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THREE EXTENSIONS OF HCF AND PCF THEOREMS

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Abstract. This paper deals with three refinements and extensions of Jensen's discrete inequality applied to half or partially convex functions. Several applications are given to show the effectiveness of the proposed extensions.

Keywords: Jensen's discrete inequality, Half convex function, Partially convex function, Extensions and refinements.

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1. Introduction

If f is a convex function defined on a real interval $\mathbb{I}$ and $x_1, x_2, \ldots, x_n \in \mathbb{I}$, then

$$f(x_1)+f(x_2)+\cdots+f(x_n) \ge nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

is Jensen non-weighted discrete inequality [4,5].

Recently, we extended Jensen's discrete inequality to half convex functions [1,2] and partially convex functions [3]. Using the notation

$$\mathbb{I}_{\ge s} = \{u | u \in \mathbb{I}, u \ge s\}, \quad \mathbb{I}_{\le s} = \{u | u \in \mathbb{I}, u \le s\},$$

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the half convex function theorem (HCF-Theorem) and the partially convex function theorem (PCF-Theorem) have the following statements.

HCF-Theorem. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$ or $\mathbb{I}_{\leq s}$, where $s \in \mathbb{I}$. The inequality*

$$f(x_1)+f(x_2)+\cdots+f(x_n) \geq nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

holds for all $x_1,x_2,\ldots,x_n \in \mathbb{I}$ satisfying $x_1+x_2+\cdots+x_n = ns$ if and only if

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x,y \in \mathbb{I}$ such that $x+(n-1)y = ns$.

PCF-Theorem. *Let f be a function defined on a real interval $\mathbb{I}$, decreasing on $\mathbb{I}_{\leq s_0}$ and increasing on $\mathbb{I}_{\geq s_0}$, where $s_0 \in \mathbb{I}$. In addition, f is convex on $[s,s_0]$ or $[s_0,s]$, where $s \in \mathbb{I}$. The inequality*

$$f(x_1)+f(x_2)+\cdots+f(x_n) \geq nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

holds for all $x_1,x_2,\ldots,x_n \in \mathbb{I}$ satisfying $x_1+x_2+\cdots+x_n = ns$ if and only if

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x,y \in \mathbb{I}$ such that $x+(n-1)y = ns$.

Notice that HCF-Theorem is an immediate consequence of both the right half convex function theorem (RHCF-Theorem) and the left half convex function theorem (LHCF-Theorem).

RHCF-Theorem. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $\mathbb{I}_{\geq s}$. The inequality*

$$f(x_1)+f(x_2)+\cdots+f(x_n) \geq nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

holds for all $x_1,x_2,\ldots,x_n \in \mathbb{I}$ satisfying $x_1+x_2+\cdots+x_n \geq ns$ if and only if

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x,y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x+(n-1)y = ns$.

LHCF-Theorem. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $\mathbb{I}_{\leq s}$. The inequality*

$$f(x_1)+f(x_2)+\cdots+f(x_n) \geq nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

holds for all $x_1,x_2,\ldots,x_n \in \mathbb{I}$ satisfying $x_1+x_2+\cdots+x_n \leq ns$ if and only if

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x,y \in \mathbb{I}$ such that $x \geq s \geq y$ and $x+(n-1)y = ns$.

Remark 1.1. Let

$$g(u) = \frac{f(u)-f(s)}{u-s}, \quad h(x,y) = \frac{g(x)-g(y)}{x-y}.$$

As it is shown in [1,2,3], for many applications of these theorems, it is useful to replace the hypothesis condition

$$f(x)+(n-1)f(y) \geq nf(s) \quad \forall\, x,y \in \mathbb{I},\ x+(n-1)y = ns.$$

by the equivalent condition

$$h(x,y) \geq 0 \quad \forall\, x,y \in \mathbb{I},\ x+(n-1)y = ns.$$

An extension of HCF-Theorem to half convex functions with support lines was given by Zlatko Pavić in [6]. In what it follows, we continue this topic by giving some new refinements and extensions of these results.

2. First extensions

The theorem below, called the right partially convex function theorem (RPCF-Theorem) is an extension of PCF-Theorem. Thus, the condition in PCF-Theorem

f is decreasing on $\mathbb{I}_{\leq s_0}$ and increasing on $\mathbb{I}_{\geq s_0}$

is relaxed in RPCF-Theorem to

f is decreasing on $\mathbb{I}_{\leq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}_{\geq s_0}$.

RPCF-Theorem. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s,s_0]$, where $s,s_0 \in \mathbb{I}$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\le s_0}$ and satisfies*

$$\min_{u\in\mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(x_1)+f(x_2)+\cdots+f(x_n) \ge nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

holds for all $x_1,x_2,\ldots,x_n \in \mathbb{I}$ satisfying $x_1+x_2+\cdots+x_n = ns$ if and only if

$$f(x)+(n-1)f(y) \ge nf(s)$$

for all $x,y \in \mathbb{I}$ such that $x \le s \le y$ and $x+(n-1)y = ns$.

Proof. Clearly, the necessity in RPCF-Theorem is obvious. By Lemma 2.1 below, to prove the sufficiency in RPCF-Theorem, it suffices to consider that $x_1,x_2,\ldots,x_n \in \mathbb{J}$, where $\mathbb{J} = \mathbb{I}_{\le s_0}$. Because f is convex on $\mathbb{J}_{\ge s}$, the desired inequality in RPCF-Theorem follows immediately from RHCF-Theorem applied to the interval $\mathbb{J}$.

Lemma 2.1. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s,s_0]$, where $s,s_0 \in \mathbb{I}$, $s < s_0$. In addition, $f(u)$ is decreasing on $\mathbb{I}_{\le s_0}$ and*

$$\min_{u\in\mathbb{I}} f(u) = f(s_0).$$

If the inequality

$$f(x_1)+f(x_2)+\cdots+f(x_n) \ge nf(s)$$

holds for all $x_1,x_2,\ldots,x_n \in \mathbb{I}_{\le s_0}$ such that $x_1+x_2+\cdots+x_n = ns$, then it holds for all $x_1,x_2,\ldots,x_n \in \mathbb{I}$ such that $x_1+x_2+\cdots+x_n = ns$.

Proof. For $i = 1,2,\ldots,n$, define the numbers $y_i \in \mathbb{I}_{\le s_0}$ as follows

$$y_i = \begin{cases} x_i, & x_i \le s_0 \\ s_0, & x_i > s_0. \end{cases}$$

We have $y_i \le x_i$ and $f(y_i) \le f(x_i)$ for $i = 1,2,\ldots,n$. Therefore,

$$y_1+y_2+\cdots+y_n \le x_1+x_2+\cdots+x_n = ns$$

and

$$f(y_1)+f(y_2)+\cdots+f(y_n) \le f(x_1)+f(x_2)+\cdots+f(x_n).$$

Thus, it suffices to show that

$$f(y_1)+f(y_2)+\cdots+f(y_n) \ge nf(s)$$

for all $y_1,y_2,\ldots,y_n \in \mathbb{I}_{\le s_0}$ such that $y_1+y_2+\cdots+y_n \le ns$. By hypothesis, this inequality is true for $y_1,y_2,\ldots,y_n \in \mathbb{I}_{\le s_0}$ and $y_1+y_2+\cdots+y_n = ns$. Since f is decreasing on $\mathbb{I}_{\le s_0}$, we have also $f(y_1)+f(y_2)+\cdots+f(y_n) \ge nf(s)$ for $y_1,y_2,\ldots,y_n \in \mathbb{I}_{\le s_0}$ such that $y_1+y_2+\cdots+y_n \le ns$.

Similarly, the following theorem, called the left partially convex function theorem (LRPCF-Theorem) is an extension of PCF-Theorem.

LPCF-Theorem. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s_0,s]$, where $s_0,s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\ge s_0}$ and satisfies*

$$\min_{u\in\mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(x_1)+f(x_2)+\cdots+f(x_n) \ge nf\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)$$

holds for all $x_1,x_2,\ldots,x_n \in \mathbb{I}$ satisfying $x_1+x_2+\cdots+x_n = ns$ if and only if

$$f(x)+(n-1)f(y) \ge nf(s)$$

for all $x,y \in \mathbb{I}$ such that $x \ge s \ge y$ and $x+(n-1)y = ns$.

Proof. The necessity in LPCF-Theorem is obvious. By Lemma 2.2 below, to prove the sufficiency in LPCF-Theorem, it suffices to consider that $x_1,x_2,\ldots,x_n \in \mathbb{J}$, where $\mathbb{J} = \mathbb{I}_{\ge s_0}$. Because f is convex on $\mathbb{J}_{\le s}$, the desired inequality in LPCF-Theorem follows immediately from LHCF-Theorem applied to the interval $\mathbb{J}$.

Lemma 2.2. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s_0,s]$, where $s_0,s \in \mathbb{I}$, $s_0 < s$. In addition, $f(u)$ is increasing on $\mathbb{I}_{\ge s_0}$ and satisfies*

$$\min_{u\in\mathbb{I}} f(u) = f(s_0).$$

If the inequality

$$f(x_1)+f(x_2)+\cdots+f(x_n)\ge nf(s)$$

holds for all $x_1,x_2,\ldots,x_n\in\mathbb{I}_{\ge s_0}$ *such that* $x_1+x_2+\cdots+x_n=ns$*, then it holds for all* $x_1,x_2,\ldots,x_n\in\mathbb{I}$ *such that* $x_1+x_2+\cdots+x_n=ns$.

Proof. For $i=1,2,\ldots,n$, define the numbers $y_i\in\mathbb{I}_{\ge s_0}$ as follows

$$y_i=\begin{cases} s_0, & x_i\le s_0\\ x_i, & x_i>s_0.\end{cases}$$

Since $y_i\ge x_i$ for $i=1,2,\ldots,n$, we have

$$y_1+y_2+\cdots+y_n\ge x_1+x_2+\cdots+x_n=ns.$$

In addition, since $f(y_i)\le f(x_i)$ for $x_i\le s_0$ and $f(y_i)=f(x_i)$ for $x_i>s_0$, we have $f(y_i)\le f(x_i)$ for $i=1,2,\ldots,n$, hence

$$f(y_1)+f(y_2)+\cdots+f(y_n)\le f(x_1)+f(x_2)+\cdots+f(x_n).$$

Thus, it suffices to show that

$$f(y_1)+f(y_2)+\cdots+f(y_n)\ge nf(s)$$

for all $y_1,y_2,\ldots,y_n\in\mathbb{I}_{\ge s_0}$ such that $y_1+y_2+\cdots+y_n\ge ns$. By hypothesis, this inequality is true for $y_1,y_2,\ldots,y_n\in\mathbb{I}_{\ge s_0}$ and $y_1+y_2+\cdots+y_n=ns$. Since f is increasing on $\mathbb{I}_{\ge s_0}$, we have also $f(y_1)+f(y_2)+\cdots+f(y_n)\ge nf(s)$ for $y_1,y_2,\ldots,y_n\in\mathbb{I}_{\ge s_0}$ such that $y_1+y_2+\cdots+y_n\ge ns$.

Remark 2.1. The inequalities in RPCF-Theorem and LPCF-Theorem turn into equalities for $x_1=x_2=\cdots=x_n=s$. In addition, the equality holds also for $x_1=x$ and $x_2=\cdots=x_n=y$ if there exist $x,y\in\mathbb{I}$, $x\ne y$, such that

$$x+(n-1)y=ns,\quad f(x)+(n-1)f(y)=nf(s).$$

The inequality in the following example cannot be proved by HCF-Theorem or PCF-Theorem, but can be proved using LPCF-Theorem.

Example 2.1. *If $x_1, x_2, \dots, x_n$ are real numbers such that $x_1 + x_2 + \cdots + x_n = n$, then*

$$\frac{x_1(x_1-1)}{4(n-1)x_1^2+n^2} + \frac{x_2(x_2-1)}{4(n-1)x_2^2+n^2} + \cdots + \frac{x_n(x_n-1)}{4(n-1)x_n^2+n^2} \geq 0,$$

with equality for $x_1 = x_2 = \cdots = x_n = 1$, and also for $x_1 = \frac{n}{2}$ and $x_2 = \cdots = x_n = \frac{n}{2(n-1)}$ (or any cyclic permutation).

To prove this inequality, we write it as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq n f(s), \quad s = 1,$$

where

$$f(u) = \frac{u(u-1)}{4(n-1)u^2+n^2}, \quad u \in \mathbb{I} = \mathbb{R}.$$

From

$$f'(u) = \frac{4(n-1)u^2 + 2n^2u - n^2}{[4(n-1)u^2+n^2]^2},$$

it follows that f is increasing on $(-\infty, s_1] \cup [s_0, \infty)$ and decreasing on $[s_1, s_0]$, where

$$s_1 = \frac{n(-n-\sqrt{n^2+4n-4})}{4(n-1)}, \quad s_0 = \frac{n(-n+\sqrt{n^2+4n-4})}{4(n-1)} \in (0,1).$$

Since

$$\lim_{u \to -\infty} f(u) = \frac{1}{4(n-1)}$$

and $f(s_0) < f(1) = 0$, we have

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

From

$$f''(u) = \frac{2g(u)}{[4(n-1)u^2+n^2]^3}, \quad g(u) = n^4 + 12n^2(n-1)u(1-u) - 16(n-1)^2u^3,$$

it follows that f is convex on $[0,1]$, because

$$g(u) \geq n^4 - 16(n-1)^2u^3 \geq n^4 - 16(n-1)^2 = (n-2)^2(n^2+4n-4) \geq 0.$$

Clearly, we cannot apply HCF-Theorem because f is not half convex. Also, we cannot apply PCF-Theorem because f is not decreasing for all $u \leq s_0$. On the other hand, all preliminary conditions in LPCF-Theorem are satisfied. Therefore, we only need to prove that $f(x) + (n-$

$1)f(y) \geq nf(1)$ for all $x,y \in \mathbb{R}$ such that $x+(n-1)y=n$. According to Remark 1.1, it suffices to show that $h(x,y) \geq 0$ for all $x,y \in \mathbb{R}$ which satisfy $x+(n-1)y=n$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{u}{4(n-1)u^2+n^2},$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{n^2-4(n-1)xy}{[4(n-1)x^2+n^2][4(n-1)y^2+n^2]}$$
$$= \frac{[2(n-1)y-n]^2}{[4(n-1)x^2+n^2][4(n-1)y^2+n^2]} \geq 0.$$

3. Second extension

The following four propositions are extensions of RHCF, LHCF, RPCF and LPCF theorems to the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where u_0 is an interior point of $\mathbb{I}$.

Proposition 3.1. *RHCF-Theorem is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}$, $u_0 < s$.*

Proposition 3.2. *LHCF-Theorem is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}$, $u_0 > s$.*

Proposition 3.3. *RPCF-Theorem is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}$, $u_0 > s_0$.*

Proposition 3.4. *LPCF-Theorem is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}$, $u_0 < s_0$.*

These propositions follow immediately from the proofs of the respective theorems. For instance, the main idea in the proof of RHCF-Theorem is to replace the desired inequality

$$f(x_1)+f(x_2)+\cdots+f(x_n) \geq nf(s), \quad x_1,x_2,\ldots,x_n \in \mathbb{I} \setminus \{u_0\},$$

with a sharper inequality in which all variables are located in $\mathbb{I}_{\geq s}$, where f is convex. More precisely, under the assumption that

$$x_1 \leq \cdots \leq x_k \leq s \leq x_{k+1} \leq \cdots \leq x_n,$$

from the hypothesis

$$f(x)+(n-1)f(y) \geq nf(s), \quad x+(n-1)y=ns, \quad x \leq s \leq y$$

it follows that

$$f(x_i)+(n-1)f(y_i)\ge nf(s),\quad x_i+(n-1)y_i=ns,\quad x_i\le s\le y_i$$

for $i=1,\dots,k$. Therefore, it suffices to prove the sharper inequality

$$\sum_{i=1}^{k}[nf(s)-(n-1)f(y_i)]+f(x_{k+1})+\cdots+f(x_n)\ge nf(s),$$

where all variables $y_1,\dots,y_k$ and $x_{k+1},\dots,x_n$ are located in $\mathbb{I}_{\ge s}$.

Example 3.1. *Let $x_1,x_2,\dots,x_n\ne -k$ be real numbers such that $x_1+x_2+\cdots+x_n=n$. If $k\ge\dfrac{n}{2\sqrt{n-1}}$, then*

$$\frac{x_1(x_1-1)}{(x_1+k)^2}+\frac{x_2(x_2-1)}{(x_2+k)^2}+\cdots+\frac{x_n(x_n-1)}{(x_n+k)^2}\ge 0,$$

with equality for $x_1=x_2=\cdots=x_n=1$. If $k=\dfrac{n}{2\sqrt{n-1}}$, then the equality holds also for $x_1=\dfrac{n}{2}$ and $x_2=\cdots=x_n=\dfrac{n}{2(n-1)}$ (or any cyclic permutation).

To prove the inequality in Example 3.1, we write it as

$$f(x_1)+f(x_2)+\cdots+f(x_n)\ge nf(s),\quad s=1,$$

where

$$f(u)=\frac{u(u-1)}{(u+k)^2},\quad u\in\mathbb{R}\setminus\{-k\}.$$

From

$$f'(u)=\frac{(2k+1)u-k}{(u+k)^3},$$

it follows that f is increasing on $(-\infty,-k)\cup[s_0,\infty)$ and decreasing on $(-k,s_0]$, where

$$s_0=\frac{k}{2k+1}<1.$$

Since

$$\lim_{u\to-\infty}f(u)=1$$

and $f(s_0)<f(1)=0$, we have

$$\min_{u\in\mathbb{I}}f(u)=f(s_0).$$

From

$$\frac{1}{2}f''(u)=\frac{k(k+2)-(2k+1)u}{(x+k)^4},$$

it follows that f is convex on $\left[0, \frac{k(k+2)}{2k+1}\right]$, hence on $[s_0, 1]$. According to LPCF-Theorem, Proposition 3.4 and Remark 1.1, it suffices to show that $h(x,y) \geq 0$ for all $x, y \in \mathbb{R} \setminus \{-k\}$ which satisfy $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u}{(u+k)^2},$$

$$h(x,y) = \frac{g(x) - g(y)}{x - y} = \frac{k^2 - xy}{(x+k)^2(y+k)^2}.$$

Since

$$k^2 - xy \geq \frac{n^2}{4(n-1)} - xy = \frac{[2(n-1)y - n]^2}{4(n-1)} \geq 0,$$

it follows that $h(x,y) \geq 0$.

4. Third extension

The following theorem is an extension of RPCF-Theorem for the case in which the condition "f is decreasing on $\mathbb{I}_{\leq s_0}$" is not satisfied.

Theorem 4.1. *Let f be a function defined on a real interval $\mathbb{I}$, convex on $[s, s_0]$ and satisfying*

$$\min_{u \geq s} f(u) = f(s_0),$$

where

$$s, s_0 \in \mathbb{I}, \quad s < s_0, \quad ns - (n-1)s_0 \leq \inf \mathbb{I}.$$

If

$$f(x) + (n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x + (n-1)y = ns$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$.

Proof. In order to prove Theorem 4.1, we define the function

$$f_0(u) = \begin{cases} f(u), & u \in \mathbb{I}_{\leq s_0} \\ f(s_0), & u \in \mathbb{I}_{\geq s_0}, \end{cases}$$

which is convex on $\mathbb{I}_{\geq s}$. Taking into account that $f_0(s) = f(s)$ and $f_0(u) \leq f(u)$ for all $u \in \mathbb{I}$, it suffices to prove that

$$f_0(x_1) + f_0(x_2) + \cdots + f_0(x_n) \geq n f_0(s)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$. According to RHCF-Theorem, we only need to show that

$$f_0(x) + (n-1) f_0(y) \geq n f_0(s)$$

for all $x, y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x + (n-1)y = ns$. The case $y > s_0$ is not possible because

$$x = ns - (n-1)y < ns - (n-1)s_0 \leq \inf \mathbb{I}$$

involves $x \notin \mathbb{I}$. For the possible case $y \leq s_0$, the inequality $f_0(x) + (n-1) f_0(y) \geq n f_0(s)$ turns into $f(x) + (n-1) f(y) \geq n f(s)$, which holds (by hypothesis) for all $x, y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x + (n-1)y = ns$.

Similarly, the following theorem is an extension of LPCF-Theorem for the case in which the condition "f is increasing on $\mathbb{I}_{\geq s_0}$" is not satisfied.

Theorem 4.2. *Let f be a function defined on a real interval $\mathbb{I}$, convex on $[s_0, s]$ and satisfying*

$$\min_{u \leq s} f(u) = f(s_0),$$

where

$$s, s_0 \in \mathbb{I}, \quad s > s_0, \quad ns - (n-1)s_0 \geq \sup \mathbb{I}.$$

If

$$f(x) + (n-1) f(y) \geq n f(s)$$

for all $x, y \in \mathbb{I}$ such that $x \geq s \geq y$ and $x + (n-1)y = ns$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq n f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$.

The proof of Theorem 4.2 is similar to the proof of Theorem 4.1.

Example 4.1. *Let* $x_1,x_2,\dots,x_n \ge \dfrac{-n}{n-2}$ *such that* $x_1+x_2+\cdots+x_n=n$*, where* $n\ge 4$*. If* $k>0$*, then*

$$\frac{1-x_1}{k+x_1^2}+\frac{1-x_2}{k+x_2^2}+\cdots+\frac{1-x_n}{k+x_n^2}\ge 0,$$

with equality for $x_1=x_2=\cdots=x_n=1$*, and also for* $x_1=\dfrac{-n}{n-2}$ *and* $x_2=\cdots=x_n=\dfrac{n}{n-2}$ (*or any cyclic permutation*).

To prove the inequality in Example 4.1, we write it as

$$f(x_1)+f(x_2)+\cdots+f(x_n)\ge nf(s),\quad s=1,$$

where

$$f(u)=\frac{1-u}{k+u^2},\quad u\in\mathbb{I}=\left[\frac{-n}{n-2},\frac{n(2n-3)}{n-2}\right].$$

From

$$f'(u)=\frac{u^2-2u-k}{(u^2+k)^2},$$

it follows that $f(u)$ is decreasing for $u\in[1,s_0]$ and increasing for $u\ge s_0$, where $s_0=1+\sqrt{1+k}$; therefore, $\min_{u\ge 1} f(u)=f(s_0)$. From

$$f''(u)=\frac{2f_1(u)}{(u^2+k)^3},$$

where

$$f_1(u)=-u^3+3u^2+3ku-k=(k+1)(3u-1)-(u-1)^3$$
$$>(k+1)(u-1)-(u-1)^3=(u-1)[k+1-(u-1)^2]\ge 0,$$

it follows that f is convex on $[1,s_0]$. By Theorem 4.1, it suffices to show that

$$ns-(n-1)s_0\le \inf\mathbb{I}$$

and

$$f(x)+(n-1)f(y)\ge nf(s)$$

for all $x,y\in\mathbb{I}$ such that $x+(n-1)y=ns$. The first condition is equivalent to

$$n-(n-1)(1+\sqrt{1+k}\,)\le\frac{-n}{n-2},$$
$$(n-1)[2-(n-2)\sqrt{1+k}\,]\le 0,$$

which is clearly true for $n \geq 4$ and $k > 0$. According to Remark 1.1, the second condition is satisfied if $h(x,y) \geq 0$ for $x, y \in \mathbb{I}$ such that $x + (n-1)y = n$. Indeed,

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-1}{u^2 + k},$$

$$h(x,y) = \frac{x + y}{(x^2 + k)(y^2 + k)} = \frac{n + (n-2)x}{(n-1)(x^2 + k)(y^2 + k)} \geq 0.$$

Notice that the inequality in Example 4.1 is an extension of the inequality from Application 4.1 in [3], where the condition for k is more restrictive, namely

$$k \geq \frac{n(3n-4)}{(n-2)^2}.$$

Remark 4.1. Theorem 4.1 is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}$ such that $u_0 < s$ or $u_0 > s_0$. Similarly, Theorem 4.2 is also valid in the case in which f is defined on $\mathbb{I} \setminus \{u_0\}$, where $u_0 \in \mathbb{I}$ such that $u_0 < s_0$ or $u_0 > s$. According to this remark, the inequality in Example 4.1 holds also for $k = 0$.

Conflict of Interests

The author declares that there is no conflict of interests.

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