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# UNIQUE COMMON FIXED POINT FOR TWO MAPPINGS WITH KANNAN-CHATTERJEA TYPE CONDITIONS ON CONE METRIC SPACES OVER BANACH ALGEBRAS WITHOUT NORMALITY

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**Abstract.** In this paper, we discuss the existence problems of points of coincidence and common fixed points for two mappings satisfying generalized Kannan-Chatterjea type conditions on cone metric spaces over Banach algebras without the assumption of normality and give some unique fixed point theorems. The obtained results generalize and improve the corresponding conclusions in the literature.

**Keywords:** Cone metric spaces with Banach algebras; Coincidence point; Common fixed point; Kannan-Chatterjea type condition.

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## 1. Introduction

In 2007, cone metric spaces were reviewed by Huang and Zhang, as a generalization of metric spaces (see [1]). The distance $d(x,y)$ of two elements $x$ and $y$ in a cone metric space $X$ is defined to be a vector in an ordered Banach space $E$, quite different from that which is defined a non-negative real numbers in general metric space. In 2011, Beg A, Azam A and Arshad M([2]) introduced the concept of topological vector space-valued cone metric spaces, where

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the ordered Banach space in the definition of cone metric spaces is replaced by a topological vector space.

Recently, some authors investigated the problems of whether cone metric spaces are equivalent to metric spaces in terms of the existence of fixed points of the mappings and successfully established the equivalence between some fixed point results in metric spaces and in (topological vector space-valued) cone metric spaces, see [3-6]. Actually, they showed that any cone metric space $(X,d)$ is equivalent to a usual metric space $(X,d^*)$, where the real-metric function $d^*$ is defined by a nonlinear scalarization function $\xi_e$ (see [4]) or by a Minkowski function $q_e$(see[5]). After that, some other interesting generalizations were developed, see [7].

In 2013, Liu and Xu [8] introduced the concept of cone metric spaces over Banach algebras, replacing a Banach space $E$ by a Banach algebra $\mathscr{A}$ as the underlying spaces of cone metric spaces. And the authors in [8-10] discussed and obtained Banach fixed point theorem, Kannan type fixed point theorem, Chatterjea type fixed point theorem and ćirić type fixed point theorem in cone metric spaces over Banach algebras. Especially, the authors in [10] gave an example to show that fixed point results of mappings in this new space are indeed more different than the standard results of come metric spaces presented in literature.

In this paper, constructing three different contractive conditions, we obtain unique point of coincidence and unique common fixed point theorems for two mappings on cone metric spaces over Banach algebras without the assumption of normality and give unique fixed point theorems. The obtained results generalize and improve the corresponding conclusions in the literature.

## 2. Preliminaries

Let $\mathscr{A}$ always be a Banach algebra. That is, $\mathscr{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties(for all $x,y,z \in \mathscr{A}$, $\alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;
2. $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\| xy \| \leq \| x \| \| y \|$ .

Here, we shall assume that $\mathscr{A}$ has a unit (i.e., a multiplicative identity) $e$ such that $ex = xe = x$ for all $x \in \mathscr{A}$. an element $x \in \mathscr{A}$ said to be invertible if there is an inverse element $y \in \mathscr{A}$ such that $xy = yx = e$. The inverse of $x$ is denoted by $x^{-1}$. For more detail, we refer to [11].

We say that $\{x_1, x_2, \cdots, x_n\} \subset \mathscr{A}$ commute if $x_i x_j = x_j x_i$ for all $i, j \in \{1, 2, \cdots, n\}$.

**Proposition 2.1.** **([11])** *Let $\mathscr{A}$ be a Banach algebra with a unit $e$, and $x \in \mathscr{A}$. If the spectral radius $r(x)$ of $x$ is less than $1$, i.e.,*

$$r(x) = \lim_{n\to\infty} \| x^n \|^{\frac{1}{n}} = \inf_{n\to\infty} \| x^n \|^{\frac{1}{n}} < 1.$$

*Then $(e-x)$ is invertible. Actually,*

$$(e-x)^{-1} = \sum_{i=0}^{+\infty} x^i.$$

**Remark 2.1.** 1) $r(x) \leq \| x \|$ for any $x \in \mathscr{A}$(see [11]).

2) In Proposition 2.1, if $r(x) < 1$ is replaced by $\| x \| < 1$, then the conclusion remains true.

A subset $P$ of a Banach algebra $\mathscr{A}$ is called a cone if

1. $P$ is nonempty closed and $\{0, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers $\alpha, \beta$;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{0\}$.

Where 0 denotes the null of the Banach algebra $\mathscr{A}$.

For a given cone $P \subset \mathscr{A}$, we can define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. $x < y$ stand for $x \leq y$ and $x \neq y$. While $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of $P$. A cone $P$ is called solid if $\operatorname{int} P \neq \emptyset$.

The cone $P$ is called normal if there is a number $M > 0$ such that for all $x, y \in \mathscr{A}$.

$$0 \leq x \leq y \implies \| x \| \leq M \| y \| .$$

The least positive number satisfying the above is called the normal constant of $P$.

Here, we always assume that $P$ is a solid and $\leq$ is the partial ordering with respect to $P$.

**Definition 2.1.** **([1, 9-10])** Let $X \neq \emptyset$. Suppose that the mapping $d : X \times X \to \mathscr{A}$ satisfies

1. $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

2. $d(x,y) = d(y,x)$ for all $x,y \in X$;

3. $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then $d$ is called a cone metric on $X$ and $(X,d)$ is called a cone metric space(over a Banach algebra $\mathscr{A}$).

**Remark 2.2.** The examples of cone metric spaces(over a Banach algebra $\mathscr{A}$) can be found in [8-10].

**Definition 2.2.** ([1, 8]) Let $(X,d)$ be a cone metric space over a Banach algebra $\mathscr{A}$, $x \in X$ and $\{x_n\}$ a sequence in $X$. Then:

1. $\{x_n\}$ converges to $x$ whenever for each $c \in \mathscr{A}$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n,x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

2. $\{x_n\}$ is Cauchy sequence whenever for each $c \in \mathscr{A}$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n,x_m) \ll c$ for all $n,m \geq N$.

3. $(X,d)$ is a complete cone metric space if every Cauchy sequence is convergent.

**Definition 2.3.** ([12-13]) Let $P$ is a solid cone in a Banach space $\mathscr{A}$. A sequence $\{u_n\} \subset P$ is a $c$-sequence if for each $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.

**Proposition 2.2.** ([12]) *Let $P$ is a solid cone in a Banach space $\mathscr{A}$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in $P$. If $\{x_n\}$ and $\{y_n\}$ are c-sequences and $\alpha,\beta > 0$, then $\{\alpha x_n + \beta y_n\}$ is a c-sequence.*

**Proposition 2.3.** ([12]) *Let $P$ is a solid cone in a Banach algebra $\mathscr{A}$ and $\{x_n\}$ a sequence in $P$. Then the following conditions are equivalent:*

*(1) $\{x_n\}$ is a c-sequence;*

*(2) for each $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $x_n < c$ for all $n \geq n_0$;*

*(3) for each $c \gg 0$ there exists $n_1 \in \mathbb{N}$ such that $x_n \leq c$ for all $n \geq n_1$.*

**Proposition 2.4.** ([10]) *Let $P$ is a solid cone in a Banach algebra $\mathscr{A}$. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in $P$. Then $\{ku_n\}$ is a c-sequence.*

**Proposition 2.5.** ([10]) *Let $\mathscr{A}$ be a Banach algebra with a unit e, P a cone in $\mathscr{A}$ and $\leq$ be the semi-order generated by the cone P. The following assertions hold true:*

*(i) For any $x, y \in \mathscr{A}$, $a \in P$ with $x \leq y$, $ax \leq ay$;*

*(ii) For any sequences $\{x_n\}, \{y_n\} \subset \mathscr{A}$ with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, where $x, y \in \mathscr{A}$, we have $x_n y_n \to xy$ as $n \to \infty$.*

**Proposition 2.6.** ([10]) *Let $\mathscr{A}$ be a Banach algebra with a unit e, P a cone in $\mathscr{A}$ and $\leq$ be the semi-order generated by the cone P. Let $\lambda \in P$. If the spectral radius $r(\lambda)$ of $\lambda$ is less than 1, then the following assertions hold true:*

*(i) Suppose that x is invertible and that $x^{-1} > 0$ implies $x > 0$, then for any integer $n \geq 1$, we have $\lambda^n \leq \lambda \leq e$.*

*(ii) For any $u > 0$, we have $u \nleq \lambda u$, i.e., $\lambda u - u \notin P$.*

*(iii) If $\lambda \geq 0$, then $(e-\lambda)^{-1} \geq 0$.*

**Proposition 2.7.** ([10]) *Let $(X, d)$ be a complete cone metric space over a Banach algebra $\mathscr{A}$ and P a solid cone in $\mathscr{A}$. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to $x \in X$, then we have*

*(i) $\{d(x_n, x)\}$ is a c-sequence.*

*(ii) For any $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$ is a c-sequence.*

**Lemma 2.1** ([14]) *If E is a real Banach space with a cone P and if $a \leq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = 0$.*

**Lemma 2.2.** ([15]) *If E is a real Banach space with a cone P and if $0 \leq u \ll c$ for all $0 \ll c$, then $u = 0$.*

**Lemma 2.3.** ([15]) *If E is a real Banach space with a solid cone P and if $\| x_n \| \to 0$ as $n \to \infty$, then for any $0 \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.*

**Lemma 2.4.** ([10]) *If $\mathscr{A}$ is a Banach algebra and $k \in \mathscr{A}$ with $r(k) < 1$, then $\| k^n \| \to 0$ as $n \to \infty$.*

**Lemma 2.5.** ([10]) *Let $\mathscr{A}$ be a Banach algebra and $x, y \in \mathscr{A}$. If x and y commute, then*

*(i) $r(xy) \leq r(x)r(y)$;*

*(ii) $r(x+y) \leq r(x) + r(y)$;*

*(iii)* $| r(x) - r(y) | \leq r(x-y)$.

**Lemma 2.6.** ([10]) *Let $\mathscr{A}$ be a Banach algebra and $\{x_n\}$ a sequence in $\mathscr{A}$. Suppose that $\{x_n\}$ converge to $x \in \mathscr{A}$ and that $x_n$ and $x$ commute for all $n$, then $r(x_n) \to r(x)$ as $n \to \infty$.*

**Lemma 2.7.** ([10]) *If $k \in \mathscr{A}$ and $0 \leq r(k) < 1$, then $r((e-k)^{-1}) \leq (1-r(k))^{-1}$.*

**Definition 2.4.** ([16]) Two mappings $f, g : X \to X$ are weakly compatible if, for every $x \in X$, $fgx = gfx$ whenever $fx = gx$.

**Definition 2.5.** ([16]) Let $f, g : X \to X$ be two mappings. If $w = fx = gx$ for some $x, w \in X$, then $x$ is called a coincidence point of $f$ and $g$, and $w$ is a point of coincidence of $f$ and $g$.

**Lemma 2.8.** ([16]) *If $f, g : X \to X$ are weakly compatible and have a unique point of coincidence $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.*

## 3. Main results

**Lemma 3.1.** *If $P$ be a solid cone in $\mathscr{A}$ and $\{\alpha, \beta, \gamma\} \subset \mathscr{A}$ commute and $r(\gamma) < 1$, then*

$$r\Big((e-\gamma)^{-1}(\alpha+\beta)\Big) \leq \frac{r(\alpha+\beta)}{1-r(\gamma)} \leq \frac{r(\alpha)+r(\beta)}{1-r(\gamma)}.$$

**Proof.** By Proposition 2.1, we have

$$(e-\gamma)^{-1} = \sum_{i=0}^{\infty} \gamma^i.$$

Since $\{\alpha, \beta, \gamma\}$ commute, so $\{(e-\gamma)^{-1}, \alpha, \beta\}$ commute. By Lemma 2.5 and Lemma 2.7,

$$r\Big((e-\gamma)^{-1}(\alpha+\beta)\Big) \leq r\Big((e-\gamma)^{-1}\Big) r(\alpha+\beta) \leq \frac{r(\alpha+\beta)}{1-r(\gamma)} \leq \frac{r(\alpha)+r(\beta)}{1-r(\gamma)}.$$

**Remark 3.1.** Since $r(e) = 1$ and $r(0) = 0$, so Lemma 3.1 reduce to Lemma 2.7 if $\alpha = e, \beta = 0$ or $\alpha = 0, \beta = e$. Hence Lemma 3.1 ia a generalization of Lemma 2.7. And by Lemma 3.1,

$$r\Big((e-\gamma)^{-1}\Big) \leq \frac{1}{1-r(\gamma)} \leq \frac{r(\alpha)+r(\beta)}{1-r(\gamma)}$$

for all $\alpha, \beta, \gamma \in \mathscr{A}$ with $r(\gamma) < 1$ and $\alpha + \beta = e$ and $\{\alpha, \beta, \gamma\}$ commute.

**Lemma 3.2. (Cauchy Principle)** *Let $(X,d)$ be a cone metric space over a Banach algebra $\mathscr{A}$, $P$ a solid cone in $\mathscr{A}$ and $k \in P$ with $r(k) < 1$. If a sequence $\{x_n\} \subset X$ satisfies that*

$$d(x_{n+1},x_{n+2}) \leq kd(x_n,x_{n+1}), \forall n = 0,1,2,\cdots.$$

*Then $\{x_n\}$ is a Cauchy sequence.*

**Proof.** By mathematical induction,

$$d(x_{n+1},x_{n+2}) \leq k^{n+1}d(x_0,x_1), \forall n = 0,1,2,\cdots,$$

hence for any $n,p \in \mathbb{N}$, using Proposition 2.1, we have

$$d(x_n,x_{n+p}) \leq \sum_{i=n}^{n+p-1} d(x_i,x_{i+1}) \leq \sum_{i=n}^{n+p-1} k^i d(x_0,x_1) \leq k^n \sum_{i=0}^{\infty} k^i d(x_0,x_1) = k^n(e-k)^{-1}d(x_0,x_1).$$

Since $r(k) < 1$, so $\| k^n \| \to 0$ as $n \to \infty$ by Lemma 2.4, hence $\{k^n(e-k)^{-1}d(x_0,x_1)\}$ is a $c$-sequence by Lemma 2.3 and Definition 2.3 and Proposition 2.4 and Proposition 2.6. Therefore for any $c \gg 0$ there is a $N \in \mathbb{N}$ such that $k^n(e-k)^{-1}d(x_0,x_1) \ll c$ for all $n > N$, hence

$$d(x_n,x_{n+p}) \leq k^n(e-k)^{-1}d(x_0,x_1) \ll c, \forall n > N.$$

This shows that $\{x_n\}$ is a Cauchy sequence.

**Remark 3.2.** the condition '$k \in P$ with $r(k) < 1$' in Lemma 3.2 can be replaced by '$k \in P$ with $\| k \| < 1$' by Remark 2.1.

**Lemma 3.3.** *Let $(X,d)$ be a cone metric space over a Banach algebra $\mathscr{A}$, $P$ a solid cone in $\mathscr{A}$ and $\{x_n\} \subset X$ a sequence. If $\{x_n\}$ is convergent, then the limits of $\{x_n\}$ is unique.*

**Proof.** Suppose that $x,y \in X$ are both limits of $\{x_n\}$, then for each $c \gg c$ there exist $N \in \mathbb{N}$ such that for all $n > N$,

$$d(x_n,x) \ll \frac{c}{2},\ \ d(x_n,y) \ll \frac{c}{2},$$

hence for $n > N$,

$$d(x,y) \leq d(x_n,x) + d(x_n,y) \ll \frac{c}{2} + \frac{c}{2} = c.$$

so $x = y$ by Lemma 2.2, this shows that $x$ is the unique limit of $\{x_n\}$.

At first, we obtain a unique common fixed point theorem for generalized Banach-Kannan type mappings.

**Theorem 3.1.** *Let $(X,d)$ be a cone metric space over a Banach algebra $\mathscr{A}$ and $P$ a solid cone in $\mathscr{A}$ and $f,g : X \to X$ two mappings satisfying $fX \subset gX$. Suppose that for each $x,y \in X, x \neq y$,*

$$d(fx,fy) \leq \alpha d(gx,gy) + \beta[d(gx,fx) + d(gy,fy)] + \gamma u(x,y), \tag{3.1}$$

*where $u(x,y) \in \{d(gx,fy), d(gy,fx)\}$ and $\{\alpha,\beta,\gamma\} \subset P$ satisfies $r(\alpha)+2r(\beta)+2r(\gamma) < 1$ and commute. If $fX$ or $gX$ is complete, then $f,g$ have a unique point of coincidence. Furthermore, if $f$ and $g$ are weakly compatible, then $f,g$ have a unique common fixed point.*

**Proof.** Take an $x_0 \in X$ and construct two sequence $\{x_n\}$ and $\{y_n\}$ in $X$ by $fX \subset gX$ as follows

$$y_n = fx_n = gx_{n+1}, \forall n = 0,1,2,\cdots. \tag{3.2}$$

If there exists $n$ such that $x_n = x_{n+1}$, then $y_n = fx_n = gx_n$, hence $y_n$ is the point of coincidence of $f$ and $g$. So from now on, we can assume that $x_n \neq x_{n+1}, \forall n = 0,1,2,\cdots$. For any fixed $n \in \mathbb{N}$, by (3.1),

$$\begin{aligned} d(y_{n+1},y_{n+2}) &= d(fx_{n+1},fx_{n+2}) \\ &\leq \alpha d(gx_{n+1},gx_{n+2}) + \beta[d(gx_{n+1},fx_{n+1}) + d(gx_{n+2},fx_{n+2})] + \gamma u(x_{n+1},x_{n+2}), \end{aligned}$$

i.e.,

$$d(y_{n+1},y_{n+2}) \leq \alpha d(y_n,y_{n+1}) + \beta[d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2})] + \gamma u(x_{n+1},x_{n+2}), \tag{3.3}$$

where

$$u(x_{n+1},x_{n+2}) \in \{d(gx_{n+1},fx_{n+2}), d(gx_{n+2},fx_{n+1})\} = \{d(y_n,y_{n+2}),0\}.$$

If $u(x_{n+1},x_{n+2}) = d(y_n,y_{n+2})$, then using (3.3), we obtain

$$d(y_{n+1},y_{n+2}) \leq \alpha d(y_n,y_{n+1}) + \beta[d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2})] + \gamma[d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2})],$$

hence

$$[e-(\beta+\gamma)]d(y_{n+1},y_{n+2}) \leq (\alpha+\beta+\gamma)d(y_n,y_{n+1}). \tag{3.4}$$

If $u(x_{n+1},x_{n+2}) = 0$, then by (3.3),

$$[e-\beta]d(y_{n+1},y_{n+2}) \leq (\alpha+\beta)d(y_n,y_{n+1}). \tag{3.5}$$

Since $\{\alpha,\beta,\gamma\} \subset P$, so $e-(\beta+\gamma) \leq e-\beta$ and $\alpha+\beta \leq \alpha+\beta+\gamma$, hence (3.5) implies (3.4). Therefore in any case, (3.4) holds for all $n \in \mathbb{N}$. And since $r(\beta+\gamma) \leq r(\beta)+r(\gamma) \leq r(\alpha)+$

$2r(\beta)+2r(\gamma)<1$ by Lemma 2.5, so $[e-(\beta+\gamma)]$ is invertible and $[e-(\beta+\gamma)]^{-1}\succeq 0$ by Proposition 2.6 and $\frac{r(\alpha)+r(\beta)+r(\gamma)}{1-r(\beta+\gamma)}<1$. Let $k=[e-(\beta+\gamma)]^{-1}(\alpha+\beta+\gamma)$, then $r(k)\leq 1$ by Lemma 3.1, hence by (3.4), we have

$$(y_{n+1},y_{n+2})\preceq kd(y_n,y_{n+1}),\forall n\in\mathbb{N}. \tag{3.6}$$

Therefore $\{y_n\}$ is a Cauchy sequence by Lemma 3.2.

Suppose that $gX$ is complete, then there exist $u,p\in X$ such that $y_n\to u=gp$ as $n\to\infty$. Using (3.1), we have

$$d(y_n,fp)=d(fx_n,fp)\preceq \alpha d(gx_n,gp)+\beta[d(gx_n,fx_n)+d(gp,fp)]+\gamma u(x_n,p),$$

i.e.,

$$d(y_n,fp)\preceq \alpha d(y_{n-1},u)+\beta[d(y_{n-1},y_n)+d(u,fp)]+\gamma u(x_n,p), \tag{3.7}$$

where

$$u(x_n,p)\in\{d(gx_n,fp),d(gp,fx_n)\}=\{d(y_{n-1},fp),d(u,y_n)\}.$$

If $u(x_n,p)=d(y_{n-1},fp)$, then from (3.7),

$$d(y_n,fp)\preceq \alpha d(y_{n-1},u)+\beta[d(y_{n-1},y_n)+d(u,fp)]+\gamma d(y_{n-1},fp),$$

which implies that

$$d(y_n,fp)\preceq \alpha d(y_{n-1},u)+\beta[d(y_{n-1},y_n)+d(u,y_n)+d(y_n,fp)]+\gamma[d(y_{n-1},y_n)+d(y_n,fp)],$$

hence

$$[e-(\beta+\gamma)]d(y_n,fp)\preceq \alpha d(y_{n-1},u)+\beta[d(y_{n-1},y_n)+d(u,y_n)]+\gamma d(y_{n-1},y_n),$$

so

$$d(y_n,fp)\preceq[e-(\beta+\gamma)]^{-1}\{\alpha d(y_{n-1},u)+\beta[d(y_{n-1},y_n)+d(u,y_n)]+\gamma d(y_{n-1},y_n)\},\forall n. \tag{3.8}$$

Since $\{y_n\}$ is Cauchy and $y_n\to u$ as $n\to\infty$, so the right-hand side of (3.8) is a $c$-sequence by Proposition 2.2, Proposition 2.4 and Proposition 2.7, hence for each $c\gg 0$ there exists $N_1\in\mathbb{N}$ such that for all $n\geq N_1$,

$$d(y_n,fp)\preceq[e-(\beta+\gamma)]^{-1}\{\alpha d(y_{n-1},u)+\beta[d(y_{n-1},y_n)+d(u,y_n)]+\gamma d(y_{n-1},y_n)\}\ll c. \tag{3.9}$$

If $u(x_n, p) = d(u, y_n)$, then from (3.7),

$$d(y_n, fp) \leq \alpha d(y_{n-1}, u) + \beta[d(y_{n-1}, y_n) + d(u, fp)] + \gamma d(u, y_n),$$

which implies that

$$d(y_n, fp) \leq \alpha d(y_{n-1}, u) + \beta[d(y_{n-1}, y_n) + d(u, y_n) + d(y_n, fp)] + \gamma d(u, y_n)],$$

hence

$$[e-\beta]d(y_n, fp) \leq \alpha d(y_{n-1}, u) + \beta[d(y_{n-1}, y_n) + d(u, y_n)] + \gamma d(u, y_n)),$$

so

$$d(y_n, fp) \leq [e-\beta]^{-1}\{\alpha d(y_{n-1}, u) + \beta[d(y_{n-1}, y_n) + d(u, y_n)] + \gamma d(u, y_n)\}, \forall n. \tag{3.10}$$

Similarly, the right-hand side of (3.10) is a $c$-sequence, hence for each $c \gg 0$ there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$d(y_n, fp) \leq [e-\beta]^{-1}\{\alpha d(y_{n-1}, u) + \beta[d(y_{n-1}, y_n) + d(u, y_n)] + \gamma d(u, y_n)\} \ll c. \tag{3.11}$$

Summing up (3.9) and (3.11) implies that for each $c \gg 0$ there exists $N \geq \max\{N_1, N_2\}$ such that

$$d(y_n, fp) \ll c, \ \forall n > N. \tag{3.12}$$

Hence $y_n \to fp$ as $n \to \infty$, therefore $u = gp = fp$ by Lemma 3.3. If $fX$ is complete, then there exist $u, p_0, p \in X$ such that $y_n \to u = fp_0 = gp$. Similarly, we obtain $u = fp = gp$. Hence in any case, $u = fp = gp$, so $u$ is a point of coincidence of $f$ and $g$. If $v$ is also a point of coincidence of $f$ and $g$, then there exists $q \in X$ such that $v = fq = gq$. By (3.1),

$$d(u, v) = d(fp, fq) \leq \alpha d(gp, gq) + \beta[d(gp, fp) + d(gq, fq)] + \gamma u(p, q) = \alpha d(u, v) + \gamma u(p, q), \tag{3,13}$$

where $u(p, q) \in \{d(gp, fq), d(gq, fp)\} = \{d(u, v)\}$. Hence (3.13) reduce to

$$d(u, v) \leq (\alpha + \gamma)d(u, v) \Longrightarrow [e - (\alpha + \gamma)]d(u, v) \leq 0.$$

Since $r(\alpha + \gamma) < 1$ implies that $[e - (\alpha + \gamma)]^{-1} \geq 0$, so $d(u, v) = 0$, i.e., $u = v$. Hence $u$ is the unique point of coincidence point of $f$ and $g$. If $f$ and $g$ are weakly compatible, then $u$ is the unique common fixed point of $f$ and $g$ by Lemma 2.8.

Using Theorem 3.1, we give two fixed point theorems:

**Corollary 3.1.** *Let $(X,d)$ be a cone metric space over a Banach algebra $\mathscr{A}$ and $P$ a solid cone in $\mathscr{A}$ and $f : X \to X$ a mapping satisfying $fX = f^2X$. Suppose that for each $x,y \in X$,*

$$d(fx, fy) \leq \alpha d(f^2x, f^2y) + \beta[d(f^2x, fx) + d(f^2y, fy)] + \gamma u(x,y),$$

*where $u(x,y) \in \{d(f^2x, fy), d(f^2y, fx)\}$ and $\{\alpha, \beta, \gamma\} \subset P$ satisfies $r(\alpha) + 2r(\beta) + 2r(\gamma) < 1$ and commute. If $fX$ is complete, then $f$ has a unique fixed point.*

**Corollary 3.2.** *Let $(X,d)$ be a complete cone metric space over a Banach algebra $\mathscr{A}$ and $P$ a solid cone in $\mathscr{A}$ and $g : X \to X$ a onto mapping. Suppose that for each $x,y \in X, x \neq y$,*

$$d(x,y) \leq \alpha d(gx, gy) + \beta[d(gx,x) + d(gy,y)] + \gamma u(x,y),$$

*where $u(x,y) \in \{d(gx,y), d(gy,x)\}$ and $\{\alpha, \beta, \gamma\} \subset P$ satisfies $r(\alpha) + 2r(\beta) + 2r(\gamma) < 1$ and commute. Then $g$ has a unique fixed point.*

Modifying the proof of Theorem 3.1, we can obtain the following unique common fixed point theorem for generalized Banach-Chatterjea-type mappings:

**Theorem 3.2.** *Let $(X,d)$ be a cone metric space over a Banach algebra $\mathscr{A}$ and $P$ a solid cone in $\mathscr{A}$ and $f, g : X \to X$ two mappings satisfying $fX \subset gX$. Suppose that for each $x,y \in X, x \neq y$,*

$$d(fx, fy) \leq \alpha d(gx, gy) + \beta v(x,y) + \gamma[d(gx, fy) + d(gy, fx)], \tag{3.14}$$

*where $u(x,y) \in \{d(gx, fx), d(gy, fy)\}$ and $\{\alpha, \beta, \gamma\} \subset P$ satisfies $r(\alpha) + 2r(\beta) + 2r(\gamma) < 1$ and commute. If $fX$ or $gX$ is complete, then $f,g$ have a unique point of coincidence. Furthermore, if $f$ and $g$ are weakly compatible, then $f,g$ have a unique common fixed point.*

**Proof.** Adopting up the method similar to the proof of Theorem 3.1, we can construct two sequence $\{x_n\}$ and $\{y_n\}$ in $X$ satisfying

$$y_n = fx_n = gx_{n+1}, \forall n = 0, 1, 2, \cdots$$

and prove that the limit $u$ of the Cauchy sequence $\{y_n\}$ is a point of coincidence of $f$ and $g$. If $v$ is also a point of coincidence of $f$ and $g$, there there exist $p, q \in X$ satisfying $u = fp = gp$ and

$v = fq = gq$. By (3.14),

$$d(u,v) = d(fp,fq) \leq \alpha d(gp,gq) + \beta v(p,q) + \gamma[d(gp,fq) + d(gq,fp)] = (\alpha+2\gamma)d(u,v) + \beta v(p,q),$$

where $v(p,q) \in \{d(gp,fp), d(gq,fq)\} = 0$. Hence we have

$$[e-(\alpha+2\gamma)]d(u,v) \leq 0.$$

But $r(\alpha+2\gamma) \leq r(\alpha)+2r(\gamma) < 1$, hence $[e-(\alpha+2\gamma)]^{-1} \geq 0$. So $d(u,v) = 0$ by Proposition 2.6, therefore $u$ is the unique point of coincidence of $f$ and $g$. If $f$ and $g$ are weakly compatible, then $u$ is the unique common fixed point of $f$ and $g$ by Lemma 2.8.

**Remark 3.3.** If $\beta = \gamma = 0$ or $\alpha = \gamma = 0$ in Theorem 3.1 and $\beta = \gamma = 0$ or $\alpha = \beta = 0$ in Theorem 3.2 respectively, then Theorem 3.1 and Theorem 3.2 reduce to the corresponding results in [17]; if $g = 1_X$ and $\beta = \gamma = 0$ or $\alpha = \gamma = 0$ in Theorem 3.1 and $g = 1_X$ and $\beta = \gamma = 0$ or $\alpha = \beta = 0$ in Theorem 3.2 respectively, then Theorem 3.1 and Theorem 3.2 reduce to the corresponding results in [8,10], i.e., Banach contraction principle, Kannam fixed point theorem and Chatterjea fixed point theorem on cone metric spaces with Banach algebras. If $g = 1_X$ and $\alpha = \beta = 0$ in Theorem 3.1 and $g = 1_X$ and $\alpha = \gamma = 0$ in Theorem 3.2 respectively, then Theorem 3.1 and Theorem 3.2 are both particular forms of the corresponding result( i,e., Ćirić type fixed point theorem) in [9].

Now, we give an unique common fixed point theorem for Banach-semi-Ćirić type mappings.

**Theorem 3.3** *Let $(X,d)$ be a cone metric space over a Banach algebra $\mathscr{A}$ and $P$ a solid cone in $\mathscr{A}$ and $f,g : X \to X$ two mappings satisfying $fX \subset gX$. Suppose that for each $x,y \in X, x \neq y$,*

$$d(fx,fy) \leq \alpha d(gx,gy) + \beta v(x,y) + \gamma u(x,y), \tag{3.15}$$

*where $v(x,y) \in \{d(gx,fx), d(gy,fy)\}$, $u(x,y) \in \{d(gx,fy), d(gy,fx)\}$ and $\{\alpha,\beta,\gamma\} \subset P$ satisfies $r(\alpha)+2r(\beta)+2r(\gamma) < 1$ and commute. If $fX$ or $gX$ is complete, then $f,g$ have a unique point of coincidence. Furthermore, if $f,g$ is weakly compatible, then $f,g$ have a unique common fixed point.*

**Proof.** Consider two sequences $\{x_n\}$ and $\{y_n\}$ in Theorem 3.1 and Theorem 3.2. By (3.15),

$$d(y_{n+1},y_{n+2}) = d(fx_{n+1},fx_{n+2}) \leq \alpha d(y_n,y_{n+1}) + \beta v(x_{n+1},x_{n+2}) + \gamma v(x_{n+1},x_{n+2}), \tag{3.16}$$

where

$$v(x_{n+1},x_{n+2}) \in \{d(gx_{n+1},fx_{n+1}),d(gx_{n+2},fx_{n+2})\} = \{d(y_n,y_{n+1}),d(y_{n+1},y_{n+2})\}, \quad (3.17)$$

$$u(x_{n+1},x_{n+2}) \in \{d(gx_{n+1},fx_{n+2}),d(gx_{n+2},fx_{n+1})\} = \{d(y_n,y_{n+2}),0\}. \quad (3.18)$$

Case 1. If $v(x_{n+1},x_{n+2}) = d(y_n,y_{n+1})$ and $u(x_{n+1},x_{n+2}) = d(y_n,y_{n+2})$, then by (3.16),

$$(e-\gamma)d(y_{n+1},y_{n+2}) \preceq (\alpha+\beta+\gamma)d(y_n,y_{n+1}), \quad (3.19)$$

Case 2. If $v(x_{n+1},x_{n+2}) = d(y_n,y_{n+1})$ and $u(x_{n+1},x_{n+2}) = 0$, then by (3.16),

$$d(y_{n+1},y_{n+2}) \preceq (\alpha+\beta)d(y_n,y_{n+1}), \quad (3.20)$$

Case 3. If $v(x_{n+1},x_{n+2}) = d(y_{n+1},y_{n+2})$ and $u(x_{n+1},x_{n+2}) = d(y_n,y_{n+2})$, then by (3.16),

$$[e-(\beta+\gamma)]d(y_{n+1},y_{n+2}) \preceq (\alpha+\gamma)d(y_n,y_{n+1}), \quad (3.21)$$

Case 4. If $v(x_{n+1},x_{n+2}) = d(y_{n+1},y_{n+2})$ and $u(x_{n+1},x_{n+2}) = 0$, then by (3.16),

$$(e-\beta)d(y_{n+1},y_{n+2}) \preceq \alpha d(y_n,y_{n+1}). \quad (3.22)$$

Since $e-(\beta+\gamma) \preceq e-\beta \preceq e$, $e-(\beta+\gamma) \preceq e-\gamma \preceq e$, $\alpha \preceq \alpha+\beta \preceq \alpha+\beta+\gamma$, $\alpha \preceq \alpha+\gamma \preceq \alpha+\beta+\gamma$, so combining (3.19)-(3.22), we have that for each $n$,

$$[e-(\beta+\gamma)]d(y_{n+1},y_{n+2}) \preceq (\alpha+\beta+\gamma)d(y_n,y_{n+1}). \quad (3.23)$$

So we obtain (3.6), hence $\{y_n\}$ is a Cauchy sequence.

Suppose that $gX$ is complete, then there exist $u,p \in X$ such that $y_n \to u = gp$ as $n \to \infty$. Using (3.15), we have

$$d(y_n,fp) = d(fx_n,fp) \preceq \alpha d(y_{n-1},u) + \beta v(x_n,p) + \gamma u(x_n,p), \quad (3.24)$$

where

$$v(x_n,p) \in \{d(y_{n-1},y_n),d(u,fp)\},\ u(x_n,p) \in \{d(y_{n-1},fp),d(u,y_n)\}.$$

Case i. If $v(x_n,p) = d(y_{n-1},y_n)$, $u(x_n,p) = d(y_{n-1},fp)$, then by (3.24),

$$d(y_n,fp) \preceq \alpha d(y_{n-1},u) + \beta d(y_{n-1},y_n) + \gamma d(y_{n-1},fp),$$

hence we have

$$[e-(\beta+\gamma)]d(y_n,fp) \le (e-\gamma)d(y_n,fp) \le \alpha d(y_{n-1},u)+\beta d(y_{n-1},y_n)+\gamma d(y_{n-1},y_n),$$

so

$$d(y_n,fp) \le [e-(\beta+\gamma)]^{-1}[\alpha d(y_{n-1},u)+\beta d(y_{n-1},y_n)+\gamma d(y_{n-1},y_n)]. \tag{3.25}$$

Case ii. If $v(x_n,p)=d(y_{n-1},y_n)$, $u(x_n,p)=d(u,y_n)$, then by (3.24),

$$[e-(\beta+\gamma)]d(y_n,fp) \le d(y_n,fp) \le \alpha d(y_{n-1},u)+\beta d(y_{n-1},y_n)+\gamma d(u,y_n),$$

so

$$d(y_n,fp) \le [e-(\beta+\gamma)]^{-1}[\alpha d(y_{n-1},u)+\beta d(y_{n-1},y_n)+\gamma d(u,y_n)]. \tag{3.26}$$

Case iii. If $v(x_n,p)=d(u,fp)$, $u(x_n,p)=d(y_{n-1},fp)$, then by (3.24),

$$d(y_n,fp) \le \alpha d(y_{n-1},u)+\beta d(u,fp)+\gamma d(y_{n-1},fp),$$

hence we have

$$[e-(\beta+\gamma)]d(y_n,fp) \le \alpha d(y_{n-1},u)+\beta d(y_n,u)+\gamma d(y_{n-1},y_n),$$

so

$$d(y_n,fp) \le [e-(\beta+\gamma)]^{-1}[\alpha d(y_{n-1},u)+\beta d(y_n,u)+\gamma d(y_{n-1},y_n)]. \tag{3.27}$$

Case iv. If $v(x_n,p)=d(u,fp)\}$, $u(x_n,p)=d(u,y_n)$, then by (3.24),

$$d(y_n,fp) \le \alpha d(y_{n-1},u)+\beta d(u,fp)+\gamma d(u,y_n),$$

hence we have

$$[e-(\beta+\gamma)]d(y_n,fp) \le (e-\beta)d(y_n,fp) \le \alpha d(y_{n-1},u)+\beta d(y_n,u)+\gamma d(u,y_n),$$

so

$$d(y_n,fp) \le [e-(\beta+\gamma)]^{-1}[\alpha d(y_{n-1},u)+\beta d(y_n,u)+\gamma d(u,y_n)]. \tag{3.28}$$

Using Proposition 2.2, Proposition 2.4 and Proposition 2.7, we know the right-hand sides of (3.25)-(3.28) are all $c$-sequences, hence $\{d(y_n,fp)\}$ is also a $c$-sequence by summing up cases (i)-(iv). So $y_n \to fp$ as $n \to \infty$, hence $fp=u=gp$. If $fX$ is complete, then there exist $u,p_0,p \in X$ such that $y_n \to u = fp_0 = gp$. Similarly, we obtain $u=fp=gp$. Hence in any case, $u=fp=$

$gp$, so $u$ is a point of coincidence of $f$ and $g$. Suppose there exist $v,q \in X$ such that $v = fq = gq$. By (3.15),

$$d(u,v) = d(fp,fq) \leq \alpha d(gp,gq) + \beta v(p,q) + \gamma u(p,q) = \alpha d(u,v) + \beta v(p,q) + \gamma u(p,q), \tag{3.29}$$

where

$$v(p,q) \in \{d(gp,fp), d(gq,fq)\} = \{0\},\ u(p,q) \in \{d(gp,fq), d(gq,fp)\} = \{d(u,v)\}.$$

Hence from (3.29),

$$[e-(\alpha+\gamma)]d(u,v) \leq 0.$$

Which implies that $u = v$ since $r(\alpha+\gamma) \leq r(\alpha)+r(\gamma) < 1$ implies $[e-(\alpha+\gamma)]$ is invertible and $[e-(\alpha+\gamma)]^{-1} \geq 0$. Hence $u$ is the unique point of coincidence of $f$ and $g$. Finally, If $f$ and $g$ are weakly compatible, then $u$ is the unique common fixed point of $f$ and $g$ by Lemma 2.8.

## Conflict of Interests

The author declares that there is no conflict of interests.

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