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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR MN-CONVEX FUNCTIONS

G. ZABANDAN

Department of Mathematics, Faculty of Mathematical Sciences and Computer,

Kharazmi University, 599 Taleghani Avenue, Tehran, Iran

Abstract. In this paper, we established new Hermite-Hadamard inequalities for MN-convex functions, where $M,N = A,G,H$. Some examples are given.

Keywords: Hermite-Hadamard inequality; MN-convex; Integral inequality.

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1. Introduction

Let $f:[a,b]\longrightarrow\mathbb{R}$ be a convex function, then the following inequality:

$$f(\frac{a+b}{2})\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{*}$$

is known as the Hermite-Hadamard inequality. In this paper we study several convexity, and deduce sharp integral inequalities similar to Hermite-Hadamard inequality. First we need the following definitions.

Definition 1.1. A function $M:(0,\infty)\times(0,\infty)\longrightarrow(0,\infty)$ called a mean if

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1) $M(x,y) = M(y,x)$,

2) $M(x,x) = x$,

3) $x < M(x,y) < y$, wherenver $x < y$,

4) $M(ax,ay) = \alpha M(x,y)$ for all $a > 0$.

Example 1.2.

1) The Arithmetic Mean $M(x,y) = A(x,y) = \dfrac{x+y}{2}$

2) The Geometric Mean $M(x,y) = G(x,y) = \sqrt{xy}$.

3) The Harmonic Mean $M(x,y) = H(x,y) = \dfrac{1}{A(\frac{1}{x},\frac{1}{y})}$.

4) The Logarithmic Mean $M(x,y) = L(x,y) = \begin{cases} \dfrac{x-y}{\ln x - \ln y} & x \neq y \\ x & x = y \end{cases}$.

Definition 1.3. Let I be a subinterval of $(0,\infty)$ and $f : I \longrightarrow (0,\infty)$ be a continuous function. f is called MN-convex if $f(M(x,y)) \leq N(f(x), f(y))$ for all $x, y \in I$.

Note that this definition reduces to usual convexity when $M = N = A$.

The authors in[1] theorem 2.4 showed that for $M, N = A, G, H$, the nine possible MN-convexity property reduces to a ordinary convexity by a simple change of variable.

Since f is continous, this the MN-convexity of f, that is

$$f(M(x,y);1-t,t) \leq N(f(x), f(y);1-t,t)$$

for every $x, y \in I, t \in [0,1]$ (see [6]). For example

(1) f is AG-convex, if for every $x, y \in I$, $t \in [0,1]$

$$f(tx + (1-t)y) \leq f^t(x) f^{1-t}(y)$$

(2) f is GH-convex, if for every $x, y \in I$, $t \in [0,1]$

$$f(x^t y^{1-t}) \leq \frac{f(x)f(y)}{(1-t)f(x) + tf(y)}$$

(3) f is HA-convex, if for every $x, y \in I$, $t \in [0,1]$

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

and etc.

Remark 1.4. Since $H(x,y) \leq G(x,y) \leq A(x,y)$, it follows that

(1) f is AH-convex$\Longrightarrow f$ if AG-convex $\Longrightarrow f$ is AA-convex.

(2) f is GH-convex $\Longrightarrow f$ is GG-convex $\Longrightarrow f$ is GA-convex.

(3) f is HH-convex $\Longrightarrow f$ is HG-convex $\Longrightarrow f$ is HA-convex.

Throughout of this paper $[a,b] \subset (0,\infty)$ and f is positive function.

2. Main results

Theorem 2.1. *Let f be a AH-convex function on $[a,b]$. Then the following inequalities hold:*

$$f(A(a,b)) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{G^2(f(a),f(b))}{L(f(a),f(b))}$$

Proof. Since f is AH-convex, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 f(tb+(1-t)a)dt &\leq \int_0^1 \frac{f(a)f(b)}{tf(a)+(1-t)f(b)}dt \\
&= \int_0^1 \frac{f(a)f(b)}{(f(a)-f(b))t+f(b)}dt \\
&= \frac{f(a)f(b)}{f(a)-f(b)}\ln((f(a)-f(b))t+f(b))\Big|_0^1 \\
&= \frac{f(a)f(b)}{f(a)-f(b)}(\ln f(a)-\ln f(b)) = \frac{G^2(f(a),f(b))}{L(f(a),f(b))}.
\end{aligned}$$

For the proof of left side, by AH-convexity of f, we have

$$\begin{aligned}
f(\frac{a+b}{2}) &= f\left(\frac{(1-t)a+tb+(1-t)b+ta}{2}\right) \\
&\leq \frac{2f(ta+(1-t)b)f(tb+(1-t)a)}{f(ta+(1-t)b)+f(tb+(1-t)a)}.
\end{aligned}$$

By integrating on $[0,1]$ and change of variable we get

$$\begin{aligned}
f(A(a,b)) = f(\frac{a+b}{2}) &\leq \int_0^1 \frac{2f(ta+(1-t)b)f(tb+(1-t)a)}{f(ta+(1-t)b)+f(tb+(1-t)a)}dt \\
&= \frac{1}{b-a}\int_a^b \frac{2f(x)f(a+b-x)}{f(a+b-x)+f(x)}dx.
\end{aligned}$$

Since $H(a,b) \le A(a,b)$ and $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, it follows that

$$\le \frac{1}{b-a}\int_a^b \frac{f(x)+f(a+b-x)}{2}dx = \frac{1}{b-a}\int_a^b f(x)dx.$$

□

Theorem 2.2. *Let f be a AG-convex function on $[a,b]$. Then the following inequalities hold:*

$$f(A(a,b)) \le \frac{1}{b-a}\int_a^b f(x)dx \le L(f(a),f(b))$$

Proof. By change of variable and AG-convexity of f, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &= \int_0^1 f(tb+(1-t)a)dt \le \int_0^1 f^t(b)f^{1-t}(a)dt \\ &= f(a)\int_0^1 \left(\frac{f(b)}{f(a)}\right)^t dt = \frac{f(a)}{\ln f(b)-\ln f(a)}\left(\frac{f(b)}{f(a)}\right)^t \Big|_0^1 \\ &= \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)} = L(f(b),f(a)).\end{aligned}$$

On the other hand, AG-convexity of f follows that

$$\begin{aligned}f(A(a,b)) = f(\frac{a+b}{2}) &= f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right) \\ &\le \sqrt{f(ta+(1-t)b)f(tb+(1-t)a)}.\end{aligned}$$

By integrating on $[0,1]$ and Holder's inequality we obtain

$$\begin{aligned}f(\frac{a+b}{2}) &\le \int_0^1 \sqrt{f(ta+(1-t)b}\sqrt{f(tb+(1-t)a}\,dt \\ &\le \left(\int_0^1 f(ta+(1-t)b)dt\right)^{\frac{1}{2}}\left(\int_0^1 f(tb+(1-t)a)dt\right)^{\frac{1}{2}} \\ &= \frac{1}{b-a}\int_a^b f(x)dx,\end{aligned}$$

because

$$\int_0^1 f(ta+(1-t)b)dt = \int_0^1 f(tb+(1-t)a)dt = \frac{1}{b-a}\int_a^b f(x)dx.$$

□

Corollary 2.3. *If f is a AH-convex function on $[a,b]$, then*

$$f(A(a,b)) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{G^2(f(a),f(b))}{L(f(a),f(b))} \le L(f(a),f(b)) \le A(f(a),f(b)).$$

The proof is obvious by Theorems 2.1, 2.2 and Remark 1.4.

Theorem 2.4. *Let f be a GH-convex function on $[a,b]$. Then the following inequalities hold:*

$$f(G(a,b)) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leq \frac{G^2(f(a),f(b))}{L(a,b)}.$$

Proof. By change of variable and GH-convexity of f, we have

$$\begin{aligned}
\int_a^b \frac{f(x)}{x}dx &= \int_0^1 \frac{f(a^{1-t}b^t)}{a^{1-t}b^t}a(\frac{b}{a})^t \ln\frac{b}{a}\,dt = \ln\frac{b}{a}\int_0^1 f(b^t a^{1-t})dt \\
&\leq \ln\frac{b}{a}\int_0^1 \frac{f(a)f(b)}{(1-t)f(b)+tf(a)}dt = \ln\frac{b}{a}\int_0^1 \frac{f(a)f(b)}{t(f(a)-f(b))+f(b)}dt \\
&= \frac{f(a)f(b)\ln\frac{b}{a}}{f(a)-f(b)}\ln[t(f(a)-f(b))+f(b)]|_0^1 \\
&= \frac{f(a)f(b)\ln\frac{b}{a}}{f(a)-f(b)}[\ln f(a) - \ln f(b)].
\end{aligned}$$

So

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leq \frac{G^2(f(a),f(b))}{L(f(a),f(b))}.$$

On the other hand we have

$$f(\sqrt{ab}) = f\left(\sqrt{(a^{1-t}b^t)(a^t b^{1-t})}\right) \leq \frac{2f(a^{1-t}b^t)f(a^t b^{1-t})}{f(a^{1-t}b^t)+f(a^t b^{1-t})}.$$

By integrating on $[0,1]$, we get

$$\begin{aligned}
f(\sqrt{ab}) &\leq \int_0^1 \frac{2f(a^{1-t}b^t)f(a^t b^{1-t})}{f(a^{1-t}b^t)+f(a^t b^{1-t})}dt \\
&\leq \frac{1}{2}\int_0^1 f(a^{1-t}b^t)dt + \frac{1}{2}\int_0^1 f(a^t b^{1-t})dt \\
&= \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx.
\end{aligned}$$

Because $H(a,b) \leq A(a,b)$ and $\int\limits_0^1 f(a^{1-t}b^t)dt = \int\limits_0^1 f(a^t b^{1-t})dt = \frac{1}{\ln b - \ln a}\int\limits_a^b \frac{f(x)}{x}dx.$ □

Theorem 2.5. *Let f be a GG-convex function on $[a,b]$. Then the following inequalities hold:*

i) $\frac{G(a,b)f(G(a,b))}{L(a,b)} \leq \frac{1}{b-a}\int\limits_a^b f(x)dx \leq \frac{L(bf(b),af(a))}{L(a,b)}$

ii) $f(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a}\int\limits_a^b \frac{f(x)}{x}dx \leq L(f(a),f(b)).$

Proof. (i) By change of variable and GG-cocnvexity of f we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &= \frac{1}{b-a}\int_a^1 f(a^{1-t}b^t)a(\ln\frac{b}{a})(\frac{b}{a})^t dt\\ &\le \frac{a\ln\frac{b}{a}}{b-a}\int_0^1 [f(a)]^{1-t}[f(b)]^t(\frac{b}{a})^t dt\\ &= \frac{af(a)\ln\frac{b}{a}}{b-a}\int_0^1\left(\frac{bf(b)}{af(a)}\right)^t dt\\ &= \frac{af(a)\ln\frac{b}{a}}{b-a}.\frac{1}{\ln\frac{bf(b)}{af(a)}}\left(\frac{bf(b)}{af(a)}\right)^t\Big|_0^1\\ &= \frac{af(a)(\ln b-\ln a)}{(b-a)(\ln bf(b)-\ln af(a))}.\frac{bf(b)-af(a)}{af(a)}\\ &= \frac{L(bf(b),af(a))}{L(a,b)}.\end{aligned}$$

So

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{L(bf(b),af(a))}{L(a,b)}.$$

On the other hand, by easy calculations we see that

$$\int_a^b f(x)dx = a\ln\frac{b}{a}\int_0^1 f(b^t a^{1-t})(\frac{b}{a})^t dt = b\ln\frac{b}{a}\int_0^1 f(b^{1-t}a^t)(\frac{a}{b})^t dt,$$

So

$$\begin{aligned}\int_a^b f(x)dx &= \left(\int_a^b f(x)dx\right)^{\frac{1}{2}}\left(\int_a^b f(x)dx\right)^{\frac{1}{2}}\\ &= \sqrt{ab}\ln\frac{b}{a}\left(\int_0^1 f(b^t a^{1-t})(\frac{b}{a})^t dt\right)^{\frac{1}{2}}\left(\int_0^1 f(b^{1-t}a^t)(\frac{a}{b})^t dt\right).\end{aligned}$$

By using Holder's inequality, we get

$$\ge \sqrt{ab}\ln\frac{b}{a}\left(\int_0^1\sqrt{f(b^t a^{1-t})}.\sqrt{f(b^{1-t}a^t)}\right)dt$$

Since f is GG-convex, it follows that

$$\begin{aligned}&\ge \sqrt{ab}\ln\frac{b}{a}\left(\int_0^1 f(\sqrt{b^t a^{1-t}b^{1-t}a^t})\right)dt\\ &= \sqrt{ab}\ln\frac{b}{a}\int_0^1 f(\sqrt{ab})\ dt = \sqrt{ab}\ln\frac{b}{a}f(\sqrt{ab}).\end{aligned}$$

So

$$\frac{1}{b-a}\int_a^b f(x)dx \geq \frac{\sqrt{ab}(\ln b - \ln a)}{b-a} f(\sqrt{ab})$$
$$= \frac{G(a,b)}{L(a,b)} f(G(a,b)).$$

(ii) We have

$$\int_a^b \frac{f(x)}{x}dt = \ln\frac{b}{a}\int_0^1 f(a^{1-t}b^t)dx \leq \ln\frac{b}{a}\int_0^1 (f(a))^{1-t}(f(b))^t dt$$
$$= (\ln\frac{b}{a})f(a)\int_0^1 (\frac{f(b)}{f(a)})^t dt = (\ln\frac{b}{a})f(a)\frac{1}{\ln\frac{f(b)}{f(a)}}(\frac{f(b)}{f(a)})^t\Big|_0^1$$
$$= (\ln\frac{b}{a})\frac{f(b)-f(a)}{\ln f(b) - \ln f(a)},$$

so

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} \leq \frac{f(b)-f(a)}{\ln f(b) - \ln f(a)} = L(f(a), f(b)).$$

On the other hand we have

$$f(\sqrt{ab}) = f(\sqrt{(b^t a^{1-t})(b^{1-t}a^t)}) \leq \sqrt{f(b^t a^{1-t})f(b^{1-t}a^t)}$$

By integrating on $[0,1]$ we obtain

$$f(\sqrt{ab}) \leq \int_0^1 \sqrt{f(b^t a^{1-t})f(b^{1-t}a^t)}dt \leq \int_0^1 \frac{f(b^t a^{1-t}) + f(b^{1-t}a^t)}{2}dt$$
$$= \frac{1}{2}\int_0^1 f(b^t a^{1-t})dt + \frac{1}{2}\int_0^1 f(b^{1-t}a^t)dt = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx,$$

because

$$\int_0^1 f(b^t a^{1-t})dt = \int_0^1 f(a^t b^{1-t})dt = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx.$$

□

Theorem 2.6. *Let f be a GA-convex function on $[a,b]$. Then the following inequalities hold:*

$$f(G(a,b)) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leq A(f(a), f(b)).$$

Proof. By change of variable and GA-convexity of f, we have

$$\int_a^b \frac{f(x)}{x}dx = \int_0^1 \frac{f(b^t a^{1-t})}{b^t a^{1-t}} a(\frac{b}{a})^t \ln\frac{b}{a}\, dt = \ln\frac{b}{a}\int_0^1 f(b^t a^{1-t})dt$$
$$\leq \ln\frac{b}{a}\int_0^1 (tf(b)+(1-t)f(a))dt = \ln\frac{b}{a}\int_0^1 [(f(b)-f(a))t+f(a)]dt$$
$$= \ln\frac{b}{a}\left[\frac{f(b)-f(a)}{2}+f(a)\right] = (\ln\frac{b}{a})\frac{f(b)+f(a)}{2}$$

so

$$\frac{1}{\ln b-\ln a}\int_a^b f(x)dx \leq A(f(a),f(b)).$$

On the other hand we have

$$f(\sqrt{ab}) = f\left(\sqrt{a^t b^{1-t} a^{1-t} b^t}\right) \leq \frac{f(a^t b^{1-t})+f(a^{1-t}b^t)}{2}$$

Thus

$$f(\sqrt{ab}) \leq \frac{1}{2}\int_0^1 f(a^t b^{1-t})dt + \frac{1}{2}\int_0^1 f(a^{1-t}b^t)dt$$

set $a^t b^{1-t} = x$, it follows that $dt = \frac{1}{\ln a-\ln b}(\frac{dx}{x})$. So

$$\int_0^1 f(a^t b^{1-t})dt = \frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx.$$

By a similar way we see that

$$\int_0^1 f(a^{1-t}b^t)dt = \frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx.$$

Hence

$$f(G(a,b)) = f(\sqrt{ab}) \leq \frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx.$$

□

Corollary 2.7. *If f is a GH-convex function on $[a,b]$, then*

$$f(G(a,b)) \leq \frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx \leq \frac{G^2(f(a),f(b))}{L(a,b)} \leq L(f(a),f(b)) \leq A(f(a),f(b)).$$

The proof is clear by theorems 2.4, 2.5, 2.6 and Remark 1.4.

Theorem 2.8. *Let f be a HH-convex function on $[a,b]$. Then the following inequalities hold:*

$$\frac{f(H(a,b))}{G^2(a,b)} \leq \frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{G^2(f(a),f(b))}{G^2(a,b)L(f(a),f(b))}$$

Proof. By change of variable $x = \dfrac{ab}{tb+(1-t)a}$, $dx = \dfrac{ab(a-b)dt}{(tb+(1-t)a)^2}$ we get

$$\begin{aligned}
\frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}dx &= \frac{1}{b-a}\int_1^0 f\left(\frac{ab}{tb+(1-t)a}\right)\frac{ab(a-b)}{(t(b-a)+a)^2}\frac{(t(b-a)+a)^2}{a^2b^2}dt \\
&= \frac{1}{ab}\int_0^1 f(\frac{ab}{tb+(1-t)a})dt \le \frac{1}{ab}\int_0^1 \frac{f(a)f(b)}{tf(a)+(1-t)f(b)}dt \\
&= \frac{f(a)f(b)}{ab}\int_0^1 \frac{dt}{t(f(a)-f(b))+f(b)} \\
&= \frac{f(a)f(b)}{ab(f(a)-f(b))}\ln[t(f(a)-f(b))+f(b)]\Big|_0^1 \\
&= \frac{f(a)f(b)}{ab(f(a)-f(b))}(\ln f(a)-\ln f(b)) = \frac{G^2(f(a),f(b))}{G^2(a,b)L(f(a),f(b))}.
\end{aligned}$$

For the proof of left side we have

$$\begin{aligned}
f(\frac{2ab}{a+b}) = f\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right) &= f\left(\frac{2}{(\frac{t}{a}+\frac{1-t}{b})+(\frac{t}{b}+\frac{1-t}{a})}\right) \\
&\le \frac{2f\left(\frac{ab}{tb+(1-t)a}\right)f\left(\frac{ab}{ta+(1-t)b}\right)}{f\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{ta+(1-t)b}\right)} \\
&\le \frac{1}{2}f\left(\frac{ab}{tb+(1-t)a}\right)+\frac{1}{2}f\left(\frac{ab}{ta+(1-t)b}\right)
\end{aligned}$$

because $H(a,b) \le A(a,b)$. By integrating we obtain

$$\begin{aligned}
f(\frac{2ab}{a+b}) &\le \frac{1}{2}\int_0^1 f(\frac{ab}{tb+(1-t)a})dt + \frac{1}{2}\int_0^1 f(\frac{ab}{ta+(1-t)b})dt \\
&= \frac{ab}{2(b-a)}\int_a^b \frac{f(x)}{x^2}dx + \frac{ab}{2(b-a)}\int_a^b \frac{f(x)}{x^2} = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2},
\end{aligned}$$

so

$$\frac{f(H(a,b))}{G^2(a,b)} \le \frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}dx.$$

□

Theorem 2.9. *Let f be a HG-convex function on $[a,b]$. Then the following inequalities hold:*

$$\frac{f(H(a,b))}{G^2(a,b)} \le \frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{L(f(a),f(b))}{G^2(a,b)}.$$

Proof. By change of variable $x = \dfrac{ab}{at+(1-t)b}$, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}dx &= \frac{1}{b-a}\int_0^1 f\left(\frac{ab}{at+(1-t)b}\right)\frac{ab(b-a)}{(t(a-b)+b)^2}\frac{(t(a-b)+b)^2}{a^2b^2}dt \\
&= \frac{1}{ab}\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt \le \frac{1}{ab}\int_0^1 f^t(b)f^{1-t}(a)dt \\
&= \frac{f(a)}{ab}\int_0^1 \left(\frac{f(b)}{f(a)}\right)^t dt = \frac{f(a)}{ab}.\frac{1}{\ln f(b)-\ln f(a)}\left(\frac{f(b)}{f(a)}\right)^t\Big|_0^1 \\
&= \frac{f(a)-f(b)}{ab(\ln f(a)-\ln f(b))} = \frac{L(f(a),f(b))}{G^2(a,b)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
f\left(\frac{2ab}{a+b}\right) = f\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right) &= f\left(\frac{2}{(\frac{t}{a}+\frac{1-t}{b})+(\frac{t}{b}+\frac{1-t}{a})}\right) \\
&\le \sqrt{f\left(\frac{ab}{tb+(1-t)a}\right)f\left(\frac{ab}{ta+(1-t)b}\right)} \\
&\le \frac{1}{2}f\left(\frac{ab}{tb+(1-t)a}\right)+\frac{1}{2}f\left(\frac{ab}{ta+(1-t)b}\right),
\end{aligned}$$

so

$$\begin{aligned}
f\left(\frac{2ab}{a+b}\right) &\le \frac{1}{2}\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt + \frac{1}{2}\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt \\
&= \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}.
\end{aligned}$$

Thus

$$\frac{f(H(a,b))}{G^2(a,b)} \le \frac{1}{b-a}\int_a^b \frac{f(x)}{x^2} \le \frac{L(f(a),f(b))}{G^2(a,b)}.$$

□

Theorem 2.10. *Let f be a HA-convex function on $[a,b]$. Then the following inequalities hold:*

$$\frac{f(H(a,b))}{G^2(a,b)} \le \frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{A(f(a),f(b))}{G^2(a,b)}.$$

Proof. We have

$$\begin{aligned}\frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}dx &= \frac{1}{ab}\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt \\ &\le \frac{1}{ab}\int_0^1 (tf(b)+(1-t)f(a))dt \\ &= \frac{1}{ab}\int_0^1 [t(f(b)-f(a))+f(a)]dt \\ &= \frac{1}{ab}\left[\frac{f(b)-f(a)}{2}+f(a)\right] = \frac{f(a)+f(b)}{2ab} \\ &= \frac{A(f(a),f(b))}{G^2(a,b)}.\end{aligned}$$

On the other hand, we have

$$f\left(\frac{2ab}{a+b}\right) \le \frac{1}{2}f\left(\frac{ab}{tb+(1-t)a}\right)+\frac{1}{2}f\left(\frac{ab}{ta+(1-t)b}\right).$$

Hence

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right) &\le \frac{1}{2}\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt+\frac{1}{2}\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt \\ &= \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2},\end{aligned}$$

so

$$\frac{f(H(a,b))}{G^2(a,b)} \le \frac{1}{b-a}\int_a^b \frac{f(x)}{x^2} \le \frac{A(f(a),f(b))}{G^2(a,b)}.$$

□

Corollary 2.11. *If f is a HH-convex function on $[a,b]$, then*

$$\frac{f(H(a,b))}{G^2(a,b)} \le \frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{G^2(f(a),f(b))}{G^2(a,b)L(f(a),f(b))} \le \frac{L(f(a),f(b))}{G^2(a,b)} \le \frac{A(f(a),f(b))}{G^2(a,b)},$$

or

$$f(H(a,b)) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{G^2(f(a),f(b))}{L(f(a),f(b))} \le L(f(a),f(b)) \le A(f(a),f(b)).$$

It is clear by Theorems 2.8, 2.9, 2.10 and Remark 1.4.

Example 2.12.

1) $f(x) = \Gamma(x)$ is AG-convex on $(0,\infty)$. By theorem 2.2 we have

$$\Gamma\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \Gamma(x)dx \le \frac{\Gamma(b)-\Gamma(a)}{\ln\Gamma(b)-\ln\Gamma(a)}$$

Especially for $b = a+1$ we get

$$\Gamma(a+\frac{1}{2}) \le \int_a^{a+1} \Gamma(x)dx \le \frac{(a-1)\Gamma(a)}{\ln a}$$

2) $f(x) = \Gamma(x)$ is GG-convex on $[1,\infty)$. By theorem 2.5 (i), (ii) we have

$$\frac{\sqrt{ab}\Gamma(\sqrt{ab})}{L(a,b)} \le \frac{1}{b-a}\int_a^b \Gamma(x)dx \le \frac{L(b\Gamma(b),a\Gamma(a))}{L(a,b)}$$

and

$$\Gamma(\sqrt{ab}) \le \frac{1}{\ln b-\ln a}\int_a^b \frac{\Gamma(x)}{x}dx \le L(\Gamma(a),\Gamma(b)).$$

3) $f(x) = \sec x$ is AH-convex on $(0,\frac{\pi}{2})$. Theorem 2.1 follows that

$$\sec(\frac{\pi}{6}) < \frac{3}{\pi}\int_0^{\frac{\pi}{3}} \frac{dx}{\cos x} < \frac{(\sec(0)\sec(\frac{\pi}{3})(\ln\sec\frac{\pi}{3}-\ln\sec 0)}{\sec(\frac{\pi}{3})-\sec(0)}$$

$$\frac{2}{\sqrt{3}} < \frac{3}{\pi}\frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}\Big|_0^{\frac{\pi}{3}} < \frac{2(\ln 2-\ln 1)}{2-1} = 2\ln 2$$

so

$$\frac{2}{\sqrt{3}} < \frac{3}{2\pi}\ln\frac{2+\sqrt{3}}{2-\sqrt{3}} < 2\ln 2,$$

or

$$\frac{2}{\sqrt{3}} < \frac{3}{\pi}\ln(2+\sqrt{3}) < 2\ln 2.$$

4) $f(x) = \psi(x)$ $(\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)})$ is GA-convex on $(0,\infty)$. Because $(x\psi'(x))' = \psi'(x) + x\psi''(x) < 0$, hence $x\psi'(x)$ is decreasing and $\psi(x)$ is GA-concave (see [1], Corollary 2.5 and [2], Lemma 2.9). So by theorem 2.6 we have

$$\psi(\sqrt{ab}) > \frac{1}{\ln b-\ln a}\int_a^b \frac{\psi(x)}{x}dx > \frac{\psi(a)+\psi(b)}{2}$$

5) $f(x)=e^{x^2}$ is HH-convex on $[0,\frac{\sqrt{3}}{2}]$. Because $\frac{x^2 f'(x)}{f^2(x)}=\frac{2x^3}{e^{x^2}}$ and

$$\left(\frac{x^2 f'(x)}{f^2(x)}\right)'=\frac{2x^2(3-2x^2)}{e^{x^2}}>0$$

(see [1], Corollary 2.5).

So by theorem 2.8 we have

$$e^{(\frac{2ab}{a^2+b^2})^2}<\frac{ab}{b-a}\int_a^b \frac{e^{x^2}}{x^2}dx<\frac{e^{a^2+b^2}(a^2-b^2)}{e^{a^2}-e^{b^2}}\quad (0<a<b<\frac{\sqrt{3}}{2})$$

6) $f(x)=\arctan x$ is HA-convex on $(0,\infty)$. Because

$$(x^2 f'(x))'=(\frac{x^2}{1+x^2})'=\frac{2x}{(1+x^2)^2}>0\quad \text{(see [1], Corollary 2.5)}$$

So by theorem 2.10 we have

$$\arctan\frac{2ab}{a^2+b^2}<\frac{ab}{b-a}\int_a^b \frac{\arctan x}{x^2}dx<\frac{\arctan a+\arctan b}{2}$$

By easy Calculations we see that

$$\begin{aligned}\int_a^b \frac{\arctan x}{x^2}dx &= \frac{\arctan a}{a}-\frac{\arctan b}{b}+\int_a^b \frac{dx}{x(1+x^2)}\\ &= \frac{\arctan a}{a}-\frac{\arctan b}{b}+(\ln x-\frac{1}{2}\ln(1+x^2))\Big|_a^b\\ &= \frac{\arctan a}{a}-\frac{\arctan b}{b}+\ln\frac{b}{a}-\ln\frac{\sqrt{1+b^2}}{\sqrt{1+a^2}}.\end{aligned}$$

Therefore

$$\arctan\frac{2ab}{a^2+b^2}<\frac{b\arctan a-a\arctan b}{b-a}+\frac{ab}{b-a}\ln\frac{b\sqrt{1+a^2}}{a\sqrt{1+b^2}}<\frac{\arctan a+\arctan b}{2}.$$

Conflict of Interests

The author declares that there is no conflict of interests.

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