FIXED POINT THEOREMS FOR MULTIVALUED CONTRACTIVE MAPPINGS IN G-SYMMETRIC SPACES

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Abstract. In this paper, we proved some fixed point theorems for multivalued contraction maps and the existence of common fixed points for a pair of occasionally weakly compatible maps satisfying some multivalued contractive conditions in G-symmetric spaces. The results extend and generalize some results in the literature.

1. Introduction

The fixed point theorem for multivalued contraction mappings which takes each point of a metric space (X,d) into a closed subset K of X has gained enormous applications in areas such as control theory, optimization, differential equations and economics. Nadler [18] proved the existence of the fixed points for multivalued contraction mapping in complete metric spaces. Dugundji and Granas [6] proved that a single-valued weakly contractive mapping of a complete metric space into itself has a unique fixed point. Kaneko [12] gave a partial generalization of

the theorem of Dugundji and Granas to the multivalued mappings. Reich [19] proposed the following problem.

Conjecture. Let (X,d) be a complete metric space. Suppose that $T : X \rightarrow CB(X)$ satisfies $H(Tx,Ty) \leq k(d(x,y))d(x,y)$, for all $x,y \in X$, $x \neq y$, where $k : (0,\infty) \rightarrow [0,\infty)$ and $\limsup_{r \rightarrow t^+} k(r) < 1$, for all $0 < t < \infty$. Then T has a fixed point in X. This conjecture was proved by Mizoguchi and Takahashi [15]. Daffer and Kaneko [5] provided an alternative and somewhat more straight forward proof for the theorem of Mizoguchi and Takahashi. Abbas et. al.[1] proved fixed point theorems for multivalued mappings satisfying generalized contractive conditions in ordered generalized metric spaces.

The notion of metric spaces has been generalized by many authors. In particular, Mustafa and Sims [17] generalized the notion of metric spaces. Eke and Olaleru [7] introduced the concept of G-partial metric spaces which generalized the notion of G-metric spaces. Cartan [3] also generalized the notion of metric spaces by omitting the triangle inequality axiom of the metric spaces. Recently, Eke and Olaleru [8] extend the concept of symmetric spaces to G-symmetric spaces by omitting the rectangle inequality axiom of the G-metric spaces. In the same reference, the existence of common fixed points for singled-valued occasionally weakly compatible mappings in G-symmetric spaces is proved. Also, Eke and Olaleru [9] proved existence of common fixed point for pair of hybrid contractive mappings in G-symmetric spaces. In this work, some fixed point theorems for multivalued contraction mappings in G-symmetric spaces are proved.

2. Preliminaries

The following definitions and results are needed in the sequel.

Definition 2.1. [17] Let X be a nonempty set, and let $G : X \times X \times X \rightarrow R^+$ be a function satisfying:

(G1) $G(x,y,z) = 0$ if $x = y = z$,

(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$,

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x)$ (symmetry in all three variables),

(G5) $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then, the function G is called a generalised metric, or more specifically a G-metric on X, and the pair (X,G) is a G-metric space.

Definition 2.2. [3] A symmetric on a set X is a real valued function d on $X \times X$ such that

(i) $d(x,y) \ge 0$ and $d(x,x) = 0$ if and only if $x = y$; and

(ii) $d(x,y) = d(y,x)$.

Wilson [21] also gave two more axioms of a symmetric d on X as:

(W_1) Given $\{x_n\}$, x and y in X, $d(x_n,x) \to 0$ and $d(x_n,y) \to 0$ imply that $x = y$;

(W_2) Given $\{x_n\}$, $\{y_n\}$ and $x \in X$, $d(x_n,x) \to 0$ and $d(x_n,y_n) \to 0$ imply that $d(y_n,x) \to 0$.

Definition 2.3. [13] A mapping $T : X \to 2^X$ is called a multivalued mapping. A point $x \in X$ is called a fixed point of T if $x \in Tx$.

Definition 2.4. [11] Let X be a given nonempty set. Assume that $g : X \to X$ and $T : X \to 2^X$. If $w = gx \in Tx$ for some $x \in X$ then x is called a coincidence point of g and T and w is a point of coincidence of g and T.

Definition 2.5. [11] Maps $g : X \to X$ and $T : X \to 2^X$ are said to be weakly compatible if $gx \in Tx$ for each $x \in X$ implies $gTx \subseteq Tgx$.

Definition 2.6. [11] Maps $g : X \to X$ and $T : X \to 2^X$ are said to be occasionally weakly compatible mappings if and only if there exists some point x in X such that $gx \in Tx$ and $gTx \subseteq Tgx$. An occasionally weakly compatible map is weakly compatible but not vice - verse.

The following definitions and motivations are found in [8].

Definition 2.7. A G-symmetric on a set X is a function $G_d : X \times X \times X \to R^+$ such that for all $x,y,z \in X$ the following conditions are satisfied:

$G_d(1)$ $G_d(x,y,z) \ge 0$ and $G_d(x,y,z) = 0$, if $x = y = z$;

$G_d(2)$ $0 < G_d(x,x,y)$ for all $x,y \in X$ with $x \ne y$,

$G_d(3)$ $G_d(x,x,y) \le G_d(x,y,z)$ for all $x,y,z \in X$ with $y \ne z$,

$G_d(4)$ $G_d(x,y,z) = G_d(y,z,x) = G_d(z,x,y) = ,...,$ (symmetry in all three variables).

It should be observed that our notion of a G-symmetric space is the same as that of G-metric space (Definition 2.1) without the rectangular property (G_5).

Example 2.8. Let $X = [0,1]$ equipped with a G-symmetric defined by $G_d(x,y,z) = (x-y)^2 + (y-z)^2 + (z-x)^2$ for all $x,y,z \in X$. Then (X,G_d) is a G-symmetric space.

This does not satisfy the rectangle inequality property of a G-metric space, hence it is not a G-metric space. The analogue of axioms of Wilson [21] in G-symmetric space is as follows:

(W_3) Given $\{x_n\}$, x and y in X; $G_d(x_n,x,x) \to 0$ and $G_d(x_n,y,y) \to 0$ imply that $x = y$.

(W_4) Given $\{x_n\}$, $\{y_n\}$ and an x in X; $G_d(x_n,x,x) \to 0$ and $G_d(x_n,y_n,y_n) \to 0$ imply that $G_d(y_n,x,x) \to 0$.

(W_5) Let (X,G_d) be a complete G-symmetric space. For any sequence $\{x_n\}$ in X, $\lim_{n,m\to\infty} G_d(x_n,x_m,x_m) = 0$ iff $\lim_{n\to\infty} G_d(x_n,x_{n+1},x_{n+1}) = 0$.

Definition 2.9. Let (X,G_d) be a G-symmetric space.

(i) (X,G_d) is G_d- complete if for every G_d- Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} G_d(x_n,x,x) = 0$.

(ii) $f : X \to X$ is G_d-continuous if $\lim_{n\to\infty} G_d(x_n,x,x) = 0$ implies $\lim_{n\to\infty} G_d(fx_n,fx,fx) = 0$.

Let (X,G_d) be a G-symmetric space. We denote the family of all nonempty closed and bounded subsets of X by CB(X).

Definition 2.10. Two mappings $T,S : X \to CB(X)$ are said to be weak-contraction if there exists $0 < \alpha < 1$ such that

$$H_{G_d}(Tx,Sy,Sy) \leq \alpha M(x,y,y)$$

and

$$H_{G_d}(Tx,Tx,Sy) \leq \alpha M(x,x,y)$$

for all $x,y \in X$, where

$$M(x,y,y) = max\{G_d(x,y,y), G_d(x,Tx,Tx), G_d(y,Sy,Sy), G_d(x,Sy,Sy), G_d(y,Tx,Tx)\}$$

and

$$M(x,x,y) = max\{G_d(x,x,y), G_d(x,x,Tx), G_d(y,y,Sy), G_d(x,x,Sy), G_d(y,y,Tx)\}.$$

Definition 2.11. A mapping $T : X \to CB(X)$ are said to be weak-contraction if there exists $0 < \alpha < 1$ such that

$H_{G_d}(Tx,Ty,Ty) \leq \alpha N(x,y,y)$

and

$H_{G_d}(Tx,Tx,Ty) \leq \alpha N(x,x,y)$

for all $x,y \in X$, where

$N(x,y,y) = max\{G_d(x,y,y), G_d(x,Tx,Tx), G_d(y,Ty,Ty), G_d(x,Ty,Ty), G_d(y,Tx,Tx)\}$

and

$N(x,x,y) = max\{G_d(x,x,y), G_d(x,x,Tx), G_d(y,y,Ty), G_d(x,x,Ty), G_d(y,y,Tx)\}$

where H_{G_d} denotes the Hausdorff G-symmetric on CB(X) induced by G_d, that is

$H_{G_d}(A,B,B) = max\{Sup_{x \in A} G_d(x,B,B), Sup_{y \in B} G_d(y,A,A)\},$

for all $A,B \in CB(X)$.

Moutawakil [16] gave a generalization of the well-known Nadler multivalued contraction fixed point to the setting of symmetric spaces. Chandra and Bhatt [4] proved a fixed point theorem for generalized contractions under restrictive conditions in symmetric spaces. Joshi et al. [10] proved some fixed point theorems for multivalued contractive mappings in symmetric spaces. Bhatt et al. [2] established some fixed point theorems for occasionally weakly compatible mappings in symmetric spaces. Jungck and Rhoades [11] proved some fixed point theorems for occasionally weakly compatible maps satisfying certain contractive conditions in symmetric spaces. Rouhani and Moradi [20] established the existence and uniqueness of fixed and coincidence points of multivalued generalized φ-weak contractive mappings on complete metric spaces. The aim of this paper, is to prove some existence and uniqueness of fixed point of multivalued contraction maps and common fixed theorems for occasionally weakly compatible mappings satisfying multivalued weak - contractive conditions in G-symmetric spaces.

3. Main results

Theorem 3.1. *Let* (X, G_d) *be a* G_d-*complete G-symmetric space which satisfies* W_4 *and* W_5 *such that*

(i) the map $f : X \to R$ *defined by* $f(x) = G_d(x,Tx,Tx)$, $x \in X$, *is lower semi-continuous;*

(ii) $T : X \to C(X)$ *a multivalued mapping such that :*

$D(Tx, Ty, Ty) \leq kG_d(x, y, y), \, k \in [0, 1),$ (3.1)

for all $x, y \in X$. Then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $x_1 \in X$ and $\alpha \in (k, 1)$. Since Tx_1 is nonempty, there exists $x_2 \in Tx_1$ such that $G_d(x_1, x_2, x_2) = G_d(x_1, Tx_1, Tx_1)$. In view of (3.1) we obtain,

$$G_d(x_2, Tx_2, Tx_2) \leq D(Tx_1, Tx_2, Tx_2)$$

$$\leq kG_d(x_1, x_2, x_2)$$

$$< \alpha G_d(x_1, x_2, x_2).$$

Similarly there exists $x_3 \in Tx_2$ such that $G_d(x_2, x_3, x_3) = G_d(x_2, Tx_2, Tx_2)$ and

$$G_d(x_3, Tx_3, Tx_3) \leq D_{G_d}(Tx_2, Tx_3, Tx_3)$$

$$\leq kG_d(x_2, x_3, x_3)$$

$$< \alpha G_d(x_2, x_3, x_3).$$

Consequently, we obtain a sequence $\{x_n\}_{\geq 1}^{\infty}$ in X satisfying $x_{n+1} \in Tx_n$ such that $G_d(x_n, x_{n+1}, x_{n+1}) = G_d(x_n, Tx_n, Tx_n)$ and have

$$G_d(x_n, Tx_n, Tx_n) \leq D_{G_d}(Tx_{n-1}, Tx_n, Tx_n)$$

$$\leq kG_d(x_{n-1}, x_n, x_n)$$

$$< \alpha G_d(x_{n-1}, x_n, x_n)$$

$$< (\alpha)^{n-1} G_d(x_1, x_2, x_2).$$

This implies that $\lim_{n \to \infty} G_d(x_n, Tx_n, Tx_n) = 0$. Since $G_d(x_n, x_{n+1}, x_{n+1}) = G_d(x_n, Tx_n, Tx_n)$ then we have $\lim_{n \to \infty} G_d(x_n, x_{n+1}, x_{n+1}) = 0$ and with (W_5), $\{x_n\}$ is a Cauchy sequence. Hence there exists $u \in X$ such that $x_n \to u$. Also by (i) where $f(u) = G_d(u, Tu, Tu)$ we obtain $0 \leq G_d(u, Tu, Tu) \leq \lim_{n \to \infty} G_d(x_n, Tx_n, Tx_n) = 0$. Since T is closed and (X, G_d) is complete then $u \in Tu$. Thus u is a fixed point of T.

Remark 3.2. *Theorem 2.1 is an extension of Theorem 2.2 in [16] in the G-symmetric spaces.*

Corollary 3.3. *Let (X, G_d) be a G_d-bounded and G_d-complete G-symmetric space which satisfies W_4 and $T : X \to C(X)$ be a multivalued mapping such that:*

$$G_d(Tx,Ty,Ty) \leq kG_d(x,y,y), \tag{3.2}$$

for all $x, y \in X$ and $k \in [0,1)$. Then there exist $u \in X$ such that $u \in Tu$.

The proof of corollary 3.2 follows from Theorem 2.1. The following Lemma is needed in the proof of the next theorem and we refer to Nadler [18] for its proof.

Lemma 3.4. *If $A, B \in CB(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that*
$$G_d(a,b,b) \leq H_{G_d}(A,B,B) + \varepsilon.$$

Theorem 3.5. *Let T and S be multivalued weak-contraction mapping of a G-symmetric space X such that the pair $\{T,S\}$ are occasionally weakly compatible. If for all $x, y \in X$*

$$H_{G_d}(Tx, Sy, Sy) \leq \alpha M(x,y,y) \tag{3.3}$$

and

$$H_{G_d}(Tx, Tx, Sy) \leq \alpha M(x,x,y) \tag{3.4}$$

where $0 \leq \alpha < 1$ and $\beta = \alpha + \varepsilon < 1$. Then there exists a point $x \in X$ such that $x \in Tx$ and $x \in Sx$ (i.e. T and S have a common fixed point in X).

Proof. Since $\{T,S\}$ are occasionally weakly compatible then there exist $x \in X$ such that $Tx \in Sx$. Suppose there exist $x_1, x_2 \in X$ such that $x_1 \in Tx_1$ and $x_2 \in Sx_2$ with $TSx_2 \subseteq STx_1$ then using (3.3) and Lemma 3.4 we show that $x_1 = x_2$. On the contrary we assume that $x_1 \neq x_2$ and have,

$$
\begin{aligned}
G_d(x_1,x_2,x_2) &\leq H_{G_d}(Tx_1, Sx_2, Sx_2) + \varepsilon M(x_1,x_2,x_2) \\
&\leq \alpha M(x_1,x_2,x_2) + \varepsilon M(x_1,x_2,x_2) \\
&= \beta \, M(x_1,x_2,x_2) \\
&= \beta \, max\{G_d(x_1,x_2,x_2), G_d(x_1,Tx_1,Tx_1), G_d(x_2,Sx_2,Sx_2), G_d(x_1,Sx_2,Sx_2), \\
&\qquad G_d(x_2,Tx_1,Tx_1)\} \\
&\leq \beta \, max\{G_d(x_1,x_2,x_2), G_d(x_1,x_1,x_1), G_d(x_2,x_2,x_2), G_d(x_1,x_2,x_2), \\
&\qquad G_d(x_2,x_1,x_1)\} \\
&\leq \beta \, max\{G_d(x_1,x_2,x_2), G_d(x_2,x_1,x_1)\}.
\end{aligned}
$$

Case (i)

Suppose $max\{G_d(x_1,x_2,x_2), G_d(x_2,x_1,x_1)\} = G_d(x_1,x_2,x_2)$. Then $G_d(x_1,x_2,x_2) \leq \beta \, G_d(x_1,x_2,x_2)$.

Case (ii)

If $\max\{G_d(x_1,x_2,x_2),G_d(x_2,x_1,x_1)\} = G_d(x_2,x_1,x_1)$. Then

$$G_d(x_1,x_2,x_2) \leq \beta\ G_d(x_2,x_1,x_1) \tag{3.5}$$

Similarly, using (3.4), we obtain

$G_d(x_1,x_1,x_2)$

$\leq H_{G_d}(Tx_1,Tx_1,Sx_2) + \varepsilon M(x_1,x_1,x_2)$

$\leq \alpha M(x_1,x_1,x_2) + \varepsilon M(x_1,x_1,x_2)$

$= \beta\ M(x_1,x_1,x_2)$

$= \beta\ max\{G_d(x_1,x_1,x_2),G_d(x_1,x_1,Tx_1),G_d(x_2,x_2,Sx_2),G_d(x_1,x_1,Sx_2),\quad G_d(x_2,x_2,Tx_1)\}$

$\leq \beta\ max\{G_d(x_1,x_1,x_2),G_d(x_1,x_1,x_1),G_d(x_2,x_2,x_2),G_d(x_1,x_1,x_2),$

$G_d(x_2,x_2,x_1)\}$

$\leq \beta\ max\{G_d(x_1,x_1,x_2),G_d(x_2,x_2,x_1)\}$

Case (i)

Suppose $\max\ \{G_d(x_1,x_1,x_2),G_d(x_2,x_2,x_1)\} = G_d(x_1,x_1,x_2)$ then

$G_d(x_1,x_1,x_2) \leq \beta G_d(x_1,x_1,x_2)$

Case(ii)

If $\max\ \{G_d(x_1,x_1,x_2),G_d(x_2,x_2,x_1)\} = G_d(x_2,x_2,x_1)$ then

$G_d(x_1,x_1,x_2) \leq \beta\ G_d(x_2,x_2,x_1) \tag{3.6}$

Combining (3.5) and (3.6) with $G_d(4)$ yields

$G_d(x_1,x_2,x_2) \leq \beta G_d(x_1,x_2,x_2)$

Since $\beta < 1$ then $x_1 = x_2$. This implies that $x_1 \in Tx_1$ and $x_1 \in Sx_1$. Hence T and S have a common fixed point.

Remarks 3.6. *Theorem 2.5 is an analogy result of Theorem 3.1 in [20] in the setting of G-symmetric spaces.*

If $T = S$ in Theorem 2.5, then we have the following corollary.

Corollary 3.7. *Let (X, G_d) be a G_d-complete G-symmetric space and let $T : X \to CB(X)$ be a multivalued weak-contraction mapping such that for all $x, y \in X$*

$$H_{G_d}(Tx, Ty, Ty) \leq \alpha N(x, y, y), \tag{3.7}$$

where $0 \leq \alpha < 1$. Then there exists a point $x \in X$ such that $x \in Tx$.

Example 3.8. *Let $X = [0, 1]$ be endowed with the G-symmetric . Let $S, T : X \to CB(X)$ be defined by $Tx = [0, \frac{x}{6}]$ and $Sy = [\frac{y}{6}]$.*

$$
\begin{aligned}
H_{G_d}(Tx, Sy, Sy) &= max\{(\frac{y}{6} - \frac{x}{6})^2, (\frac{y}{6})^2\} \\
&\leq \frac{1}{3} max\{(y - x)^2, (y - \frac{y}{6})^2\} \\
&\leq \frac{1}{3} max\{G_d(x, y, y), G_d(y, Sy, Sy)\} \\
&\leq \frac{1}{3} M(x, y, y).
\end{aligned}
$$

So, *T* and *S* have a common fixed point $(x = 0)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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