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APPROXIMATING POSITIVE SOLUTIONS OF NONLINEAR FIRST ORDER ORDINARY QUADRATIC DIFFERENTIAL EQUATIONS WITH MAXIMA

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Abstract. In this paper, we consider the initial value problem for first order nonlinear quadratic differential equations with maxima and we study the existence and approximation of the solutions. The main results are related to a recent hybrid fixed point theorem of Dhage (2014) in partially ordered normed linear spaces.

Keywords: Partially ordered normed linear spaces; Differential equations with maxima; Hybrid fixed point theorem; Approximation of solutions.

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1. Introduction

The study of fixed point theorems for the contraction mappings in partially ordered metric spaces is initiated by Ran and Reurings [13] which are further continued by Nieto and Rodringuez-Lopez [7] and applied to boundary value problems of nonlinear first order ordinary

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differential equations for proving the existence results under certain monotonic conditions. Since then many mathe- maticians have established several fixed point theorems for different classes of contraction mappings in partially ordered metric spaces (see, for example [1], [3], [4], [10], [12]). In this paper we investigate the existence of solutions of quadratic differential equations with maxima in partially ordered spaces. More precisely, we consider the following equation,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right]+\lambda\left[\frac{x(t)}{f(t,x(t))}\right] &= g\big(t,\max_{a\le\xi\le t} x(\xi)\big) \\ x(a) &= x_0 \end{aligned}\right\} \tag{1}$$

for all $t \in I = [a,b]$, a,b,x_0 and $f,g \in I\times\mathbb{R}\to\mathbb{R}$ are continuous function.

For $\lambda \in \mathbb{R}, \lambda > 0$, where $f : I\times\mathbb{R}\to\mathbb{R}/0$ and $g : I\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

By the solution of the QDE(12),we mean a function $x \in C^1(I,\mathbb{R})$ that satisfies.

(i) $t \mapsto \frac{x}{f(t,x)}$ is continuous differentiable function $x \in \mathbb{R}$ and,

(ii) x-satisfies the equation in (12) on I,where $C(I,\mathbb{R})$ is a space of continuously differentiable real valued defined on I.

2. Preliminaries

We need the following notions and results.

Definition 2.1. A mapping $\mathscr{A} : X \to X$ is called *isotone* or *monotone nondecreasing* if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathscr{A}x \preceq \mathscr{A}y$ for all $x,y \in X$.

Definition 2.2. An operator $\mathscr{A}$ on a normed linear space X into itself is called *compact* if $\mathscr{A}(X)$ is a relatively compact subset of X. $\mathscr{A}$ is called *totally bounded* if for any bounded subset S of X, $\mathscr{A}(S)$ is a relatively compact subset of X. If $\mathscr{A}$ is continuous and totally bounded, then it is called *completely continuous* on X.

Definition 2.3. [Dhage 4] A mapping $\mathscr{A} : X \to X$ is called *partially continuous* at a point $a \in X$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathscr{A}x-\mathscr{A}a\| < \varepsilon$ whenever x is comparable to a and $\|x-a\| < \delta$. $\mathscr{A}$ called partially continuous on X if it is partially continuous at every point of it.

It is clear that if $\mathscr{A}$ is partially continuous on X, then it is continuous on every chain C contained in X.

Definition 2.4. [Dhage 4] An operator $\mathscr{A}$ on a partially normed linear space X into itself is called *partially bounded* if $A(C)$ is bounded for every chain C in X. $\mathscr{A}$ is called *uniformly partially bounded* if all chains $\mathscr{A}(C)$ in X are bounded by a unique constant. $\mathscr{A}$ is called *partially compact* if $\mathscr{A}(C)$ is a relatively compact subset of X for all totally ordered sets or chains C in X. $\mathscr{A}$ is called *partially totally bounded* if for any totally ordered and bounded subset C of X, $\mathscr{T}(C)$ is a relatively compact subset of X. If $\mathscr{A}$ is partially continuous and partially totally bounded, then it is called *partially completely continuous* on X.

Definition 2.5. [Dhage 4] The order relation $\preceq$ and the metric d on a non-empty set X are said to be *compatible* if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(X,\preceq,\|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Definition 2.6. Let $(X,\preceq,\|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathscr{A}:X\to X$ is called partially nonlinear $\mathscr{D}$-Lipschitz if there exists a D-function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\|\mathscr{A}x-\mathscr{A}y\|\le\psi(\|x-y\|) \tag{2}$$

for all comparable elements $x,y\in X$, where $\psi(0)=0$. If $\psi(r)=kr$, $k>0$, then $\mathscr{A}$ is called a partially Lipschitz with a Lipschitz constant k. If $k<1$, $\mathscr{A}$ is called a partially contraction with contraction constant k. Finally, $\mathscr{A}$ is called nonlinear $\mathscr{D}$-contraction if it is a nonlinear $\mathscr{D}$-Lipschitz with $\psi(r)<r$ for $r>0$.

Theorem 2.7. *Let $\big(X,\preceq,\|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in X. Let $A,B:X\to X$ be two nondecreasing operators such that*

(a) *A is partially bounded and partially nonlinear $\mathscr{D}$-Lipschitz with $\mathscr{D}$-function ψ_A,*

(b) *B is partially continuous and uniformly partially compact, and*

(c) $M\ \psi_{a<r,r>0}$,*where* $M = \sup\{\|B(C)\|\ C$ *is chain in* $x\ \}$

(d) *there exists an element* $x_0 \in X$ *such that* $x_0 \preceq Ax_0Bx_0$ *or* $x_0 \succeq Ax_0Bx_0$.

Then the operator equation $AxBx = x$ *has a solution* x^* *in* X *and the sequence* $\{x_n\}$ *of successive iterations defined by* $x_{n+1} = Ax_nBx_n$, $n = 0, 1, \ldots$, *converges monotonically to* x^*.

3. Main results

The QDE (1) is considered in the function space $C(I;\mathbb{R})$ of continuous real-valued functions defined on I. We define a norm $\|\cdot\|$ and the order relation $\preceq$ in $C(I;\mathbb{R})$ by

$$\|x\| = \sup_{t\in I} |x(t)| \tag{3}$$

and

$$x \leq y \Leftrightarrow x(t) \leq y(t) \tag{4}$$

for all $t \in I$ respectively. Clearly, $C(I;\mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach algebra $C(I;\mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzella-Ascolli theorem.

Lemma 3.1. *Let* $C((I,\mathbb{R}), \leq, \|.\|)$ *be a partially ordered Banach space with the norm* $\|\cdot\|$ *and the order relation* $\leq$ *defined by (3) and (4) respectively. Then* $\|\cdot\|$ *and* $\leq$ *are compatible in every partially compact subset of* $C(I;\mathbb{R})$.

Proof. Let S be a partially compact subset of $C(I;\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq x_3(t) \cdots \cdot$$

for each $t \in \mathbb{R}_+$ Suppose that a subsequence $\{x_{nk}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x$in S. Then the subsequence $\{x_{nk}\}_{n\in\mathbb{N}}$ of the monotone real sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t) \in \mathbb{R}$ for each $t \in \mathbb{R}_+$. This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzella-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result $\leq$ and $\|.\|$ are compatible in S. This completes the proof. We need the following definition in what follows.

Definition 3.2. A function $u \in C^1(I,\mathbb{R})$ is said to be a lower solution of the QDE (1) if the function $t \mapsto \frac{u(t)}{f(t,u(t)}$ is continuously differentiable and satisfies

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{u(t)}{f(t,u(t))}\right]+\lambda\left[\frac{u(t)}{f(t,u(t))}\right] \leq g\big(t,\max_{a\leq\xi\leq t} u(\xi)\big) \\ u(a)=x_0 \end{aligned}\right\} \tag{5}$$

all $u \in I$. We consider the following set of assumptions:

(C_0) $t \mapsto \frac{x}{f(t,x)}$ is injection for each $t \in I$,
(C_1) f defines a function $f : I\times\mathbb{R} \to \mathbb{R}$,
(C_2) There exist a constant $M_f > 0$ such that $0 < f(t,x) \leq M_f$ for all $t \in I$ and $x \in \mathbb{R}$,
(C_3) There exist a D-Function ϕ, such that, $o \leq f(t,x)-f(t,y) \leq \phi(x-y)$
for all $t \in I$ and $x,y \in \mathbb{R}$,$x \leq y$,
(C_4) g defines a function $g : I\times\mathbb{R} \to \mathbb{R}$,
(C_5) There exists a constant $M_g > 0$ such that $g(t,\max_{a\leq\xi\leq t} x(\xi)) \leq M_g$ for all $t \in I$,
(C_6) $g(t,x)$ is increaseing in x for all $t \in I$,
(C_7) The QDE (1)has a lower solution $u \in C^1(I,\mathbb{R})$.

Lemma 3.3. *Suppose that hypothesis (C_0) holds. Then a function $x \in C(I,\mathbb{R})$ is a solution of the QDE (1), if and only if it is a solution of the nonlinear quadratic integral equation (in short QIE),*

$$x(t) = [f(t,x(t))]\left(\frac{ce^{-\lambda t}}{f(a,x_0)}+\int_a^t e^{-\lambda(t-s)}g(s,\max_{a\leq s\leq t} x(s)\right)ds \tag{6}$$

for all $t \in I = [a,b]$,where $C = x_0e^{\lambda a}$.

Theorem 3.4. *Assume that hypotheses(C_1)-(C_7) hold.Furthermore, assume that*

$$\left(\frac{x_0}{f(a,x_0)}+M_g b\right)\phi(r) < r, r < 0 \tag{7}$$

then the QDE (12) has a positive solution x^* defined on I and the sequence $\{x_n\}$ of successive approximations defined by

$$x_{n+1}(t) = [f(t,x_n(t))]\left(\frac{ce^{-\lambda t}}{f(a,x_0)} + \int_a^t e^{-\lambda(t-s)} g(s, \max_{a\le s\le t} x_n(s)\right)ds \tag{8}$$

for $t \in \mathbb{R}$, where $x_1 = u$, converges monotonically to x^*.

Proof. Set $X = C(I,\mathbb{R})$ and define two operators A and B on X by

$$Ax(t) = f(t,x(t)), t \in I \tag{9}$$

and

$$Bx(t) = \frac{ce^{-\lambda t}}{f(a,x_0)} + \int_a^t e^{-\lambda(t-s)} g(s, \max_{a\le s\le t} x(s))ds, t \in I. \tag{10}$$

From the continuity of the integral, it follows that A and B define the maps $A,B : X \to X$. The QDE (1) is equivalent to the operator equation

$$Ax(t)Bx(t) = x(t), t \in I. \tag{11}$$

We shall show that the operators A and A satisfy all the conditions of Theorem (2.1). This is achieved in the series of following steps.

Step I: *A and B are nondecreasing on X.*

Let $x,y \in X$ be such that $x \le y$. Then by hypothesis (C_3), we obtain

$$Ax(t) = f(t,x(t)) \ge f(t,y(t)) = Ay(t)$$

for all $t \in I$. This shows that A is nondecreasing operator on X into X. Similarly using hypothesis (C3), it is shown that the operator B is also nondecreasing on X into itself. Thus, A and B are nondecreasing positive operators on X into itself

Step II: *A is partially bounded and partially D-Lipschitz on X.*

Let $x \in X$ be arbitrary. Then by (C2),

$$|Ax(t)| = |f(t,x(t))| \le M_f$$

for all $t \in I$. Taking supremum over t, we obtain $\|Ax\| \le M_f$ and so, A is bounded. This further implies that A is partially bounded on E.

Next, let $x,y \in I$ be such that $x \geq y$. Then
$|Ax(t) - Ay(t)| = |f(t,x(t) - f(t,y(t)| \leq \phi(|x(t) - y(t)|)$ for all $t \in I$ Taking supremum over t, we obtain $\|Ax - Ay\| \leq \phi(\|x - y\|)$ for all $x,y \in X$, $x \geq y$. Hence, A is a partial nonlinear *D*-Lipschitz on X which further implies that A is a partially continuous on X.

Step III: *B is a partially continuous on* X.

Let $\{x_n\}$ be a sequence in a chain C of X such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \frac{ce^{-\lambda t}}{f(a,x_0)} + \int_a^t e^{-\lambda(t-s)} g(s, \max_{a\leq s\leq t} x(s))ds \\
&= \frac{ce^{-\lambda t}}{f(a,x_0)} + \int_a^t e^{-\lambda(t-s)} [\lim_{n\to\infty} g(s, \max_{a\leq s\leq t} x(s))]ds \\
&= \frac{ce^{-\lambda t}}{f(a,x_0)} + \int_a^t e^{-\lambda(t-s)} g(s, \max_{a\leq s\leq t} x(s))ds \\
&= Bx(t).
\end{aligned}$$

for all $t \in I$. This shows that Bx_n converges monotonically to Bx pointwise on I. Next, we will show that $\{Bx_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in X. Let $t_1, t_2 \in I$ with $t_1 < t_2$. Then

$$\begin{aligned}
\left|Bx_n(t_2) - By_n(t_1)\right| \leq & \left|\frac{ce^{-\lambda t_1}}{f(a,x_0)} - \frac{ce^{-\lambda t_2}}{f(a,x_0)}\right| \\
& + \Big| \int_a^{t_1} e^{-\lambda(t_1-s)} g(s, \max_{a\leq s\leq t} x_n(s))ds - \int_a^{t_1} e^{-\lambda(t_2-s)} g(s, \max_{a\leq s\leq t} x_n(s))ds \Big| \\
& + \Big| \int_a^{t_1} e^{-\lambda(t_1-s)} g(s, \max_{a\leq s\leq t} x_n(s))ds - \int_a^{t_2} e^{-\lambda(t_2-s)} g(s, \max_{a\leq s\leq t} x_n(s))ds \Big| \\
& \leq \Big| \frac{ce^{-\lambda t_1}}{f(a,x_0)} - \frac{ce^{-\lambda t_2}}{f(a,x_0)} \Big| + \Big| \int_a^{t_1} \left|e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)}\right| |g(s, \max_{a\leq s\leq t} x_n(s))| ds \Big| \\
& + \Big| \int_{t_2}^{t_1} |g(s, \max_{a\leq s\leq t} x_n(s))ds| \Big| \\
& \leq \left| \Big| \frac{ce^{-\lambda t}}{f(a,x_0)} - \frac{ce^{-\lambda t}}{f(a,x_0)} \Big| + M_g \int_a^b \left| e^{-\lambda(t-s)} - e^{-\lambda(t-s)} \right| \right| ds \\
& + M_g |t_1 - t_2| \\
& \to 0 \qquad as \qquad t_2 - t_1
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $Bx_n \to Bx$ is uniform and hence B is partially continuous on X.

Step IV: *B is uniformly partially compact operator on X.*

Let C be an arbitrary chain in X. We show that $B(C)$ is a uniformly bounded and equicontinuous set in X. First, we show that $B(C)$ is uniformly bounded. Let $y \in B(C)$ be any element. Then there is an element $x \in C$, such that $y = Bx$. Now, by hypothesis (C2),

$$\begin{aligned}
|y(t)| &\leq \left| \frac{ce^{-\lambda t}}{f(a,x_0)} + \int_a^t e^{-\lambda(t-s)} g(s, \max_{a \leq s \leq t} x_n(s)) ds \right| \\
&\leq \left| \frac{ce^{-\lambda t}}{f(a,x_0)} \right| + \left| \int_a^t e^{-\lambda(t-s)} g(s, \max_{a \leq s \leq t} x_n(s)) ds \right| \\
&\leq \left| \frac{x_0}{f(a,x_0)} \right| + \left| \int_a^b |g(s, \max_{a \leq s \leq t} x_n(s)) ds| \right| \\
&\leq \left| \frac{a}{f(a,x_0)} \right| + M_g b = M
\end{aligned}$$

for all $t \in I$. Taking supremum over t, we obtain $\|y\| = \|Bx\| \leq M$ for all $y \in B(C)$. Hence, B(C) is a uniformly bounded subset of X. Moreover, $\|B(C)\| \leq M$ for all chains C in X. Hence, B is a uniformly partially bounded operator on X.

Next, we will show that $B(C)$ is an equicontinuous set in X. Let $t_1, t_2 \in I$ with $t_1 < t_2$. Then, for any $y \in B(C)$, one has

$$\begin{aligned}
|y(t_2) - y(t_1)| &= |Bx(t_2) - Bx(t_1)| \\
&+ \left| \int_a^{t_1} e^{-\lambda(t_1-s)} g(s, \max_{a \leq s \leq t} x(s)) ds - \int_a^{t_1} e^{-\lambda(t_2-s)} g(s, \max_{a \leq s \leq t} x(s)) ds \right| \\
&+ \left| \int_a^{t_1} e^{-\lambda(t_1-s)} g(s, \max_{a \leq s \leq t} x(s)) ds - \int_a^{t_2} e^{-\lambda(t_2-s)} g(s, \max_{a \leq s \leq t} x(s)) ds \right| \\
&\leq \left| \frac{ce^{-\lambda t_1}}{f(a,x_0)} - \frac{ce^{-\lambda t_2}}{f(a,x_0)} \right| + \left| \int_a^{t_1} \left| e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)} \right| |g(s, \max_{a \leq s \leq t} x(s))| ds \right| \\
&+ \left| \int_{t_2}^{t_1} |g(s, \max_{a \leq s \leq t} x(s)) ds| \right| \\
&\leq \left| \frac{ce^{-\lambda t_1}}{f(a,x_0)} - \frac{ce^{-\lambda t_2}}{f(a,x_0)} \right| + M_g \int_a^b \left| e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)} \right| ds \\
&+ M_g |t_1 - t_2| \\
&\to 0 \qquad as \qquad t_2 - t_1 \to 0
\end{aligned}$$

uniformly for all $y \in B(C)$. Hence$B(C)$ is an equicontinuous subset of X. Now, $B(C)$ is a uniformly bounded and equicontinuous set of functions in X, so it is compact. Consequently, B is a uniformly partially compact operator on X into itself.

Step V: *u satisfies the operator inequality $u \leq AuBu$.*

By hypothesis (C7), the QDE (1) has a lower solution u defined on I. Then, we have

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] + \lambda\left[\frac{x(t)}{f(t,x(t))}\right] &= g\big(t, \max_{a\leq\xi\leq t} x(\xi)\big) \\ x(a) &= x_0 \end{aligned}\right\} \tag{12}$$

for all $t \in I$. Multiplying the above inequality (11) by the integrating factor $e^{\lambda}t$, we obtain

$$\left(e^{\lambda t}\frac{u(t)}{f(t,u(t))}\right)' \leq e^{\lambda t} g(t,u(t)) \tag{13}$$

for all $t \in I$. A direct integration of (12) from a to t yields

$$u(t) \leq [f(t,u(t)]\left(\frac{ce^{-\lambda t}}{f(a,x_0)} + \int_a^{t_1} e^{-\lambda(t-s)} g(s, \max_{a\leq u\leq t} u(s))ds\right) \tag{14}$$

for all $t \in J$. From definitions of the operators A and B, it follows that $u(t) \leq Au(t)Bu(t)$, for all $t \in I$. Hence$u \leq AuBu$.

Step VI: *D-fuction ϕ is satisfies the growth condition $M\phi_A(r), r > 0$*

Finally, the D-function ϕ of the operator A satisfies the inequality given in hypothesis (d) of Theorem 2.1. Now from the estimate given in Step IV, it follows that

$$M\phi_A(r) \leq \left(\frac{x_0}{f(a,x_0)} + M_g b\right)\phi(r) < r,$$

for all $r > 0$.

Thus, A and B satisfy all the conditions of Theorem 2.1 and we apply it to conclude that the operator equation $AxBx = x$ has a solution. Consequently the integral Equation(6) and the QDE (1) has a solution x^* defined on I. Furthermore, the sequence$\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by(7) converges monotonically to x^*. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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