γ and the refinement of Hermite-Hadmad inequality, the following inequalities will be obtained

$$\begin{aligned}
\sqrt{a}\Gamma(a+\frac{1}{2}) &\leq \sqrt{2\pi a}\ a^a e^{-a} \\
&\leq \sqrt{a}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{k}{m})}\right) \\
&\leq \Gamma(a+1) \leq \sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{2k+1}{m+1})} \\
&\leq \sqrt{2\pi}a^{\frac{a}{2}}(a+1)^{\frac{a+1}{2}}e^{-a-\frac{1}{2}} \\
&\leq \sqrt{\frac{a+1}{a}}\Gamma(a+1)
\end{aligned}$$

and we will deduce the Stirling's formula and improvement of it.

Moreover in this paper we will get the Wallis' inequality and Wallis' formula of order m

$$\frac{mn\prod_{k=0}^{m-1}\Gamma(n+\frac{k}{m})}{(n!)^{m-1}\sqrt{(2\pi)^{m-1}m}} < \frac{(mn)!}{(n!)^m m^{mn}} < \frac{\sqrt{m}}{\sqrt{(2\pi n)^{m-1}}},$$

$$\lim_{n\to\infty}\frac{(n!)^{2m}m^{2mn}}{((mn)!)^2(mn+1)^{m-1}} = \frac{(2\pi)^{m-1}}{m^m}$$

and we show that the Wallis' inequality and Waills' formula of order 2 are classic Wallis' inequality and formula.

Finally we confirm these results by the infinite product representation of the sine [1]

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right)$$

for $m = 2,3,4$.

2. **Main results**

First we bring two easy and well known Lemmas.

Lemma 2.1. *For all $a \in \mathbb{R}^+$ the following identities hold:*

(i) $\int_0^1 \ln\Gamma(x)dx = \ln\sqrt{2\pi}$,

(ii) $\int_0^{a+1} \ln\Gamma(x)dx = -a + a\ln a + \ln\sqrt{2\pi}$.

Proof. (i) By (1.4) we have

$$\ln\Gamma(x) = (x-1)\ln 2 - \ln\sqrt{\pi} + \ln\Gamma(\frac{x}{2}) + \ln\Gamma(\frac{x+1}{2}).$$

By integrating, we obtain

$$\int_0^1 \ln\Gamma(x)dx = \ln 2\int_0^1 (x-1)dx - \ln\sqrt{\pi} + \int_0^1 \ln\Gamma(\frac{x}{2})dx + \int_0^1 \ln\Gamma(\frac{x+1}{2})dx.$$

Since

$$\int_0^1 \ln\Gamma(\frac{x}{2})dx = 2\int_0^{\frac{1}{2}} \ln\Gamma(t)dt$$

and

$$\int_0^1 \ln\Gamma(\frac{x+1}{2})dx = 2\int_{\frac{1}{2}}^1 \ln\Gamma(t)dt,$$

we deduce that

$$\int_0^1 \ln\Gamma(x)dx = -\frac{1}{2}\ln 2 - \ln\sqrt{\pi} + 2\int_0^1 \ln\Gamma(x)dx.$$

Hence, we have

$$\int_0^1 \ln\Gamma(x)dx = \frac{1}{2}\ln 2 + \frac{1}{2}\ln\pi = \ln\sqrt{2\pi}.$$

(ii) We have

$$\left(\int_a^{a+1} \ln\Gamma(x)dx\right)' = (-a + a\ln a)' = \ln a.$$

Hence $\int_a^{a+1} \ln\Gamma(x)dx + a - a\ln a$ is constant. By part (i) we obtain

$$\int_a^{a+1} \ln\Gamma(x)dx + a - a\ln a = \ln\sqrt{2\pi}.$$

Lemma 2.2.

$$\lim_{a\to\infty} \frac{\sqrt{a}\,\Gamma(a+\frac{1}{2})}{\Gamma(a+1)} = 1 \quad (a > 0).$$

Proof. By identity (1.6) we have

$$\begin{aligned}\frac{\sqrt{a}\,\Gamma(a+\frac{1}{2})}{\Gamma(a+1)} &= \frac{\sqrt{a}B(a+\frac{1}{2},\frac{1}{2})}{\Gamma(\frac{1}{2})}\\ &= \frac{1}{\Gamma(\frac{1}{2})}\sqrt{a}\int_0^1 x^{\frac{1}{2}-1}(1-x)^{a+\frac{1}{2}-1}dx\\ &= \frac{1}{\Gamma(\frac{1}{2})}\sqrt{a}\int_0^1 x^{-\frac{1}{2}}(1-x)^{a-\frac{1}{2}}dx.\end{aligned}$$

Take $x=\frac{t}{a}$, so $dt=adx$. Since $\lim\limits_{a\to\infty}(1-\frac{t}{a})^{a-\frac{1}{2}}=e^{-t}$, we get

$$\begin{aligned}\lim_{a\to\infty}\frac{\sqrt{a}\,\Gamma(a+\frac{1}{2})}{\Gamma(a+1)} &= \frac{1}{\Gamma(\frac{1}{2})}\lim_{a\to\infty}\sqrt{a}\int_0^a \frac{t^{-\frac{1}{2}}}{a^{-\frac{1}{2}}}(1-\frac{t}{a})^{a-\frac{1}{2}}\frac{1}{a}dt\\ &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}dt = \frac{1}{\Gamma(\frac{1}{2})}\Gamma(\frac{1}{2})=1.\end{aligned}$$

In the following theorems, we obtain the Stirling's formula and improvment of it.

Theorem 2.3. *For all $m\in\mathbb{N}$ and $a>0$ the following inequalities hold:*

$$\begin{aligned}(i)\qquad \sqrt{a}\,\Gamma(a+\frac{1}{2}) &\le \sqrt{a}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{2k+1}{2m+2})}\right)\le\sqrt{2\pi a}\,a^a e^{-a}\\ &\le \sqrt{a}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{k}{m})}\right)\le\Gamma(a+1)\end{aligned}$$

and we have

$$\begin{aligned}(ii)\qquad \lim_{a\to\infty}\frac{\sqrt{a}\left(\sqrt[m+1]{\prod\limits_{k=0}^{m}\Gamma(a+\frac{2k+1}{2m+2})}\right)}{\Gamma(a+1)} &= \lim_{a\to\infty}\frac{\sqrt{a}\left(\sqrt[m+1]{\prod\limits_{k=0}^{m}\Gamma(a+\frac{k}{m})}\right)}{\Gamma(a+1)}\\ &= \lim_{a\to\infty}\frac{\sqrt{2\pi a}\,a^a e^{-a}}{\Gamma(a+1)}=1.\end{aligned}$$

Proof. Since $f(x)=\ln\Gamma(x)$ is convex on $(0,\infty)$, by the right side of inequalities (1.8), we have $\int_a^{a+1}\ln\Gamma(x)dx\le\frac{1}{m+1}\sum\limits_{k=0}^{m}\ln\Gamma(a+\frac{k}{m})\le\frac{\ln\Gamma(a)+\ln\Gamma(a+1)}{2}=\ln\sqrt{\Gamma(a)\Gamma(a+1)}$.

By Lemma 2.1, we get

$$-a+a\ln a+\ln\sqrt{2\pi}\le\frac{1}{m+1}\ln\prod_{k=0}^{m}\Gamma(a+\frac{k}{m})\le\ln\sqrt{a(\Gamma(a))^2}.$$

It follows that

$$\ln a^a\sqrt{2\pi}e^{-a} \leq \ln \sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{k}{m})} \leq \ln\sqrt{a}\,\Gamma(a).$$

Since e^x is increasing, we obtain

$$\sqrt{2\pi}a^a e^{-a} \leq \sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{k}{m})} \leq \sqrt{a}\,\Gamma(a).$$

By multiplying both side of inequalities $\sqrt{a}$, we get

$$(*)\qquad \sqrt{2\pi a}\,a^a e^{-a} \leq \sqrt{a}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{k}{m})}\right) \leq a\Gamma(a) = \Gamma(a+1).$$

On the other hand, by the left side of inequalities (1.8) we have

$$\ln(\frac{2a+1}{2}) \leq \frac{1}{m+1}\sum_{k=0}^{m}\ln\Gamma(a+\frac{2k+1}{2m+2}) \leq \int_a^{a+1}\ln\Gamma(x)dx$$

so

$$\ln\Gamma(a+\frac{1}{2}) \leq \ln \sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{2k+1}{2m+2})} \leq \int_a^{a+1}\ln\Gamma(x)dx = \ln\sqrt{2\pi}\,a^a e^{-a}.$$

Hence

$$\Gamma(a+\frac{1}{2}) \leq \sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{2k+1}{2m+2})} \leq \sqrt{2\pi}a^a e^{-a}.$$

Multiplying both side of inequalities by $\sqrt{a}$, we get

$$(**)\qquad \sqrt{a}\,\Gamma(a+\frac{1}{2}) \leq \sqrt{a}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{2k+1}{2m+2})}\right) \leq \sqrt{2\pi a}\,a^a e^{-a}$$

By (*) and (**) the proof of (i) is complete.

For the proof of (ii) divide all of inequalities (i) by $\Gamma(a+1)$

$$\frac{\sqrt{a}\,\Gamma(a+\frac{1}{2})}{\Gamma(a+1)} \leq \frac{\sqrt{a}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(n+\frac{2k+1}{2m+2})}\right)}{\Gamma(a+1)} \leq \frac{\sqrt{2\pi a}\,a^a e^{-a}}{\Gamma(a+1)}$$
$$\leq \frac{\sqrt{a}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{k}{m})}\right)}{\Gamma(a+1)} \leq 1.$$

Now the assertion is clear by Lemma 2.2.

Theorem 2.4. *For all $m \in \mathbb{N}$ and $a > 0$ the following inequalities hold:*

$$(i) \quad \Gamma(a+1) \leq \sqrt[m+1]{\prod_{k=0}^{m} \Gamma(a+\frac{2k+1}{m+1})} < \sqrt{2\pi} a^{\frac{a}{2}} (a+1)^{\frac{a+1}{2}} e^{-a-\frac{1}{2}}$$
$$\leq \sqrt[m+1]{\prod_{k=0}^{m} \Gamma(a+\frac{2k}{m})} < \sqrt{\frac{a+1}{a}} \Gamma(a+1).$$

$$(ii) \quad \lim_{a\to\infty} \frac{\sqrt[m+1]{\prod_{k=0}^{m} \Gamma(a+\frac{2k+1}{m+1})}}{\Gamma(a+1)} = \lim_{a\to\infty} \frac{\sqrt[m+1]{\prod_{k=0}^{m} \Gamma(a+\frac{2k}{m})}}{\Gamma(a+1)}$$
$$= \lim_{a\to\infty} \frac{\sqrt{2\pi}\, a^{\frac{a}{2}} (a+1)^{\frac{a+1}{2}} e^{-a-\frac{1}{2}}}{\Gamma(a+1)} = 1.$$

Proof. (i) By Lemma 2.1 we have

$$\frac{1}{2}\int_{a}^{a+2} \ln\Gamma(x)dx = \frac{1}{2}\left[\int_{a}^{a+1} \ln\Gamma(x)dx + \int_{a+1}^{a+2} \ln\Gamma(x)dx\right]$$
$$= \frac{1}{2}\left[\ln a^a \sqrt{2\pi} e^{-a} + \ln (a+1)^{a+1} \sqrt{2\pi} e^{-(a+1)}\right]$$
$$= \ln \sqrt{2\pi a^a (a+1)^{a+1} e^{-2a-1}}.$$

Now by the right side of inequalities (1.8), we get

$$\frac{1}{2}\int_{a}^{a+2} \ln\Gamma(x)dx = \ln \sqrt{2\pi a^a (a+1)^{a+1} e^{-2a-1}} \leq \ln \sqrt[m+1]{\prod_{k=0}^{m} \Gamma(a+\frac{2k}{m})}$$
$$\leq \ln \sqrt{\Gamma(a)\Gamma(a+2)}$$

so

$$\sqrt{2\pi a^a (a+1)^{a+1} e^{-2a-1}} \leq \sqrt[m+1]{\prod_{k=0}^{m} \Gamma(a+\frac{2k}{m})}$$
$$(*) \qquad \leq \sqrt{(a+1)\Gamma(a)\Gamma(a+1)} = \sqrt{\frac{a+1}{a}} \Gamma(a+1).$$

On the other hand, by the left side of inequalities (1.8), we have

$$\begin{aligned}\ln\Gamma(\frac{2a+2}{2}) &\leq \frac{1}{m+1}\sum_{k=0}^{m}\ln\Gamma(a+\frac{2k+1}{m+1})\\ &\leq \frac{1}{2}\int_{n}^{n+2}\ln\Gamma(x)dx\\ &= \ln\sqrt{2\pi a^{a}(a+1)^{a+1}e^{-2a-1}}\end{aligned}$$

so

$$(**)\qquad \Gamma(a+1) \leq \sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{2k+1}{m+1})} \leq \sqrt{2\pi a^{a}(a+1)^{a+1}e^{-2a-1}}.$$

By $(*)$ and $(**)$ the proof is complet. The proof of (ii) is clear by (i).

Corollary 2.5. *For all $m \in \mathbb{N}$ and $a > 0$ the following inequalities hold:*

$$\begin{aligned}\sqrt{a}\,\Gamma(a+\frac{1}{2}) \leq \sqrt{2\pi a}\,a^{a}e^{-a} \leq \sqrt{a}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{k}{m})}\right) &\leq \Gamma(a+1)\\ \leq \sqrt[m+1]{\prod_{k=0}^{m}\Gamma(a+\frac{2k+1}{m+1})} \leq \sqrt{2\pi}a^{\frac{a}{2}}(a+1)^{\frac{a+1}{2}}e^{-a-\frac{1}{2}}&\\ \leq \sqrt{\frac{a+1}{a}}\Gamma(a+1)&\end{aligned}$$

Remark 2.6. In the above corollary, let a be a natural number n. Then we have

$$\begin{aligned}\sqrt{n}\,\Gamma(n+\frac{1}{2}) \leq \sqrt{2\pi n}\,n^{n}e^{-n} \leq \sqrt{n}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(n+\frac{k}{m})}\right) &\leq n!\\ \leq \sqrt[m+1]{\prod_{k=0}^{m}\Gamma(n+\frac{2k+1}{m+1})} \leq \sqrt{2\pi}\,n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}e^{-n-\frac{1}{2}}&\\ \leq \sqrt{\frac{n+1}{n}}n!.&\end{aligned}$$

We can write $\sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})}(m=1,2,\dots)$ instead of $\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(n+\frac{2k+1}{m+1})}$ $(m=0,1,2,\dots)$. So

$$\begin{aligned}\sqrt{n}\,\Gamma(n+\frac{1}{2}) \le \sqrt{2\pi n}\,n^n e^{-n} \le \sqrt{n}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(n+\frac{k}{m})}\right) \le n!\\ \le \sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})} \le \sqrt{2\pi}n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}e^{-n-\frac{1}{2}}\\ \le \sqrt{\frac{n+1}{n}}n!.\end{aligned}$$

Now let $x_m = \sqrt{n}\left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(n+\frac{k}{m})}\right)$ and $y_m = \sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})}$. For $m=1,2,3$ we have

$$\begin{aligned}&x_1 = y_1 = n!\\ &x_2 = \sqrt{n}\sqrt[3]{\Gamma(n)\Gamma(n+\frac{1}{2})\Gamma(n+1)}, \quad y_2 = \sqrt{n+\frac{1}{2}}\,\Gamma(n+\frac{1}{2})\\ &x_3 = \sqrt{n}\sqrt[4]{\Gamma(n)\Gamma(n+\frac{1}{3})\Gamma(n+\frac{2}{3})\Gamma(n+1)},\\ &y_3 = \sqrt[3]{n+\frac{2}{3}}\sqrt[3]{\Gamma(n+\frac{1}{3})\Gamma(n+\frac{2}{3})\Gamma(n+1)}.\end{aligned}$$

Since $\{x_m\}$ is decreasing and $\{y_m\}$ is increasing (see [5], [6]), we obtain the following inequalities

$$\begin{aligned}\sqrt{n}\,\Gamma(n+\frac{1}{2}) < \sqrt{2\pi n}\,n^n e^{-n} < \cdots < \sqrt{n}\,\sqrt[4]{\Gamma(n)\Gamma(n+\frac{1}{3})\Gamma(n+\frac{2}{3})}\\ < \sqrt{n}\,\sqrt[3]{\Gamma(n)\Gamma(n+\frac{1}{2})\Gamma(n+1)} < n!\end{aligned}$$

and

$$\begin{aligned}n! < \sqrt{n+\frac{1}{2}}\,\Gamma(n+\frac{1}{2}) < \sqrt[3]{n+\frac{2}{3}}\,\sqrt[3]{\Gamma(n+\frac{1}{3})\Gamma(n+\frac{2}{3})\Gamma(n+1)}\\ < \cdots < \sqrt{2\pi}n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}e^{-n-\frac{1}{2}} < \sqrt{1+\frac{1}{n}}\,n!.\end{aligned}$$

3. Wallis' inequality and Wallis' formula

Theorem 3.1. *For all $m \in \mathbb{N}$ and $n \in \mathbb{N}$ the following inequalities hold:*

$$\frac{mn\prod_{k=0}^{m-1}\Gamma(n+\frac{k}{m})}{(n!)^{m-1}\sqrt{(2\pi)^{m-1}m}\left(\sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})}\right)} < \frac{(mn)!}{(n!)^m m^{mn}} < \frac{\sqrt{m}}{\sqrt{(2\pi n)^{m-1}}}.$$

Proof By formula (1.5) and remark 2.6, we have

$$\sqrt{n}\ \sqrt[m+1]{(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mn}\Gamma(mn)\Gamma(n+1)} = \sqrt{n}\ \left(\sqrt[m+1]{\prod_{k=0}^{m}\Gamma(n+\frac{k}{m})}\right) < n!.$$

So

$$n^{\frac{m+1}{2}}(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mn}\Gamma(mn)n! < (n!)^{m+1} \Longrightarrow$$

$$n^{\frac{m+1}{2}}(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mn}\Gamma(mn) < (n!)^{m} \Longrightarrow$$

$$\frac{\Gamma(mn)}{(n!)^m m^{mn}} < \frac{1}{n^{\frac{m+1}{2}}(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}}} \Longrightarrow$$

$$\frac{mn\Gamma(mn)}{(n!)^m m^{mn}} < \frac{mn}{n^{\frac{m+1}{2}}(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}}} = \frac{\sqrt{m}}{n^{\frac{m-1}{2}}(2\pi)^{\frac{m-1}{2}}} = \frac{\sqrt{m}}{\sqrt{(2\pi n)^{m-1}}}.$$

Hence

$$\frac{(mn)!}{(n!)^m m^{mn}} < \frac{\sqrt{m}}{\sqrt{(2\pi n)^{m-1}}}.$$

For the proof of left side again by formula (1.5) we have

$$\prod_{k=0}^{m-1}\Gamma(n+\frac{k}{m}) = (2\pi)^{\frac{m-1}{2}}\frac{\sqrt{m}\,\Gamma(mn)}{m^{mn}} = (2\pi)^{\frac{m-1}{2}}\frac{\sqrt{m}(mn)!}{m^{mn}(mn)}.$$

So

$$\frac{(mn)!}{m^{mn}} = \frac{mn}{(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}}}\prod_{k=0}^{m-1}\Gamma(n+\frac{k}{m}) = \frac{mn}{\sqrt{(2\pi)^{m-1}m}}\prod_{k=0}^{m-1}\Gamma(n+\frac{k}{m}).$$

Hence by remark 2.6 and above identity we obtain

$$\frac{mn\prod_{k=0}^{m-1}\Gamma(n+\frac{k}{m})}{(n!)^{m-1}\sqrt{(2\pi)^{m-1}m}\left(\sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})}\right)} = \frac{(mn)!}{(n!)^{m-1}m^{mn}\left(\sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})}\right)}$$

$$< \frac{(mn)!}{(n!)^{m-1}m^{mn}n!} = \frac{(mn)!}{(n!)^m m^{mn}}.$$

The proof is complete.

Definition 3.2. We call the inequality

$$(3\cdot 1)\qquad \frac{mn\prod_{k=0}^{m-1}\Gamma(n+\frac{k}{m})}{(n!)^{m-1}\sqrt{(2\pi)^{m-1}m}\left(\sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})}\right)} < \frac{(mn)!}{(n!)^m m^{mn}} < \frac{\sqrt{m}}{\sqrt{(2\pi n)^{m-1}}}$$

Wallis' inequality of order m.

For $m=2$ we have

$$\frac{2n\prod_{k=0}^{1}\Gamma(n+\frac{k}{2})}{(n!)\sqrt{2\pi\cdot 2}\left(\sqrt{\prod_{k=0}^{1}\Gamma(n+\frac{2k+1}{2})}\right)} < \frac{(2n)!}{(n!)^2 2^{2n}} < \frac{\sqrt{2}}{\sqrt{2\pi n}}.$$

Thus

$$\frac{1}{\sqrt{\pi n+\frac{\pi}{2}}} < \frac{(2n)!}{(n!)^2 2^{2n}} < \frac{1}{\sqrt{\pi n}},$$

which is the Wallis' inequality. In other words, the Wallis' inequality of order 2 is the classic Wallis inequality.

In the following theorem we extend Wallis' formula for order m.

Theorem 3.3. *For all $m\in\mathbb{N}$ we have*

$$\lim_{n\to\infty}\frac{(mn)!(mn+1)^{\frac{m-1}{2}}}{(n!)^m m^{mn}} = \frac{m^{\frac{m}{2}}}{(2\pi)^{\frac{m-1}{2}}}.$$

Proof. multiply both sides of Wallis' inequality of order m by $(mn-1)^{\frac{m-1}{2}}$,

$$\frac{mn(mn+1)^{\frac{m-1}{2}}\prod_{k=0}^{m-1}\Gamma(n+\frac{k}{m})}{(n!)^{m-1}\sqrt{(2\pi)^{m-1}m}\left(\sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})}\right)} < \frac{(mn)!(mn+1)^{\frac{m-1}{2}}}{(n!)^m m^{mn}}$$

$$< \frac{\sqrt{m}(mn+1)^{\frac{m-1}{2}}}{\sqrt{(2\pi n)^{m-1}}}.$$

It is obvious that

$$(*)\qquad \lim_{n\to\infty}\frac{\sqrt{m}(mn+1)^{\frac{m-1}{2}}}{\sqrt{(2\pi n)^{m-1}}} = \frac{\sqrt{m}m^{\frac{m-1}{2}}}{(2\pi)^{\frac{m-1}{2}}} = \frac{m^{\frac{m}{2}}}{(2\pi)^{\frac{m-1}{2}}}.$$

On the other hand, we have

$$\lim_{n\to\infty}\frac{mn(mn+1)^{\frac{m-1}{2}}\prod_{k=0}^{m-1}\Gamma(n+\frac{k}{m})}{(n!)^{m-1}\sqrt{(2\pi)^{m-1}.m}\left(\sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})}\right)}=\lim_{n\to\infty}\frac{n^{\frac{m+1}{2}}\prod_{k=0}^{m}\Gamma(n+\frac{k}{m})}{(n!)^{m+1}}$$

$$\lim_{n\to\infty}\frac{n!}{\sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})}}\lim_{n\to\infty}\frac{mn(mn+1)^{\frac{m-1}{2}}}{\sqrt{(2\pi)^{m-1}m}.n^{\frac{m+1}{2}}}.$$

The first and second limits are equal to 1 by theorems 2.3 and 2.4. For the third limit we have

$$\begin{aligned}\lim_{n\to\infty}\frac{mn(mn+1)^{\frac{m-1}{2}}}{(2\pi)^{\frac{m-1}{2}}.m^{\frac{1}{2}}.n^{\frac{m+1}{2}}}&=\lim_{n\to\infty}\frac{m^{\frac{1}{2}}(mn+1)^{\frac{m-1}{2}}}{(2\pi)^{\frac{m-1}{2}}.n^{\frac{m-1}{2}}}\\&=\frac{m^{\frac{1}{2}}.m^{\frac{m-1}{2}}}{(2\pi)^{\frac{m-1}{2}}}=\frac{m^{\frac{m}{2}}}{(2\pi)^{\frac{m-1}{2}}}.\end{aligned}$$

So

$$(**)\qquad \lim_{n\to\infty}\frac{mn(mn+1)^{\frac{m-1}{2}}\prod_{k=0}^{m-1}\Gamma(n+\frac{k}{m})}{(n!)^{m-1}\sqrt{(2\pi)^{m-1}m}\left(\sqrt[m]{\prod_{k=0}^{m-1}\Gamma(n+\frac{2k+1}{m})}\right)}=\frac{m^{\frac{m}{2}}}{(2\pi)^{\frac{m-1}{2}}}.$$

Now the assertion is obvious by (*) and (**).

Definition 3.4. We call the following formulas

$$(3\cdot 2)\qquad \lim_{n\to\infty}\frac{(n!)^m m^{mn}}{(mn)!(mn+1)^{\frac{m-1}{2}}}=\frac{(2\pi)^{\frac{m-1}{2}}}{m^{\frac{m}{2}}}$$

or

$$(3\cdot 3)\qquad \lim_{n\to\infty}\frac{(n!)^{2m}m^{2mn}}{((mn)!)^2(mn+1)^{m-1}}=\frac{(2\pi)^{m-1}}{m^m}$$

Wallis' formula of order m.

Remark 3.5. Taking $m=2$ in (3.3) we obtain

$$\lim_{n\to\infty}\frac{(n!)^4 2^{4n}}{((2n)!)^2(2n+1)}=\frac{2\pi}{2^2}=\frac{\pi}{2}.$$

It is the classic Wallis' formula. On the other hand, we know the Wallis' formula follows from the infinite product representation of the sine [1].

$$\sin x = x\prod_{n=1}^{\infty}(1-\frac{x^2}{(\pi n)^2}) \tag{3·4}$$

Taking $x=\frac{\pi}{2}$ in (3.4) gives

$$1=\sin\frac{\pi}{2}=\frac{\pi}{2}\prod_{n=}^{\infty}(1-\frac{\frac{\pi^2}{4}}{\pi^2n^2})=\frac{\pi}{2}\prod_{n=1}^{\infty}\frac{4n^2-1}{4n^2}$$

so

$$\frac{2}{\pi}=\prod_{n=1}^{\infty}\frac{4n^2-1}{4n^2} \quad \text{or} \quad \frac{\pi}{2}=\prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}.$$

Hence

$$\frac{\pi}{2}=\lim_{n\to\infty}\frac{(2\cdot4\cdot6\cdot\ldots2n)^2}{(1\cdot3)(3\cdot5)\ldots(2n-1)(2n+1)}=\lim_{n\to\infty}\frac{2^{4n}(n!)^2}{((2n)!)^2(2n+1)}$$

so (3.3) and (3.4) have the same results.

Now we verify (3.2) and (3.4) have the same results for $m=3$ and $x=\frac{\pi}{3}$, respectivly. Taking $m=3$ in (3.2) and $x=\frac{\pi}{3}$ in (3.4) we get

$$\lim_{n\to\infty}\frac{(n!)^3 3^{3n}}{(3n)!(3n+1)}=\frac{2\pi}{3\sqrt{3}}$$

and

$$\frac{\sqrt{3}}{2}=\sin\frac{\pi}{3}=\frac{\pi}{3}\prod_{n=0}^{\infty}(1-\frac{\frac{\pi^2}{9}}{\pi^2n^2})=\frac{\pi}{3}\prod_{n=1}^{\infty}\frac{9n^2-1}{9n^2}.$$

So

$$\frac{3\sqrt{3}}{2\pi}=\prod_{n=1}^{\infty}\frac{9n^2-1}{9n^2} \quad \text{or} \quad \frac{2\pi}{3\sqrt{3}}=\prod_{n=1}^{\infty}\frac{(3n)^2}{(3n-1)(3n+1)}.$$

Hence

$$\begin{aligned}\frac{2\pi}{3\sqrt{3}}&=\lim_{n\to\infty}\frac{(3\cdot6\cdot9\ldots3n)^2}{(2\cdot4)(5\cdot7)(8\cdot10)\ldots(3n-1)(3n+1)}\\&=\lim_{n\to\infty}\frac{3^{2n}(n!)^2(3\cdot6\cdot9\ldots3n)}{(3n)!(3n+1)}=\lim_{n\to\infty}\frac{3^{3n}(n!)^3}{(3n)!(3n+1)}.\end{aligned}$$

Finally we show that (3.3) and (3.4) have the same results for $m=4$ and $x=\frac{\pi}{4}$, respectively.

Taking $m=4$ in (3.3) and $x=\frac{\pi}{4}$ in (3.4) we obtain

$$\lim_{n\to\infty}\frac{(n!)^8 4^{8n}}{((4n)!)^2(4n+1)^3}=\frac{(2\pi)^3}{4^4}=\frac{\pi^3}{32}$$

and

$$\frac{\sqrt{2}}{2} = \sin\frac{\pi}{4} = \frac{\pi}{4}\prod_{n=1}^{\infty}\frac{16n^2-1}{16n^2}.$$

So

$$\frac{2\sqrt{2}}{\pi} = \prod_{n=1}^{\infty}\frac{16n^2-1}{16n^2} \quad \text{or} \quad \frac{\pi}{2\sqrt{2}} = \prod_{n=1}^{\infty}\frac{(4n)^2}{(4n-1)(4n+1)}.$$

Hence

$$\begin{aligned}\frac{\pi}{2\sqrt{2}} &= \lim_{n\to\infty}\frac{(4.8.12\ldots 4n)^2}{(3.5)(7.9)(11.13)\ldots(4n-1)(4n+1)}\\ &= \lim_{n\to\infty}\frac{(4^n.n!)^2(2.4.6\ldots 4n)}{(4n)!(4n+1)} = \lim_{n\to\infty}\frac{4^{2n}(n!)^2 2^{2n}(2n)!}{(4n)!(4n+1)}.\end{aligned}$$

So

$$\begin{aligned}\frac{\pi^2}{8} &= \lim_{n\to\infty}\frac{4^{4n}(n!)^4 2^{4n}((2n)!)^2}{((4n)!)^2(4n+1)^2}\\ &= \lim_{n\to\infty}\frac{(n!)^4 4^{4n}}{(4n)!(4n+1)^{\frac{3}{2}}}\lim_{n\to\infty}\frac{2^{4n}((2n)!)^2}{(4n)!\sqrt{4n+1}}\\ &= \lim_{n\to\infty}\frac{(n!)^4 4^{4n}}{(4n)!(4n+1)^{\frac{3}{2}}}\lim_{p\to\infty}\frac{2^{2p}(p!)^2}{(2p)!\sqrt{2p+1}}\\ &= \sqrt{\frac{\pi}{2}}\lim_{n\to\infty}\frac{(n!)^4 4^{4n}}{(4n)!(4n+1)^{\frac{3}{2}}}.\end{aligned}$$

Since the second limit is equal to $\sqrt{\frac{\pi}{2}}$ by Wallis' formula of order 2. Therefore, we have

$$\lim_{n\to\infty}\frac{(n!)^4 4^{4n}}{(4n)!(4n+1)^{\frac{3}{2}}} = \frac{\frac{\pi^2}{8}}{\frac{\sqrt{\pi}}{\sqrt{2}}} = \frac{\pi^{\frac{3}{2}}}{4\sqrt{2}}.$$

It follows that $\lim_{n\to\infty}\frac{(n!)^8 4^{8n}}{((4n)!)^2(4n+1)^3} = \frac{\pi^3}{32}$.

Conflict of Interests

The author declares that there is no conflict of interests.

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