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SOME INEQUALITIES FOR THE GAMMA k-FUNCTION

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Abstract. The main objective of this paper is to present the k-analogue of inequalities for the Euler gamma function and psi function in terms of a new symbol $k > 0$.

Keywords: Gamma k-function; Digamma k-function; Euler gamma function; Inequality.

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1. Introduction

Recently, Diaz and Pariguan [2] introduced the generalized gamma k-function as

(1) $$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ k > 0, x \in \mathbb{C}\setminus kZ^-,$$

where $(x)_{n,k}$, is called the Pochhammer k-symbol and is defined as

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k)$$

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for $n \geq 1$. They have also introduced and proved some identities of the said functions and deduced an integral representation of gamma k-function as,

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, \quad Re(x) > 0, k > 0. \tag{2}$$

Mubeen et al. [9] have defined k-hypergeometric differential equation and gave twenty four solutions of said k-hypergeometric differential equation. Many researchers [4]-[8] have worked on the generalized gamma k-function and discussed the following properties for $k > 0$ and $n \in \mathbb{N}$:

$$\Gamma_k(x+k) = x\Gamma_k(x),$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)},$$

$$\Gamma_k(k) = 1,$$

$$\Gamma_k(\alpha k) = k^{\alpha-1}\Gamma(\alpha),\, \alpha \in \mathbb{R}^+,$$

$$\Gamma_k(nk) = k^{n-1}(n-1)!,$$

$$\Gamma_k((2n+1)\frac{k}{2}) = k^{\frac{2n-1}{2}}\frac{(2n)!\sqrt{\pi}}{2^n n!},$$

$$\Gamma_k(x) = x^{-1}k^{\frac{x}{k}}e^{-\frac{x}{k}\gamma}\prod_{n=1}^{\infty}(\frac{nk}{x+nk})e^{\frac{x}{nk}},$$

where γ is Euler's or Mascheroni's constant and its value is given by

$$\gamma = \lim_{n\to\infty}\sum\frac{1}{n} - \ln(n) = 0.5772156649....$$

Kokologiannaki [3] gave some properties and inequalities for the above gamma k-function. In [12], the same auther gave some power product bounds for the gamma k-function and beta k-function. Brahim et al. [13] established some new inequalities for the gamma, beta and psi q-k functions by using q-integral inequalities. Zhang et.al. [14] extended a double inequality for the gamma function to the gamma k-function and the Riemann zeta k-function by using methods in the theory of majorization. Rehman et al. [10, 11] presented some inequalities involving gamma k-function and beta k-functions via some classical inequalities like the Chebychev inequality for synchronous (asynchronous) mappings, and the Grüss and the Ostrowski's inequality. They also gave proof of the log-convexity of these k-functions by using the Hölder inequality. Beside these, the researchers [15]-[18] have proved bounds, inequalities and monotonicity properties for the functions $\Gamma_k(x)$ and $\beta_k(x,y)$ and for functions involving them.

2. Main results

The logarithmic derivative of $\Gamma_k(x)$ is called digamma k-function or psi k-function. It is denoted by $\psi_k(x)$ and is given by (See [8])

$$\psi_k(x) = \frac{\partial}{\partial x}\log\Gamma_k(x), \tag{3}$$

where $x,k>0$. The series representation of $\psi_k(x)$ [8] is given by the relation

$$\psi_k(x) = \frac{\ln k-\gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x}{nk(x+nk)}. \tag{4}$$

It can also be written as

$$\psi_k(x) = \frac{\ln k-\gamma}{k} + \sum_{n=0}^{\infty}\frac{(x-k)}{(nk+k)(x+nk)}. \tag{5}$$

In this present paper, we are going to deduce the k-analogue of inequalities involving the gamma and digamma functions with the same conditions on parameters which have been proved in [1]. In order to prove our main results, we need the following lemmas.

Lemma 2.1. *Let $x\in(0,1)$ and p, q be two positive real numbers such that $p>q$. Then*

$$\psi_k(p+qx) > \psi_k(q+px). \tag{6}$$

Proof. It is easy to verify that $p+qx>0$, $q+px>0$. Then by equation (5) we obtain the following inequality:

$$\begin{aligned}\psi_k(p+qx)-\psi_k(q+px) &= \sum_{n=0}^{\infty}\frac{(p+qx-k)}{(nk+k)(p+qx+nk)} - \sum_{n=0}^{\infty}\frac{(q+px-k)}{(nk+k)(q+px+nk)}\\ &= \sum_{n=0}^{\infty}\frac{(p-q)(1-x)}{(p+qx+nk)(q+px+nk)}\\ &> 0,\end{aligned}$$

because $x\in(0,1)$ and $p>q$. □

Lemma 2.2. *Let $x \in (0,1)$ and $p > q$ be two positive real numbers such that $\psi_k(q+px) > 0$. Also let r, s be two positive real numbers such that $qr > ps > 0$. Then*

$$qr\psi_k(p+qx) - ps\psi_k(q+px) > 0. \tag{7}$$

Proof. Since $\psi_k(q+px) > 0$, therefore by inequality (6), $\psi_k(p+qx) > 0$. As $qr > ps$ and by using lemma 2.1 , we have

$$qr\psi_k(p+qx) > ps\psi_k(p+qx) > ps\psi_k(q+px)$$

$$\Rightarrow \qquad qr\psi_k(p+qx) - ps\psi_k(q+px) > 0.$$

□

Theorem 2.3. *Let f_k be a function defined by*

$$f_k(x) = \frac{\Gamma_k(p+qx)^{\frac{r}{k}}}{\Gamma_k(q+px)^{\frac{s}{k}}}, \tag{8}$$

where $x \in (0,1)$, $p > q > 0$, r, s are positive real numbers such that $qr > ps > 0$ and $\psi_k(q+px) > 0$. Then f_k is an increasing function on $(0,1)$, and the following double inequality holds:

$$\frac{\Gamma_k(p)^{\frac{r}{k}}}{\Gamma_k(q)^{\frac{s}{k}}} < \frac{\Gamma_k(p+qx)^{\frac{r}{k}}}{\Gamma_k(q+px)^{\frac{s}{k}}} < \frac{\Gamma_k(p+q)^{\frac{r}{k}}}{\Gamma_k(p+q)^{\frac{s}{k}}}. \tag{9}$$

Proof. Consider a function $g_k(x)$ defined by

$$\begin{aligned} g_k(x) &= \log f_k(x) \\ &= \tfrac{1}{k}[r\log\Gamma_k(p+qx) - s\log\Gamma_k(q+px)]. \end{aligned}$$

Differentiating it with respect to x, we get

$$g_k'(x) = \frac{1}{k}[qr\psi_k(p+qx) - ps\psi_k(q+px)]. \tag{10}$$

Since $k > 0$ and by inequality (7)

$$g_k'(x) > 0.$$

This implies that $g_k(x)$ is increasing on $(0,1)$. Hence, $f_k(x)$ is increasing on $(0,1)$. Now since $x \in (0,1)$,

$$f_k(0) < f_k(x) < f_k(1)$$

$$\Rightarrow \qquad \frac{\Gamma_k(p)^{\frac{r}{k}}}{\Gamma_k(q)^{\frac{s}{k}}} < \frac{\Gamma_k(p+qx)^{\frac{r}{k}}}{\Gamma_k(q+px)^{\frac{s}{k}}} < \frac{\Gamma_k(p+q)^{\frac{r}{k}}}{\Gamma_k(p+q)^{\frac{s}{k}}}.$$

□

Lemma 2.4. *Let $x > 1$ and p, q be two positive real numbers such that $q > p$. Then*

$$\psi_k(p+qx) > \psi_k(q+px). \tag{11}$$

Proof. As

$$\begin{aligned} \psi_k(p+qx) - \psi_k(q+px) &= \sum_{n=0}^{\infty} \tfrac{(p-q)(1-x)}{(p+qx+nk)(q+px+nk)} \\ &> 0, \end{aligned}$$

because $x > 1$and $q > p$. □

Lemma 2.5. *Let $x > 1$ and p, q $(q > p)$ be two positive real numbers such that $\psi_k(q+px) > 0$. Also let r, s be two positive real numbers such that $qr > ps > 0$. Then*

$$qr\psi_k(p+qx) - ps\psi_k(q+px) > 0. \tag{12}$$

Proof. Since $\psi_k(q+px) > 0$, therefore by inequality (11), $\psi_k(p+qx) > 0$. As $qr > ps$ and by using lemma 2.4 , we have

$$qr\psi_k(p+qx) > ps\psi_k(p+qx) > ps\psi_k(q+px)$$

$$\Rightarrow \qquad qr\psi_k(p+qx) - ps\psi_k(q+px) > 0.$$

□

Theorem 2.6. *Let f_k be a function defined by*

$$f_k(x) = \frac{\Gamma_k(p+qx)^{\frac{r}{k}}}{\Gamma_k(q+px)^{\frac{s}{k}}}, \tag{13}$$

where $x > 1$, $q > p > 0$, r, s are positive real numbers such that $qr > ps > 0$ and $\psi_k(q+px) > 0$. Then f_k is an increasing function on $(0,1)$.

Proof. Consider a function $g_k(x)$ defined by

$$g_k(x) = \log f_k(x).$$

By following the steps of theorem , we arrive at

$$g'_k(x) = \frac{1}{k}[qr\psi_k(p+qx) - ps\psi_k(q+px)]. \tag{14}$$

Since $k > 0$, so by inequality (12) for $x > 1$

$$g'_k(x) > 0.$$

This implies that $g_k(x)$ is increasing for $x > 1$. Hence, $f_k(x)$ is increasing for $x > 1$. □

Lemma 2.7. *Let $x \in (0,1)$ and p, q $(p > q)$ be two positive real numbers such that $\psi_k(p+qx) < 0$. Also let r, s be two positive real numbers such that $ps > qr > 0$. Then*

$$qr\psi_k(p+qx) - ps\psi_k(q+px) > 0. \tag{15}$$

Proof. Since $\psi_k(p+qx) < 0$ and $qr > 0$, imply $qr\psi_k(p+qx) < 0$. Therefore by lemma 2.1, we have the following inequality

$$0 > qr\psi_k(p+qx) > ps\psi_k(p+qx) > ps\psi_k(q+px)$$

$$\Rightarrow \qquad qr\psi_k(p+qx) - ps\psi_k(q+px) > 0.$$

□

Theorem 2.8. *Let f_k be a function defined by*

$$f_k(x)=\frac{\Gamma_k(p+qx)^{\frac{r}{k}}}{\Gamma_k(q+px)^{\frac{s}{k}}}, \tag{16}$$

where $x\in(0,1)$, p, q $(q>p)$ are positive real numbers such that $\psi_k(p+qx)<0$ and r,s are positive real numbers such that $ps>qr>0$. Then f_k is an increasing function on $(0,1)$.

Proof. Consider a function $g_k(x)$ defined by

$$g_k(x)=\log f_k(x).$$

By following the steps of theorem , we arrive at

$$g_k'(x)=\frac{1}{k}[qr\psi_k(p+qx)-ps\psi_k(q+px)]. \tag{17}$$

Since $k>0$, so by inequality (15) for $x\in(0,1)$

$$g_k'(x)>0.$$

This implies that $g_k(x)$ is increasing for $x\in(0,1)$. Hence, $f_k(x)$ is increasing for $x\in(0,1)$. □

Similarly, by following the steps and methods used in lemma 2.7 and theorem 2.8, the following lemma and theorem can be proved.

Lemma 2.9 *Let $x>1$ and p, q $(q>p)$ be two positive real numbers such that $\psi_k(p+qx)<0$. Also let r, s be two positive real numbers such that $ps>qr>0$. Then*

$$qr\psi_k(p+qx)-ps\psi_k(q+px)>0. \tag{18}$$

Theorem 2.10. Let f_k be a function defined by

$$f_k(x)=\frac{\Gamma_k(p+qx)^{\frac{r}{k}}}{\Gamma_k(q+px)^{\frac{s}{k}}}, \tag{19}$$

where $x>1$, $q>p$ and r, s are positive real numbers such that $ps>qr>0$ and $\psi_k(p+qx)<0$. Then f_k is an increasing function on $(1,+\infty)$.

Remarks 2.11. If we use $k=1$ in all the lemmas and theorems, then we get the corresponding lemmas and theorems which were proved in [1].

Conflict of Interests

The authors declare that there is no conflict of interests.

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