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GRÜSS TYPE INEQUALITIES IN SEMI-INNER PRODUCT C^* -MODULES AND APPLICATIONS

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Abstract. In this paper, some Grüss type inequalities in semi-inner product modules over C^* -algebras for n -tuples of vectors are established. Applications for the approximation of the discrete Fourier and the Melin transforms of bounded linear operators on a Hilbert space are provided.

Keywords: Grüss inequality; Semi-inner product C^* -modules; C^* -algebra; Bounded linear operator.

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1. Introduction

The development of mathematical inequalities (Schwarz, triangle, Bessel, Grüss, Gram, Hadamard, Landau, Čebyšev, Holder, Minkowsky, etc.) has experienced a surge, having been stimulated by their applications in different branches of pure and applied Mathematics. These inequalities have been frequently used as powerful tools in obtaining bounds or estimating the errors for various approximation formulae occurring in the domains mentioned above. Therefore, any new advancement related to these fundamental facts will have a flow of important consequences in the mathematical fields where these inequalities have been used before.

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For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{R}$, consider the Čebyšev functional:

$$T(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{b-a} \int_a^b f(t)dt \frac{1}{b-a} \int_a^b g(t)dt.$$

In 1934, G. Grüss [7] showed that

$$(1) \quad |T(f, g)| \leq \frac{1}{4}(M-m)(N-n),$$

provided m, M, n, N are real numbers with the property $-\infty < m \leq f \leq M < \infty$ and $-\infty < n \leq g \leq N < \infty$ a.e. on $[a, b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity and is achieved for

$$f(x) = g(x) = \operatorname{sgn}\left(x - \frac{a+b}{2}\right).$$

The discrete version of (1) states that: If $a \leq a_i \leq A$, $b \leq b_i \leq B$, ($i = 1, \dots, n$) where a, A, b, B, a_i, b_i are real numbers, then

$$(2) \quad \left| \frac{1}{n} \sum_{i=1}^n a_i b_i - \frac{1}{n} \sum_{i=1}^n a_i \cdot \frac{1}{n} \sum_{i=1}^n b_i \right| \leq \frac{1}{4}(A-a)(B-b),$$

where the constant $\frac{1}{4}$ is the best possible for an arbitrary $n \geq 1$. Some refinements of the discrete version of Grüss inequality (2) are given in [1, 8]. In [4] some new inequalities of Schwarz and Buzano type for n -tuples of vectors and applications for norm and numerical radius inequalities for n -tuples of bounded linear operators are given.

In the recent years, the Grüss inequality (1) has been investigated, applied and generalized by many authors in different areas of mathematics, among others in inner product spaces [3], in the approximation of integral transforms [11] and the references therein, in semi-inner $*$ -modules for positive linear functionals and C^* -seminorms [5], for positive maps [13], in inner product modules over H^* -algebras and C^* -algebras [10]. A further extension of Grüss type inequality for Bochner integrals of vector-valued functions in Hilbert C^* -modules is given in [6].

For an entire chapter devoted to the history of this inequality see [12] where further references are given. We recall some of the most important Grüss type discrete inequalities for inner product spaces that are available in [2].

Theorem 1.1. Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over \mathbb{K} ; ($\mathbb{K} = \mathbb{C}, \mathbb{R}$), $x_i, y_i \in H$, $p_i \geq 0$ ($i = 1, \dots, n$) ($n \geq 2$) with $\sum_{i=1}^n p_i = 1$. If $x, X, y, Y \in H$ are such that

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0$$

for all $i \in \{1, \dots, n\}$, or, equivalently,

$$\left\| x_i - \frac{x+X}{2} \right\| \leq \frac{1}{2} \|X - x\| \quad \text{and} \quad \left\| y_i - \frac{y+Y}{2} \right\| \leq \frac{1}{2} \|Y - y\|$$

for all $i \in \{1, \dots, n\}$, then the following inequality holds

$$(3) \quad \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right| \leq \frac{1}{4} \|X - x\| \|Y - y\|.$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Theorem 1.2. Let $(H; \langle \cdot, \cdot \rangle)$ and \mathbb{K} be as above and $\bar{x} = (x_1, \dots, x_n) \in H^n$, $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$ and $\bar{p} = (p_1, \dots, p_n)$ a probability vector. If $x, X \in H$ are such that

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad \text{for all } i \in \{1, \dots, n\},$$

or, equivalently,

$$\left\| x_i - \frac{x+X}{2} \right\| \leq \frac{1}{2} \|X - x\| \quad \text{for all } i \in \{1, \dots, n\},$$

holds, then the following inequality holds

$$(4) \quad \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i \right\| \leq \frac{1}{2} \|X - x\| \left| \sum_{i=1}^n p_i \alpha_i - \sum_{j=1}^n p_j \alpha_j \right|$$

$$\leq \frac{1}{2} \|X - x\| \left[\sum_{i=1}^n p_i |\alpha_i|^2 - \left| \sum_{i=1}^n p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}.$$

The constant $\frac{1}{2}$ in the first and second inequalities is best possible.

Motivated by the above results in the present paper, we obtain some further generalization of Grüss type inequalities in semi-inner product modules over C^* -algebras. we give some analogue of the discrete Grüss inequality (2) for n -tuples of vectors, which are generalizations of Theorem 1.1 and Theorem 1.2. We also give some their applications for the approximation of the discrete Fourier and Melin transforms. In order to do that we need the following preliminary definitions and results.

2. Preliminaries

The theory of Hilbert spaces plays a central role in contemporary mathematics with numerous applications for linear operators, matrix analysis, partial differential equations, nonlinear analysis, approximation theory, optimization theory, numerical analysis, probability theory, statistics and other fields. Hilbert spaces have a rich geometric structure because they are endowed with an inner product that allows the introduction of the concept of orthogonality of vectors.

Hilbert C^* -modules are used as the framework for Kasparov's bivariant K-theory and form the technical underpinning for the C^* -algebraic approach to quantum groups. Hilbert C^* -modules are very useful in the following research areas: operator K-theory, index theory for operator-valued conditional expectations, group representation theory, the theory of AW^* -algebras, non-commutative geometry, and others. Hilbert C^* -modules form a category in between Banach spaces and Hilbert spaces and obey the same axioms as a Hilbert space except that the inner product takes values in a general C^* -algebra rather than the complex number \mathbb{C} . This simple generalization gives a lot of trouble. Fundamental and familiar Hilbert space properties like Pythagoras' equality, self-duality and decomposition into orthogonal complements must be given up. Moreover, a bounded module map between Hilbert C^* -modules does not need to have an adjoint; not every adjointable operator needs to have a polar decomposition. Hence to get its applications, we have to use it with great care.

Let \mathcal{A} be a C^* -algebra. A semi-inner product module over \mathcal{A} is a right module X over \mathcal{A} together with a generalized semi-inner product, that is with a mapping $\langle \cdot, \cdot \rangle$ on $X \times X$, which is \mathcal{A} -valued and has the following properties:

- (i) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in X$,
- (ii) $\langle x, ya \rangle = \langle x, y \rangle a$ for $x, y \in X, a \in \mathcal{A}$,
- (iii) $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in X$,
- (iv) $\langle x, x \rangle \geq 0$ for $x \in X$.

We will say that X is a semi-inner product C^* -module. The absolute value of $x \in X$ is defined as the square root of $\langle x, x \rangle$, and it is denoted by $|x|$.

If, in addition,

(v) $\langle x, x \rangle = 0$ implies $x = 0$,

then $\langle \cdot, \cdot \rangle$ is called a generalized inner product and X is called an inner product module over \mathcal{A} or an inner product C^* -module.

As we can see, an inner product module obeys the same axioms as an ordinary inner product space, except that the inner product takes values in a more general structure rather than in the field of complex numbers. If \mathcal{A} is a C^* -algebra and X is a semi-inner product \mathcal{A} -module, then the following Schwarz inequality holds:

$$(5) \quad \langle x, y \rangle \langle y, x \rangle \leq \|\langle x, x \rangle\| \langle y, y \rangle \quad (x, y \in X)$$

(e.g. [9, Proposition 1.1]). It follows from the Schwarz inequality (5) that

$$(6) \quad |\langle x, y \rangle|^2 \leq \|\langle x, x \rangle\| \langle y, y \rangle \quad (x, y \in X)$$

(where, for $a \in \mathcal{A}$, $|a| = (a^*a)^{\frac{1}{2}}$). Now let \mathcal{A} be a $*$ -algebra, φ a positive linear functional on \mathcal{A} , and let X be a semi-inner \mathcal{A} -module. We can define a sesquilinear form on $X \times X$ by $\sigma(x, y) = \varphi(\langle x, y \rangle)$; the Schwarz inequality for σ implies that

$$(7) \quad |\varphi(\langle x, y \rangle)|^2 \leq \varphi(\langle x, x \rangle) \varphi(\langle y, y \rangle).$$

In [5, Proposition 1, Remark 1] the authors present two other forms of the Schwarz inequality in semi-inner \mathcal{A} -module X , one for a positive linear functional φ on \mathcal{A} :

$$(8) \quad \varphi(\langle x, y \rangle \langle y, x \rangle) \leq \varphi(\langle x, x \rangle) r \langle y, y \rangle,$$

where r is the spectral radius, and another one for a C^* -seminorm γ on \mathcal{A} :

$$(9) \quad (\gamma(\langle x, y \rangle))^2 \leq \gamma(\langle x, x \rangle) \gamma(\langle y, y \rangle).$$

3. Grüss type inequalities in inner product C^* -modules

Let X be an inner product C^* -module and $x, y, e \in X$, and let $\langle e, e \rangle$ be an idempotent, we put

$$G_e(x, y) := \langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle.$$

By [5, Lemma 2] or, a straightforward calculation shows that

$$\begin{aligned}
\langle e\langle e, e \rangle - e, e\langle e, e \rangle - e \rangle &= \langle e\langle e, e \rangle, e\langle e, e \rangle \rangle - \langle e\langle e, e \rangle, e \rangle - \langle e, e\langle e, e \rangle \rangle + \langle e, e \rangle \\
&= \langle e, e \rangle \langle e, e \rangle \langle e, e \rangle - \langle e, e \rangle \langle e, e \rangle - \langle e, e \rangle \langle e, e \rangle + \langle e, e \rangle \\
&= 0,
\end{aligned}$$

therefore $e\langle e, e \rangle - e = 0$. This implies that

$$\langle e, e \rangle \langle e, x \rangle = \langle e, x \rangle, \quad \langle x, e \rangle = \langle x, e \rangle \langle e, e \rangle.$$

Lemma 3.1. *Let X be an inner product \mathcal{A} -module over C^* -algebra \mathcal{A} and $x, y, e \in X$. If $\langle e, e \rangle$ be an idempotent in \mathcal{A} , then for every $a, b \in \mathcal{A}$, we have*

$$(10) \quad (i) \quad G_e(x, x) = \langle x, x \rangle - \langle x, e \rangle \langle e, x \rangle \geq 0,$$

$$(11) \quad (ii) \quad G_e(x, x) \leq \langle x - ea, x - ea \rangle,$$

$$\begin{aligned}
(12) \quad (iii) \quad G_e(x - ea, y - eb) &= \langle x - ea, y - eb \rangle - \langle x - ea, e \rangle \langle e, y - eb \rangle \\
&= \langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle = G_e(x, y).
\end{aligned}$$

Proof. By a simple calculation, we get

$$G_e(x, x) = \langle x, x \rangle - \langle x, e \rangle \langle e, x \rangle = \langle x - e\langle e, x \rangle, x - e\langle e, x \rangle \rangle \geq 0$$

and

$$\langle x - ea, x - ea \rangle = \langle x - e\langle e, x \rangle, x - e\langle e, x \rangle \rangle + \langle a - e\langle e, x \rangle, a - e\langle e, x \rangle \rangle.$$

Therefore,

$$G_e(x, x) = \langle x - e\langle e, x \rangle, x - e\langle e, x \rangle \rangle \leq \langle x - ea, x - ea \rangle.$$

A straightforward calculation shows that

$$\begin{aligned}
G_e(x - ea, y - eb) &= \langle x - ea, y - eb \rangle - \langle x - ea, e \rangle \langle e, y - eb \rangle \\
&= \langle x, y \rangle - \langle x, e \rangle b - a^* \langle e, y \rangle + a^* \langle e, e \rangle b - [\langle x, e \rangle - a^* \langle e, e \rangle] [\langle e, y \rangle - \langle e, e \rangle b] \\
&= \langle x, y \rangle - \langle x, e \rangle b - a^* \langle e, y \rangle + a^* \langle e, e \rangle b \\
&\quad - \langle x, e \rangle \langle e, y \rangle + \langle x, e \rangle b + a^* \langle e, y \rangle - a^* \langle e, e \rangle b \\
&= \langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle = G_e(x, y).
\end{aligned}$$

Theorem 3.2. *Let X be an inner product \mathcal{A} -module over C^* -algebra \mathcal{A} and $x, y, e \in X$. If $\langle e, e \rangle$ be an idempotent in \mathcal{A} , then for every $a, b, c, d \in \mathcal{A}$, we have*

$$\begin{aligned}
(13) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| &\leq \left\| x - e \left(\frac{a+b}{2} \right) \right\| \left\| y - e \left(\frac{c+d}{2} \right) \right\| \\
&= \left\| \frac{1}{4} |e(a-b)|^2 - \operatorname{Re} \langle x - ea, eb - x \rangle \right\|^{\frac{1}{2}} \left(\frac{1}{4} |e(c-d)|^2 - \operatorname{Re} \langle y - ec, ed - y \rangle \right)^{\frac{1}{2}}.
\end{aligned}$$

furthermore, if

$$\operatorname{Re} \langle x - ea, eb - x \rangle \geq 0, \quad \operatorname{Re} \langle y - ec, ed - y \rangle \geq 0,$$

then

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} \|e(a-b)\| |e(c-d)|.$$

Proof. It is easy to show that $G_e(\cdot, \cdot)$ is an \mathcal{A} -value semi-inner product on X . Using Schwarz inequality (6), we obtain

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq \|\langle x, x \rangle - \langle x, e \rangle \langle e, x \rangle\| (\langle y, y \rangle - \langle y, e \rangle \langle e, y \rangle).$$

From (11), we get

$$\langle x, x \rangle - \langle x, e \rangle \langle e, x \rangle \leq \left\langle x - e \left(\frac{a+b}{2} \right), x - e \left(\frac{a+b}{2} \right) \right\rangle$$

and

$$\langle y, y \rangle - \langle y, e \rangle \langle e, y \rangle \leq \left\langle y - e \left(\frac{c+d}{2} \right), y - e \left(\frac{c+d}{2} \right) \right\rangle.$$

Since for any $y, x, x' \in X$

$$\frac{1}{4}|x' - x|^2 - \left| y - \frac{x' + x}{2} \right|^2 = \operatorname{Re}\langle y - x', x - y \rangle,$$

therefore, we have

$$\left| x - e \left(\frac{a+b}{2} \right) \right|^2 = \frac{1}{4}|e(a-b)|^2 - \operatorname{Re}\langle x - ea, eb - x \rangle,$$

$$\left| y - e \left(\frac{c+d}{2} \right) \right|^2 = \frac{1}{4}|e(c-d)|^2 - \operatorname{Re}\langle y - ec, ed - y \rangle.$$

The rest follows from these facts and we omit the details.

Example 3.3. Let $L^1(H)$ be the set of all trace class operators on the Hilbert space H . It is known that $L^1(H)$ is a Hilbert $B(H)$ -module with the inner product defined by $\langle X, Y \rangle := X^*Y$. If E is a trace class operator such that $|E|$ is an idempotent in $B(H)$ then for every $A, B, C, D \in B(H)$ and $X, Y \in L^1(H)$ we have

$$\begin{aligned} |X^*Y - X^*EY|^2 &\leq \left\| \frac{1}{4}|E(A-B)|^2 - \operatorname{Re}\langle X - EA, EB - X \rangle \right\|^2 \\ &\quad \times \left| \frac{1}{4}|E(C-D)|^2 - \operatorname{Re}\langle Y - EC, ED - Y \rangle \right|^2 \\ &= \left\| X - E \left(\frac{A+B}{2} \right) \right\|^2 \left| Y - E \left(\frac{C+D}{2} \right) \right|^2. \end{aligned}$$

4. Grüss type inequalities in semi-inner product C^* -modules

Before stating the main results in this section, let us fix the rest of our notation. We assume unless stated otherwise, throughout this paper \mathcal{A} is a C^* -algebra and $\bar{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ a probability vector i.e. $p_i \geq 0$ ($i = 1, \dots, n$) and $\sum_{i=1}^n p_i = 1$. If X is a semi-inner product C^* -module and $\bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_n) \in X^n$ we put

$$G_{\bar{p}}(\bar{x}, \bar{y}) := \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle.$$

Lemma 4.1. *Let X be a semi-inner product C^* -module, $a, b \in X$, $\bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_n) \in X^n$, $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$; ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $\bar{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ a probability vector, then*

$$(14) \quad \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i = \sum_{i=1}^n p_i \left(\alpha_i - \sum_{j=1}^n p_j \alpha_j \right) (x_i - a),$$

and

$$(15) \quad G_{\bar{p}}(\bar{x}, \bar{y}) = \sum_{i=1}^n p_i \langle x_i - a, y_i - b \rangle - \left\langle \sum_{i=1}^n p_i (x_i - a), \sum_{i=1}^n p_i (y_i - b) \right\rangle.$$

In particular

$$(16) \quad G_{\bar{p}}(\bar{x}, \bar{x}) = \sum_{i=1}^n p_i |x_i - a|^2 - \left| \sum_{i=1}^n p_i x_i - a \right|^2 \leq \sum_{i=1}^n p_i |x_i - a|^2.$$

Proof. For every $a \in X$ a simple calculation shows that

$$\begin{aligned} \sum_{i=1}^n p_i \left(\alpha_i - \sum_{j=1}^n p_j \alpha_j \right) (x_i - a) &= \sum_{i=1}^n p_i \alpha_i x_i - \sum_{j=1}^n p_j \alpha_j \sum_{i=1}^n p_i x_i \\ &\quad - a \sum_{i=1}^n p_i \alpha_i + a \sum_{i=1}^n p_i \sum_{j=1}^n p_j \alpha_j \\ &= \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i. \end{aligned}$$

For every $a, b \in X$, a simple calculation shows that

$$\begin{aligned} &\sum_{i=1}^n p_i \langle x_i - a, y_i - b \rangle - \left\langle \sum_{i=1}^n p_i (x_i - a), \sum_{i=1}^n p_i (y_i - b) \right\rangle \\ &= \sum_{i=1}^n p_i (\langle x_i, y_i \rangle - \langle x_i, b \rangle - \langle a, y_i \rangle + \langle a, b \rangle) \\ &\quad - \left\langle \sum_{i=1}^n p_i x_i - a, \sum_{i=1}^n p_i y_i - b \right\rangle \\ &= \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \\ &= G_{\bar{p}}(\bar{x}, \bar{y}). \end{aligned}$$

In particular for $a = b, x_i = y_i$ we have

$$\begin{aligned} G_{\bar{p}}(\bar{x}, \bar{x}) &= \sum_{i=1}^n p_i \langle x_i - a, x_i - a \rangle - \left\langle \sum_{i=1}^n p_i (x_i - a), \sum_{i=1}^n p_i (x_i - a) \right\rangle \\ &= \sum_{i=1}^n p_i |x_i - a|^2 - \left| \sum_{i=1}^n p_i (x_i - a) \right|^2 \leq \sum_{i=1}^n p_i |x_i - a|^2. \end{aligned}$$

In the following Theorem we give a generalization of Theorem 1.1 for semi-inner product C^* -modules.

Theorem 4.2. *Let X be a semi-inner product C^* -module, $a, b \in X$ and $\bar{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ a probability vector. If $\bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_n) \in X^n$, then the following inequality holds*

$$\begin{aligned} (17) \quad & \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right|^2 \\ & \leq \left\| \sum_{i=1}^n p_i |x_i - a|^2 - \left| \sum_{i=1}^n p_i x_i - a \right|^2 \right\| \left\| \left(\sum_{i=1}^n p_i |y_i - b|^2 - \left| \sum_{i=1}^n p_i y_i - b \right|^2 \right) \right\| \\ & \leq \left(\sum_{i=1}^n p_i \|x_i - a\|^2 \right) \left(\sum_{i=1}^n p_i |y_i - b|^2 \right). \end{aligned}$$

Proof. A simple calculation shows that

$$\sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle = \frac{1}{2} \sum_{i,j=1}^n p_i p_j \langle x_i - x_j, y_i - y_j \rangle,$$

therefore

$$G_{\bar{p}}(\bar{x}, \bar{x}) = \frac{1}{2} \sum_{i,j=1}^n p_i p_j \langle x_i - x_j, x_i - x_j \rangle \geq 0.$$

It is easy to show that $G_{\bar{p}}(\cdot, \cdot)$ is an \mathcal{A} -value semi-inner product on X^n , so Schwarz inequality (6) holds i.e.,

$$(18) \quad |G_{\bar{p}}(\bar{x}, \bar{y})|^2 \leq \|G_{\bar{p}}(\bar{x}, \bar{x})\| G_{\bar{p}}(\bar{y}, \bar{y}).$$

From (16), we get

$$(19) \quad \|G_{\bar{p}}(\bar{x}, \bar{x})\| = \left\| \sum_{i=1}^n p_i |x_i - a|^2 - \left| \sum_{i=1}^n p_i x_i - a \right|^2 \right\| \leq \sum_{i=1}^n p_i \|x_i - a\|^2$$

and

$$(20) \quad G_{\bar{p}}(\bar{y}, \bar{y}) = \sum_{i=1}^n p_i |y_i - b|^2 - \left| \sum_{i=1}^n p_i y_i - b \right|^2 \leq \sum_{i=1}^n p_i |y_i - b|^2.$$

From inequalities (18), (19) and (20) we obtain the inequality (17).

Since every inner-product space H can be regarded as an inner product \mathbb{C} -module, therefore the following inequality (22) is a generalization of inequality (3).

Corollary 4.3. *Let X be a semi-inner product C^* -module, $a, b \in X$ and $\bar{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ a probability vector. If $\bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_n) \in X^n$, $r \geq 0, s \geq 0$ are such that*

$$(21) \quad \|x_i - a\| \leq r, \quad \|y_i - b\| \leq s, \text{ for all } i \in \{1, \dots, n\},$$

then the following inequality holds

$$(22) \quad \left\| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle \right\| \leq rs.$$

The constant 1 coefficient of rs in the inequality (22) is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. From inequalities (17) and (21) we obtain (22). To prove the sharpness of the constant 1 in the inequality in (22), let us assume that, under the assumptions of the theorem, the inequalities hold with a constant $c > 0$, i.e.,

$$(23) \quad \|G_{\bar{p}}(\bar{x}, \bar{y})\| \leq crs.$$

Assume that $n = 2, p_1 = p_2 = \frac{1}{2}$ and e is an element of X such that $\|\langle e, e \rangle\| = 1$. We put

$$x_1 = a + re, \quad y_1 = b + se$$

$$x_2 = a - re, \quad y_2 = b - se,$$

then, obviously,

$$\|x_i - a\| \leq r, \quad \|y_i - b\| \leq s, \quad (i = 1, 2),$$

which shows that the condition (21) holds. If we replace $n, p_1, p_2, x_1, x_2, y_1, y_2$ in (23), we obtain

$$\|G_{\bar{p}}(\bar{x}, \bar{y})\| = rs \leq crs,$$

from where we deduce that $c \geq 1$, which proves the sharpness of the constant 1.

The following Remark 4.4 (ii) is a generalization of Theorem 1.2 for semi-inner product C^* -modules.

Remark 4.4.

- (i) Let \mathcal{A} be a C^* -algebra, and $\bar{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ a probability vector. If $a, b, a_i, b_i, (i = 1, 2, \dots, n) \in \mathcal{A}, r \geq 0, s \geq 0$ are such that

$$\|a_i - a\| \leq r, \|b_i - b\| \leq s, \text{ for all } i \in \{1, \dots, n\},$$

it is known that \mathcal{A} is a Hilbert C^* -module over itself with the inner product defined by $\langle a, b \rangle := a^*b$. In this case (22) implies that

$$\left\| \sum_{i=1}^n p_i a_i^* b_i - \sum_{i=1}^n p_i a_i^* \cdot \sum_{i=1}^n p_i b_i \right\| \leq rs.$$

Since

$$\|a_i^* - a^*\| \leq r, \text{ for all } i \in \{1, \dots, n\},$$

we deduce

$$\left\| \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \cdot \sum_{i=1}^n p_i b_i \right\| \leq rs.$$

- (ii) Let X be a semi-inner product C^* -module, $a \in X, \bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$ and $\bar{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ a probability vector. If $\bar{x} = (x_1, \dots, x_n) \in X^n, r \geq 0$ are such that

$$\|x_i - a\| \leq r, \text{ for all } i \in \{1, \dots, n\},$$

holds, from equality (14) we obtain

$$(24) \quad \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i \right\| \leq r \sum_{i=1}^n p_i \left| \alpha_i - \sum_{i=1}^n p_j \alpha_j \right| \\ \leq r \left[\sum_{i=1}^n p_i |\alpha_i|^2 - \left| \sum_{i=1}^n p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}.$$

The constant 1 in the first and second inequalities in (24) is best possible. Since every Hilbert space is a Hilbert \mathbb{C} -module, the inequality (24) is a generalization of (4).

5. Applications

In this section we give applications of Theorem 4.2 for the approximation of some discrete transforms such as the discrete Fourier and the Melin transforms for bounded linear operators on a Hilbert space.

Let X be a semi-inner product C^* -module on C^* -algebra \mathcal{A} and $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in X^n$. For a given $\omega \in \mathbb{R}$, define the *discrete Fourier transform*

$$\mathcal{F}_\omega(x)(m) = \sum_{k=1}^n \exp(2\omega imk) \times x_k, \quad m = 1, \dots, n.$$

The element $\sum_{k=1}^n \exp(2\omega imk) \times \langle x_k, y_k \rangle$ of \mathcal{A} is called Fourier transform of the vector

$$(\langle x_1, y_1 \rangle, \dots, \langle x_k, y_k \rangle) \in \mathcal{A}^n$$

and will be denoted by

$$\mathcal{F}_\omega(x, y)(m) = \sum_{k=1}^n \exp(2\omega imk) \times \langle x_k, y_k \rangle \quad m = 1, \dots, n.$$

We can also consider the *Mellin transform*

$$\mathcal{M}(x)(m) = \sum_{k=1}^n k^{m-1} x_k, \quad m = 1, \dots, n.$$

of the vector $x = (x_1, \dots, x_n) \in X^n$. The Mellin transform of the vector $(\langle x_1, y_1 \rangle, \dots, \langle x_k, y_k \rangle) \in \mathcal{A}^n$ is defined by

$\sum_{k=1}^n k^{m-1} \langle x_k, y_k \rangle$ and will be denoted by

$$\mathcal{M}(x, y)(m) = \sum_{k=1}^n k^{m-1} \langle x_k, y_k \rangle.$$

Example 5.1. Let A_1, \dots, A_n and B_1, \dots, B_n be bounded linear operators in $B(H_1, H_2)$. It is known that $B(H_1, H_2)$ is a Hilbert C^* -module over $B(H_1)$ with the inner product defined by $\langle X, Y \rangle := X^*Y$, therefore for every $A, B \in B(H_1)$, from the inequality (17) we obtain

$$\begin{aligned}
(25) \quad & \left| \sum_{k=1}^n \exp(2\omega imk) A_k^* B_k - \left(\frac{1}{n} \sum_{k=1}^n A_k^* \right) \left(\sum_{k=1}^n \exp(2\omega imk) B_k \right) \right|^2 \\
& \leq \left\| \sum_{k=1}^n |A_k - A|^2 - \left| \frac{1}{n} \sum_{k=1}^n A_k - A \right|^2 \right\| \\
& \quad \times \left(\sum_{k=1}^n |\exp(2\omega imk) B_k - B|^2 - \left| \frac{1}{n} \sum_{k=1}^n \exp(2\omega imk) B_k - B \right|^2 \right) \\
& \leq \left(\sum_{k=1}^n \|A_k - A\|^2 \right) \left(\sum_{k=1}^n |\exp(2\omega imk) B_k - B|^2 \right).
\end{aligned}$$

and

$$\begin{aligned}
(26) \quad & \left| \sum_{k=1}^n k^{m-1} A_k^* B_k - \left(\frac{1}{n} \sum_{k=1}^n A_k^* \right) \left(\sum_{k=1}^n k^{m-1} B_k \right) \right|^2 \\
& \leq \left\| \sum_{k=1}^n |A_k - A|^2 - \left| \frac{1}{n} \sum_{k=1}^n A_k - A \right|^2 \right\| \\
& \quad \times \left(\sum_{k=1}^n |k^{m-1} B_k - B|^2 - \left| \frac{1}{n} \sum_{k=1}^n k^{m-1} B_k - B \right|^2 \right) \\
& \leq \left(\sum_{k=1}^n \|A_k - A\|^2 \right) \left(\sum_{k=1}^n |k^{m-1} B_k - B|^2 \right).
\end{aligned}$$

Example 5.2. Let B, A, A_1, \dots, A_n be bounded linear operators on the Hilbert space H , $\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$. and I be identity operator on H from inequality (17) we obtain

$$\begin{aligned}
(27) \quad & \left| \sum_{k=1}^n p_k \alpha_k A_k - \left(\sum_{k=1}^n p_k \alpha_k \right) \left(\sum_{k=1}^n p_k A_k \right) \right|^2 = \left| \sum_{k=1}^n p_k \langle \bar{\alpha}_k I, A_k \rangle - \left\langle \sum_{k=1}^n p_k \bar{\alpha}_k I, \sum_{k=1}^n p_k A_k \right\rangle \right|^2 \\
& \leq \left\| \sum_{k=1}^n p_k |\bar{\alpha}_k I - A|^2 - \left| \sum_{k=1}^n p_k \bar{\alpha}_k I - A \right|^2 \right\| \left(\sum_{k=1}^n p_k |A_k - B|^2 - \left| \sum_{k=1}^n p_k A_k - B \right|^2 \right),
\end{aligned}$$

for $A = B = 0$ we get

$$(28) \quad \left| \sum_{k=1}^n p_k \alpha_k A_k - \left(\sum_{k=1}^n p_k \alpha_k \right) \left(\sum_{k=1}^n p_k A_k \right) \right|^2 \\ \leq \left(\sum_{k=1}^n p_k |\alpha_k|^2 - \left| \sum_{k=1}^n p_k \alpha_k \right|^2 \right) \left(\sum_{k=1}^n p_k |A_k|^2 - \left| \sum_{k=1}^n p_k A_k \right|^2 \right).$$

A simple calculation shows that (see the proof of Theorem 59 in [2]),

$$\sum_{k=1}^n \exp(2\omega imk) = \frac{\sin(\omega mn)}{\sin(\omega m)} \times \exp[\omega(n+1)im].$$

Putting $\alpha_k = \exp(2\omega imk)$, $p_k = \frac{1}{n}$, in (28), we get

$$(29) \quad \left| \sum_{k=1}^n \exp(2\omega imk) A_k - \frac{\sin(\omega mn)}{\sin(\omega m)} \exp[\omega(n+1)im] \times \frac{1}{n} \sum_{k=1}^n A_k \right|^2 \\ \leq \left[n^2 - \frac{\sin^2(\omega mn)}{\sin^2(\omega m)} \right] \left[\frac{1}{n} \sum_{k=1}^n |A_k|^2 - \frac{1}{n^2} \left| \sum_{k=1}^n A_k \right|^2 \right].$$

Also, Putting $\alpha_k = k^{m-1}$, $p_k = \frac{1}{n}$, in (28), we obtain

$$(30) \quad \left| \sum_{k=1}^n k^{m-1} A_k - S_{m-1}(n) \cdot \frac{1}{n} \sum_{k=1}^n A_k \right|^2 \\ \leq [nS_{2m-2}(n) - S_{m-1}^2(n)] \left[\frac{1}{n} \sum_{k=1}^n |A_k|^2 - \frac{1}{n^2} \left| \sum_{k=1}^n A_k \right|^2 \right], m \in \{1, \dots, n\},$$

where $S_p(n), p \in \mathbb{R}, n \in \mathbb{N}$ is the p -powered sum of the first n natural numbers, i.e.,

$$S_p(n) := \sum_{k=1}^n k^p.$$

For the following particular values of Mellin Transform (see [3, Corollary 4]), we have

$$\left| \sum_{k=1}^n k A_k - \frac{n+1}{2} \sum_{k=1}^n A_k \right|^2 \leq \left[\frac{n^2(n-1)(n+1)}{12} \right] \left[\frac{1}{n} \sum_{k=1}^n |A_k|^2 - \frac{1}{n^2} \left| \sum_{k=1}^n A_k \right|^2 \right] \\ \leq \left[\frac{n(n-1)(n+1)}{12} \right] \sum_{k=1}^n |A_k|^2,$$

and

$$\begin{aligned} & \left| \sum_{k=1}^n k^2 A_k - \frac{(n+1)(2n+1)}{6} \sum_{k=1}^n A_k \right|^2 \\ & \leq \left(\frac{n^2(n-1)(n+1)(2n+1)(8n+11)}{180} \right) \left[\frac{1}{n} \sum_{k=1}^n |A_k|^2 - \frac{1}{n^2} \left| \sum_{k=1}^n A_k \right|^2 \right] \\ & \leq \left(\frac{n(n-1)(n+1)(2n+1)(8n+11)}{180} \right) \sum_{k=1}^n |A_k|^2. \end{aligned}$$

There exist other examples for the approximation of some discrete transforms such as polynomials with coefficients in a semi-inner product C^* -module. However, the details are omitted but each of them can be proven in a similar manner as this section.

Conflict of Interests

The authors declare that there is no conflict of interests.

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