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GENERALIZATION OF WEIGHTED OSTROWSKI INTEGRAL INEQUALITY FOR TWICE DIFFERENTIABLE MAPPINGS

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Abstract. We introduce a general form of weighted integral inequality of Ostrowski type for twice differentiable mappings whose second derivatives are bounded and first derivatives are absolutely continuous. The weighted integral inequality gives us generalized result of different bounds. The weighted integral inequality is then applied to some quadrature rules in generalized way.

Keywords: Ostrowski's inequality; numerical integration.

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1. Introduction

In 1938, a Ukrainian Mathematician Alexandar Markowich Ostrowski discovered an inequality called Ostrowski inequality, which states that

Proposition 1.1. Let f : I $\subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0 (I^0 is the interior of I) and let a,b $\in \mathrm{I}^0$ with a<b. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $|f'(x)| \leq M$, $\forall x \in (a,b)$,

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for some positive constant M, then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right](b-a)M$$

for all x $\in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

Ostrowski type inequalities can be used to estimate the absolute deviation from its integral mean. It has wide applications in Numerical integration and Probability theory. They can be used to provide explicit error bounds for numerical quadrature formulae. In 1983, J. E. Pečarić and B. Savić [9] presented first weighted version of Ostrowski inequality. Due to the importance of this inequality, in the last few decades, researchers are continuously in an effort for gaining sharp bounds of Ostrowski's inequality in terms of weight. We have introduced a weighted inequality. The inequalities are then applied to Numerical quadrature rules and Probability theory.

In 1976, G. V. Milovanović and J. E. Pečarić proved a generalization of Ostrowski's inequality for n-time differentiable mappings [7] from which we would like to mention only the case of twice differentiable mappings [7, p. 470].

Proposition 1.2. Let $g : [a,b] \to \mathbb{R}$ be a twice differentiable mapping such that $g'' : (a,b) \to \mathbb{R}$ is bounded on (a,b). Then the following inequality holds $\forall\, x \in [a,b]$.

$$\left| \frac{1}{2}\left[g(x) + \frac{(x-a)g(a)+(b-x)g(b)}{b-a} \right] - \frac{1}{b-a}\int_a^b g(t)dt \right|$$
$$\le \frac{\|g''\|_\infty}{4}(b-a)^2 \left[\frac{1}{12} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right]$$

for all x $\in [a,b]$.

In year 1999 Cerone, Dragomir and Roumeliotis in [1] proved the following inequality.

Proposition 1.3. Under the assumption of Proposition 1.2, following inequality holds

$$\left| g(x) - \left(x - \frac{a+b}{2} \right) g'(x) - \frac{1}{b-a}\int_a^b g(t)dt \right|$$
$$\le \left[\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2} \right)^2 \right] \|g''\|_\infty \le \frac{(b-a)^2}{6}\|g''\|_\infty.$$

for all x $\in [a,b]$.

In the same year, Dragomir and Barnett in [3] stated the following result.

Proposition 1.4. Under the assumption of Proposition 1.2, following inequality holds $\forall\, x \in [a,b]$

$$\left| g(x) - \frac{g(b)-g(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(t)dt \right|$$
$$\leq \frac{(b-a)^2}{2}\left\{ \left[\left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} \|g''\|_\infty \leq \frac{(b-a)^2}{6}\|g''\|_\infty .$$

In [1] P. Cerone, S. S. Dragomir and J. Roumeliotis, established an integral inequality of Ostrowski type for mappings with bounded second derivatives. A similar inequality has been established by S. S. Dragomir and N. S. Barnett in [3]. In [2], S. S. Dragomir and A. Sofo, pointed out an integral inequality of Ostrowski type similar in sense that of [1] or [3].

S. S. Dragomir and A. Sofo proved the following integral inequality in [2].

Proposition 1.5. Let $g : [a,b] \to \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative $g'' \in L_\infty\, [a,b]$. Then we have the following inequality $\forall\, x \in [a,b]$

$$\left| \int_a^b g(t) - \frac{1}{2}\left[g(x) + \frac{g(a)+g(b)}{2} \right](b-a) + \frac{(b-a)}{2}\left(x - \frac{a+b}{2}\right) g'(x) \right|$$
$$\leq \|g''\|_\infty \left(\frac{1}{3}\left| x - \frac{a+b}{2} \right|^3 + \frac{(b-a)^3}{48} \right) \tag{1.1}$$

Fiza Zafar and Nazir Ahmad Mir established following general form of integral inequality in [11].

Proposition 1.6. Let all the assumption of Proposition 1.5 be valid. Then,we have the following inequality

$$\left| \frac{1}{b-a}\int_a^b g(t)dt - \frac{1}{2}\left[(1-h)g(x) + (1+h)\left(\frac{g(a)+g(b)}{2} \right) \right.\right.$$
$$\left.\left. -(1-h)\left(x - \frac{a+b}{2}\right) g'(x) - h\frac{b-a}{4}\left(g'(b) - g'(a)\right) \right] \right| \tag{1.2}$$
$$\leq \left\| g'' \right\|_\infty \frac{1}{(b-a)}\left[\frac{1}{3}\left| x - \frac{a+b}{2} \right|^3 + \frac{(b-a)^3}{48}\Psi(h) \right]$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$, where $\Psi(h) = (1-h)[2(1-h)^2 - 1] + 2h$, $h \in [0,1]$.

We established the weighted result of Proposition 1.5 of [5].

Proposition 1.7. Under the assumption of Proposition 1.5, we have the inequality:

$$\begin{aligned}
&\Bigg|\int_a^b g(t)w(t)dt - \frac{1}{2}\left[g(x) + \frac{b-a}{2}\left[g(a)w(a)+g(b)w(b)\right]\right.\\
&\left.-\int_a^b g(t)\left(t-\frac{a+b}{2}\right)w'(t)dt - \int_a^b w(t)\left(x-\frac{a+b}{2}\right)g'(x)dt\right]\Bigg|\\
&\leq \|g''\|_\infty \left[\left|\left(\int_a^x w(u)du + \int_b^x w(u)du\right)\left(\frac{(a+b)x}{4}-\frac{x^2}{4}\right)\right.\right. \qquad (1.3)\\
&\left.+\int_x^{\frac{a+b}{2}}\left(\frac{(a+b)t}{2}-\frac{t^2}{2}\right)w(t)dt\right| + \left(\int_a^{\frac{a+b}{2}} w(u)du\right)\frac{(a+b)^2}{8}\\
&\left.-\int_a^b\left(\frac{(a+b)t}{4}-\frac{t^2}{4}\right)w(t)dt\right]
\end{aligned}$$

where $w:[a,b]\to[0,\infty)$ is some probability density function, i.e., it is a positive integrable function satisfying $\int_a^b w(t)dt = 1$.

In the present paper we introduce weights on the generalization of Ostrowski-type inequality (1.1) which was proved by Fiza Zafar and Nazir Ahmad Mir in [11] and has its application in Numerical quadrature rules and Probability theory. By introducing arbitrary parameters in the weighted Peano kernels, we obtain generalization to the inequalities. We then adjust our parameters not only to reproduce the previously proven results, but also to obtain some new estimates of such inequalities. This paper is organized in the following manner. The first section is based on preliminaries where as in second section we have stated our main result by introducing weights in inequality (1.2) which are in fact probability density functions. In the last section we have stated one of its application in terms of composite quadrature rules.

2. Main Results

Throughout this section $\alpha = a + h\frac{b-a}{2}$ and $\beta = b - h\frac{b-a}{2}$ where $h\in[0,1]$.

Theorem 2.1. Under the assumption of Proposition 1.5, we have the inequality

$$\begin{aligned}
&\Bigg|\int_a^b g(t)w(t)dt - \frac{1}{2}\left[\int_\alpha^\beta w(t)dt\, g(x) + \frac{b-a}{2}\left(g(a)w(a)+g(b)w(b)\right)\right.\\
&+\int_\beta^b w(t)dt\, g(b) - \int_\alpha^a w(t)dt\, g(a) - \int_\alpha^\beta w(t)dt\left(x-\frac{a+b}{2}\right)g'(x)
\end{aligned}$$

$$
\begin{aligned}
&-\int_a^b g(t)\left(t-\frac{a+b}{2}\right)w'(t)dt-\frac{b-a}{2}\left(\int_\alpha^a w(t)dtg'(a)+\int_\beta^b w(t)dtg'(b)\right)\Bigg]\Bigg| \\
&\leq \|g''\|_\infty\Bigg[\Bigg|\left(\int_\alpha^x w(u)du+\int_\beta^x w(u)du\right)\left(\frac{(a+b)x}{4}-\frac{x^2}{4}\right) \\
&+\frac{1}{2}\int_x^{\frac{a+b}{2}}\left((a+b)t-t^2\right)w(t)dt\Bigg| \qquad (2.1)\\
&+\left(\int_\alpha^{\frac{a+b}{2}} w(u)du\right)\frac{(a+b)^2}{8}-\left(\int_a^\alpha w(u)du+\int_\beta^b w(u)du\right)\frac{ab}{4} \\
&-2\int_\alpha^\beta\left(\frac{(a+b)t}{4}-\frac{t^2}{4}\right)w(t)dt+\int_a^b\left(\frac{(a+b)t}{4}-\frac{t^2}{4}\right)w(t)dt\Bigg]
\end{aligned}
$$

where $w:[a,b]\to[0,\infty)$ is some probability density function, i.e., it is a positive integrable function satisfying $\int_a^b w(t)dt=1$.

Proof. Let us commence with the following integral identity

$$
\begin{aligned}
\int_\alpha^\beta w(t)dtf(x) &= \int_a^b w(t)f(t)dt+\int_\alpha^a w(t)dtf(a) \\
&-\int_\beta^b w(t)dtf(b)+\int_a^b P_w(x,t)f'(t)dt
\end{aligned} \tag{2.2}
$$

for all $x\in[\alpha,\beta]$, provided that f is absolutely continuous on $[a,b]$ and the kernel $P_w:[a,b]\times[a,b]\to\mathbb{R}$ is given by:

$$
P_w(x,t)=\begin{cases}\displaystyle\int_\alpha^t w(u)du, & \text{if} \quad t\in[a,x]\\[2ex] \displaystyle\int_\beta^t w(u)du, & \text{if} \quad t\in(x,b]\end{cases}
$$

where $t\in[a,b]$.

Let us consider

$$
f(x)=\left(x-\frac{a+b}{2}\right)g'(x).
$$

(2.2) implies

$$
\begin{aligned}
&\int_\alpha^\beta w(t)dt\left(x-\frac{a+b}{2}\right)g'(x) \\
&=\int_a^b w(t)\left(t-\frac{a+b}{2}\right)g'(t)dt-\frac{b-a}{2}\left(\int_\alpha^a w(t)dtg'(a)\right. \qquad (2.3)\\
&\left.+\int_\beta^b w(t)dtg'(b)\right)+\int_a^b P_w(x,t)\left[g'(t)+\left(t-\frac{a+b}{2}\right)g''(t)\right]dt.
\end{aligned}
$$

Integrating by parts, we have

$$\begin{aligned}\int_a^b \left(t-\frac{a+b}{2}\right) w(t)g'(t)dt &= \frac{b-a}{2}\left[g(a)w(a)+g(b)w(b)\right] \\ &- \int_a^b g(t)w(t)dt - \int_a^b g(t)\left(t-\frac{a+b}{2}\right)w'(t)dt,\end{aligned} \tag{2.4}$$

also

$$\begin{aligned}\int_a^b P_w(x,t)g'(t)dt &= \int_\beta^b w(t)dtg(b) - \int_\alpha^a w(t)dtg(a) \\ &+g(x)\int_\alpha^\beta w(t)dt - \int_a^b g(t)w(t)dt.\end{aligned} \tag{2.5}$$

Now using equations (2.4) and (2.5) in (2.3), we get

$$\begin{aligned}&\int_\alpha^\beta w(t)dt\left(x-\frac{a+b}{2}\right)g'(x) \\ &= \frac{b-a}{2}\left(g(a)w(a)+g(b)w(b)\right) + \int_\beta^b w(t)dtg(b) - \int_\alpha^a w(t)dtg(a) \\ &- \frac{b-a}{2}\left(\int_\alpha^a w(t)dtg'(a) + \int_\beta^b w(t)dtg'(b)\right) + \int_\alpha^\beta w(t)dtg(x) - 2\int_a^b g(t)w(t)dt \\ &- \int_a^b g(t)\left(t-\frac{a+b}{2}\right)w'(t)dt + \int_a^b P_w(x,t)\left(t-\frac{a+b}{2}\right)g''(t)dt.\end{aligned}$$

or

$$\begin{aligned}&\int_a^b w(t)g(t)dt \\ &= \frac{b-a}{4}\left(g(a)w(a)+g(b)w(b)\right) + \frac{1}{2}\int_\beta^b w(t)dtg(b) - \frac{1}{2}\int_\alpha^a w(t)dtg(a) - \frac{b-a}{4} \\ &\times\left(\int_\alpha^a w(t)dtg'(a) + \int_\beta^b w(t)dtg'(b)\right) + \frac{1}{2}\int_\alpha^\beta w(t)dt\left(g(x)-\left(x-\frac{a+b}{2}\right)g'(x)\right) \\ &- \frac{1}{2}\int_a^b g(t)\left(t-\frac{a+b}{2}\right)w'(t)dt + \frac{1}{2}\int_a^b P_w(x,t)\left(t-\frac{a+b}{2}\right)g''(t)dt,\end{aligned}$$

for all $x \in [\alpha, \beta]$ which gives us

$$\begin{aligned}&\Bigg|\int_a^b g(t)w(t)dt - \frac{1}{2}\Bigg[\int_\alpha^\beta w(t)dtg(x) + \frac{b-a}{2}\left(g(a)w(a)+g(b)w(b)\right) \\ &+ \int_\beta^b w(t)dtg(b) - \int_\alpha^a w(t)dtg(a) - \int_\alpha^\beta w(t)dt\left(x-\frac{a+b}{2}\right)g'(x) \\ &- \int_a^b g(t)\left(t-\frac{a+b}{2}\right)w'(t)dt - \frac{b-a}{2}\left(\int_\alpha^a w(t)dtg'(a) + \int_\beta^b w(t)dtg'(b)\right)\Bigg]\Bigg| \\ &= \left|\frac{1}{2}\int_a^b P_w(x,t)\left(t-\frac{a+b}{2}\right)g''(t)dt\right| \le \frac{1}{2}\int_a^b |P_w(x,t)|\left|t-\frac{a+b}{2}\right|\left|g''(t)\right|dt\end{aligned} \tag{2.6}$$

It is easy to see that

$$\int_a^b |P_w(x,t)| \left|t-\frac{a+b}{2}\right| \left|g''(t)\right| dt \le \|g''\|_\infty \int_a^b |P_w(x,t)| \left|t-\frac{a+b}{2}\right| dt, \tag{2.7}$$

where

$$\|g''\|_\infty = \sup_{t\in(a,b)} |g''(t)| < \infty.$$

Also,

$$I = \int_a^b |P_w(x,t)| \left|t-\frac{a+b}{2}\right| dt$$

or

$$I = \int_a^x \left|\int_\alpha^t w(u)du\right| \left|t-\frac{a+b}{2}\right| dt + \int_x^b \left|\int_\beta^t w(u)du\right| \left|t-\frac{a+b}{2}\right| dt. \tag{2.8}$$

Now, we have two cases:

(a) For $x \in \left[\alpha, \frac{a+b}{2}\right]$, we obtain

$$\begin{aligned} I &= \int_a^\alpha \left(\int_t^\alpha w(u)du\right)\left(\frac{a+b}{2}-t\right)dt + \int_\alpha^x \left(\int_\alpha^t w(u)du\right)\left(\frac{a+b}{2}-t\right)dt \\ &+ \int_x^{\frac{a+b}{2}} \left(\int_t^\beta w(u)du\right)\left(\frac{a+b}{2}-t\right)dt + \int_{\frac{a+b}{2}}^\beta \left(\int_t^\beta w(u)du\right)\left(t-\frac{a+b}{2}\right)dt \\ &+ \int_\beta^b \left(\int_\beta^t w(u)du\right)\left(t-\frac{a+b}{2}\right)dt. \end{aligned}$$

After some simple calculations, we obtain

$$\begin{aligned} I &= -\left(\int_\alpha^x w(u)du - \int_x^\beta w(u)du\right)\left(\frac{x^2}{2}-\frac{(a+b)x}{2}\right) + 2\int_{\frac{a+b}{2}}^\beta w(u)du \frac{(a+b)^2}{8} \\ &- \left(\int_a^\alpha w(u)du + \int_\beta^b w(u)du\right)\frac{ab}{2} + \int_a^\alpha \left(\frac{(a+b)t}{2}-\frac{t^2}{2}\right)w(t)dt \\ &+ \int_\beta^b \left(\frac{(a+b)t}{2}-\frac{t^2}{2}\right)w(t)dt + \int_x^{\frac{a+b}{2}} \left(\frac{(a+b)t}{2}-\frac{t^2}{2}\right)w(t)dt \\ &- \int_\alpha^x \left(\frac{(a+b)t}{2}-\frac{t^2}{2}\right)w(t)dt + \int_\beta^{\frac{a+b}{2}} \left(\frac{(a+b)t}{2}-\frac{t^2}{2}\right)w(t)dt \end{aligned}$$

$$
\begin{aligned}
&= -\left(\int_{\alpha}^{x} w(u)du - \int_{x}^{\beta} w(u)du\right)\left(\frac{x^2}{2} - \frac{(a+b)x}{2}\right) + 2\int_{\frac{a+b}{2}}^{\beta} w(u)du\frac{(a+b)^2}{8} \\
&-\left(\int_{a}^{\alpha} w(u)du + \int_{\beta}^{b} w(u)du\right)\frac{ab}{2} + 2\int_{x}^{\frac{a+b}{2}}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt \\
&-2\int_{\alpha}^{\beta}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt + \int_{a}^{b}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt \\
&= -\left[\left(\int_{\alpha}^{x} w(u)du - \int_{x}^{\beta} w(u)du\right)\left(\frac{x^2}{2} - \frac{(a+b)x}{2}\right) - 2\int_{x}^{\frac{a+b}{2}}\left(\frac{(a+b)t}{2}\right.\right. \\
&\left.\left.-\frac{t^2}{2}\right)w(t)dt\right] + 2\int_{\frac{a+b}{2}}^{\beta} w(u)du\frac{(a+b)^2}{8} - \left(\int_{a}^{\alpha} w(u)du + \int_{\beta}^{b} w(u)du\right)\frac{ab}{2} + \\
&-2\int_{\alpha}^{\beta}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt + \int_{a}^{b}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt
\end{aligned} \tag{2.9}
$$

for all $x \in \left[\alpha, \frac{a+b}{2}\right]$.

(b) Similarly, for $x \in \left[\frac{a+b}{2}, \beta\right]$, we obtain

$$
\begin{aligned}
I &= \int_{a}^{\alpha}\left(\int_{t}^{\alpha} w(u)du\right)\left(\frac{a+b}{2} - t\right)dt + \int_{\alpha}^{\frac{a+b}{2}}\left(\int_{\alpha}^{t} w(u)du\right)\left(\frac{a+b}{2} - t\right)dt \\
&+ \int_{\frac{a+b}{2}}^{x}\left(\int_{\alpha}^{t} w(u)du\right)\left(t - \frac{a+b}{2}\right)dt + \int_{x}^{\beta}\left(\int_{t}^{\beta} w(u)du\right)\left(t - \frac{a+b}{2}\right)dt \\
&+ \int_{\beta}^{b}\left(\int_{\beta}^{t} w(u)du\right)\left(t - \frac{a+b}{2}\right)dt
\end{aligned}
$$

After some simplification, we get

$$
\begin{aligned}
I &= \left(\int_{\alpha}^{x} w(u)du - \int_{x}^{\beta} w(u)du\right)\left(\frac{x^2}{2} - \frac{(a+b)x}{2}\right) + 2\int_{\alpha}^{\frac{a+b}{2}} w(u)du\frac{(a+b)^2}{8} \\
&- \left(\int_{a}^{\alpha} w(u)du + \int_{\beta}^{b} w(u)du\right)\frac{ab}{2} + \int_{a}^{\alpha}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt \\
&+ \int_{\beta}^{b}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt - \int_{x}^{\frac{a+b}{2}}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt \\
&- \int_{\alpha}^{\frac{a+b}{2}}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt - \int_{x}^{\beta}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&= -\left(\int_{\alpha}^{x} w(u)du - \int_{x}^{\beta} w(u)du\right)\left(\frac{x^2}{2} - \frac{(a+b)x}{2}\right) + 2\int_{\frac{a+b}{2}}^{\alpha} w(u)du\frac{(a+b)^2}{8} \\
&- \left(\int_{a}^{\alpha} w(u)du + \int_{\beta}^{b} w(u)du\right)\frac{ab}{2} - 2\int_{x}^{\frac{a+b}{2}}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt \\
&- 2\int_{\alpha}^{\beta}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt + \int_{a}^{b}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt \\
&= \left[\left(\int_{\alpha}^{x} w(u)du - \int_{x}^{\beta} w(u)du\right)\left(\frac{x^2}{2} - \frac{(a+b)x}{2}\right) - 2\int_{x}^{\frac{a+b}{2}}\left(\frac{(a+b)t}{2}\right.\right. \\
&\left.\left. - \frac{t^2}{2}\right)w(t)dt\right] + 2\int_{\frac{a+b}{2}}^{\alpha} w(u)du\frac{(a+b)^2}{8} - \left(\int_{a}^{\alpha} w(u)du + \int_{\beta}^{b} w(u)du\right)\frac{ab}{2} \\
&- 2\int_{\alpha}^{\beta}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt + \int_{a}^{b}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt
\end{aligned} \tag{2.10}
$$

for all $x \in \left[\frac{a+b}{2}, \beta\right]$.

Using equations (2.7), (2.8), (2.9) and (2.10) we obtain

$$
\begin{aligned}
&\left|\int_{a}^{b} g(t)w(t)dt - \frac{1}{2}\left[\int_{\alpha}^{\beta} w(t)dt g(x) + \frac{b-a}{2}\left(g(a)w(a) + g(b)w(b)\right)\right.\right. \\
&+ \int_{\beta}^{b} w(t)dt g(b) - \int_{\alpha}^{a} w(t)dt g(a) - \int_{\alpha}^{\beta} w(t)dt\left(x - \frac{a+b}{2}\right)g'(x) \\
&\left.\left. - \int_{a}^{b} g(t)\left(t - \frac{a+b}{2}\right)w'(t)dt - \frac{b-a}{2}\left(\int_{\alpha}^{a} w(t)dt g'(a) + \int_{\beta}^{b} w(t)dt g'(b)\right)\right]\right| \\
&\le \frac{\left\|g''\right\|_{\infty}}{2}\left[\left(\int_{\alpha}^{x} w(u)du - \int_{x}^{\beta} w(u)du\right)\left(\frac{x^2}{2} - \frac{(a+b)x}{2}\right)\right. \\
&\left. -2\int_{x}^{\frac{a+b}{2}}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt\right] + 2\int_{\alpha}^{\frac{a+b}{2}} w(u)du\frac{(a+b)^2}{8} \\
&- \left(\int_{a}^{\alpha} w(u)du + \int_{\beta}^{b} w(u)du\right)\frac{ab}{2} + -2\int_{\alpha}^{\beta}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt \\
&+ \int_{a}^{b}\left(\frac{(a+b)t}{2} - \frac{t^2}{2}\right)w(t)dt
\end{aligned}
$$

$$= \|g''\|_\infty \left[\left(\int_\alpha^x w(u)du - \int_x^\beta w(u)du \right) \left(\frac{(a+b)x}{4} - \frac{x^2}{4} \right) \right.$$
$$\left. + \int_x^{\frac{a+b}{2}} \left(\frac{(a+b)t}{2} - \frac{t^2}{2} \right) w(t)dt \right] + \int_\alpha^{\frac{a+b}{2}} w(u)du \frac{(a+b)^2}{8}$$
$$- \left(\int_a^\alpha w(u)du + \int_\beta^b w(u)du \right) \frac{ab}{4} + -2 \int_\alpha^\beta \left(\frac{(a+b)t}{4} - \frac{t^2}{4} \right) w(t)dt$$
$$+ \int_a^b \left(\frac{(a+b)t}{4} - \frac{t^2}{4} \right) w(t)dt.$$

Special Case 1. If we simply put $w(t) \equiv \frac{1}{b-a}$ in (2.1), then we will get (1.2).

Special Case 2. If we put $h = 0$, then $\alpha = a$ and $\beta = b$ in (2.1), then we get the inequality (1.3).

Remark 2.2. In (2.1), if we investigate the estimates for the end points $x = a$, $x = b$ and the midpoint x= $\frac{a+b}{2}$, we find the midpoint gives us the best estimate so inequality from first theorem, we have

$$\left| \int_a^b g(t)w(t)dt - \frac{1}{2} \left[\int_\alpha^\beta w(t)dt g(\frac{a+b}{2}) + \frac{b-a}{2} \left(g(a)w(a) + g(b)w(b) \right) \right. \right.$$
$$+ \int_\beta^b w(t)dt g(b) - \int_\alpha^a w(t)dt g(a) - \int_a^b g(t) \left(t - \frac{a+b}{2} \right) w'(t)dt$$
$$\left. \left. - \frac{b-a}{2} \left(\int_\alpha^a w(t)dt g'(a) + \int_\beta^b w(t)dt g'(b) \right) \right] \right|$$
$$\le \|g''\|_\infty \left[\left| \left(\int_\alpha^{\frac{a+b}{2}} w(u)du + \int_\beta^{\frac{a+b}{2}} w(u)du \right) \left(\frac{(a+b)^2}{16} \right) \right| \right. \tag{2.11}$$
$$+ \left(\int_\alpha^{\frac{a+b}{2}} w(u)du \right) \frac{(a+b)^2}{8} - \left(\int_a^\alpha w(u)du + \int_\beta^b w(u)du \right) \frac{ab}{4}$$
$$\left. -2 \int_\alpha^\beta \left(\frac{(a+b)t}{4} - \frac{t^2}{4} \right) w(t)dt + \int_a^b \left(\frac{(a+b)t}{4} - \frac{t^2}{4} \right) w(t)dt \right].$$

Special Case 3. If we simply put $w(t) \equiv \frac{1}{b-a}$ in (2.11), then we get the midpoint rule which gives us the best estimate.

Remark 2.3. If we choose $h = 0$, then $\alpha = a$ and $\beta = b$ in (2.11), we get the inequality

$$\begin{aligned}
&\left| \int_a^b g(t)w(t)dt - \frac{1}{2}\left[\int_a^b w(t)dt g(\frac{a+b}{2}) + \frac{b-a}{2}\left(g(a)w(a)+g(b)w(b)\right) \right.\right. \\
&\left.\left. - \int_a^b g(t)\left(t-\frac{a+b}{2}\right)w'(t)dt \right]\right| \\
&\leq \|g''\|_\infty \left[\left| \left(\int_a^{\frac{a+b}{2}} w(u)du + \int_b^{\frac{a+b}{2}} w(u)du \right)\left(\frac{(a+b)^2}{16}\right) \right| \right. \\
&\left. + \left(\int_a^{\frac{a+b}{2}} w(u)du \right)\frac{(a+b)^2}{8} - \int_a^b \left(\frac{(a+b)t}{4} - \frac{t^2}{4}\right)w(t)dt \right].
\end{aligned} \tag{2.12}$$

Special Case 4. If we simply put $w(t) \equiv \frac{1}{b-a}$ in (2.12), then we get the midpoint rule which gives us the best estimate.

Remark 2.4. If we choose $h = 1$, then $\alpha = \beta = \frac{a+b}{2}$ in (2.11), we get the inequality

$$\begin{aligned}
&\left| \int_a^b g(t)w(t)dt - \frac{1}{2}\left[\frac{b-a}{2}\left(g(a)w(a)+g(b)w(b)\right) \right.\right. \\
&+ \int_{\frac{a+b}{2}}^b w(t)dt g(b) - \int_{\frac{a+b}{2}}^a w(t)dt g(a) - \int_a^b g(t)\left(t-\frac{a+b}{2}\right)w'(t)dt \\
&\left.\left. -\frac{b-a}{2}\left(\int_{\frac{a+b}{2}}^a w(t)dt g'(a) + \int_{\frac{a+b}{2}}^b w(t)dt g'(b) \right) \right]\right| \\
&\leq \|g''\|_\infty \left[-\int_a^b w(u)du \frac{ab}{4} + \int_a^b \left(\frac{(a+b)t}{4} - \frac{t^2}{4}\right)w(t)dt \right].
\end{aligned} \tag{2.13}$$

Special Case 5. If we simply put $w(t) \equiv \frac{1}{b-a}$ in (2.13), then we get perturbed trapezoid inequality which gives us the better estimate for $\|.\|_\infty$ norm.

Remark 2.5. If we choose $h = \frac{3}{10}$, then $\alpha = \frac{17a+3b}{20}$ and $\beta = \frac{3a+17b}{20}$ in (2.11), we get the inequality

$$\begin{aligned}
&\Bigg|\int_a^b g(t)w(t)dt - \frac{1}{2}\Bigg[\int_{\frac{17a+3b}{20}}^{\frac{3a+17b}{20}} w(t)dt g(\frac{a+b}{2}) + \frac{b-a}{2}\left(g(a)w(a)+g(b)w(b)\right) \\
&+\int_{\frac{3a+17b}{20}}^{b} w(t)dt g(b) - \int_{\frac{17a+3b}{20}}^{a} w(t)dt g(a) - \int_a^b g(t)\left(t-\frac{a+b}{2}\right)w'(t)dt \\
&-\frac{b-a}{2}\left(\int_{\frac{17a+3b}{20}}^{a} w(t)dt g'(a) + \int_{\frac{3a+17b}{20}}^{b} w(t)dt g'(b)\right)\Bigg]\Bigg| \\
&\leq \|g''\|_\infty \Bigg[\Bigg|\left(\int_{\frac{17a+3b}{20}}^{\frac{a+b}{2}} w(u)du + \int_{\frac{3a+17b}{20}}^{\frac{a+b}{2}} w(u)du\right)\left(\frac{(a+b)^2}{16}\right)\Bigg| \qquad (2.14)\\
&+\left(\int_{\frac{17a+3b}{20}}^{\frac{a+b}{2}} w(u)du\right)\frac{(a+b)^2}{8} - \left(\int_a^{\frac{17a+3b}{20}} w(u)du + \int_{\frac{3a+17b}{20}}^{b} w(u)du\right)\frac{ab}{4} \\
&-2\int_{\frac{17a+3b}{20}}^{\frac{3a+17b}{20}}\left(\frac{(a+b)t}{4}-\frac{t^2}{4}\right)w(t)dt + \int_a^b\left(\frac{(a+b)t}{4}-\frac{t^2}{4}\right)w(t)dt\Bigg].
\end{aligned}$$

Special Case 6. If we simply put $w(t) \equiv \frac{1}{b-a}$ in (2.14), then we get the best estimate than the three point quadrature inequalities.

3. Applications in Numerical Integration

We may use inequality (2.1) to get the estimates of composite quadrature rule with small error, by which better results may be obtained.

Theorem 3.1. Let $I_n : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ be a partition of interval $[a,b]$, $\Delta x_i = x_{i+1} - x_i$, h $\in [0,1]$, $\alpha_i \leq \xi_i \leq \beta_i$, where $\alpha_i = x_i + h\frac{\Delta x_i}{2}$, $\beta_i = x_{i+1} - h\frac{\Delta x_i}{2}$, $i \in \{0,\ldots,n-1\}$, then

$$\int_a^b g(t)w(t)dt = S(g,g',I_n,\xi_i,h,w) + R(g,g',I_n,\xi_i,h,w)$$

where

$$
\begin{aligned}
&S(g,g',I_n,\xi_i,h,w)\\
&=\frac{1}{2}\sum_{i=0}^{n-1}\left[\int_{\alpha_i}^{\beta_i} w(t)dt g(\xi_i)+\frac{\Delta x_i}{2}\left(g(x_i)w(x_i)+g(x_{i+1})w(x_{i+1})\right)\right.\\
&+\int_{\beta_i}^{x_{i+1}} w(t)dt g(x_{i+1})-\int_{\alpha_i}^{x_i} w(t)dt g(x_i) \qquad (3.1)\\
&-\int_{\alpha_i}^{\beta_i} w(t)dt\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)g'(\xi_i)-\int_{x_i}^{x_{i+1}} g(t)\left(t-\frac{x_i+x_{i+1}}{2}\right)w'(t)dt\\
&\left.-\frac{\Delta x_i}{2}\left(\int_{\alpha_i}^{x_i} w(t)dt g'(x_i)+\int_{\beta_i}^{x_{i+1}} w(t)dt g'(x_{i+1})\right)\right]\Delta x_i
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|R(g,g',I_n,\xi,h,w)\right|\\
&\leq \|g''\|_\infty\left[\left|\left(\int_{\alpha_i}^{\xi_i} w(u)du+\int_{\beta_i}^{\xi_i} w(u)du\right)\left(\frac{(x_i+x_{i+1})\xi_i}{4}-\frac{\xi_i{}^2}{4}\right)\right.\right.\\
&\left.+\frac{1}{2}\int_{\xi_i}^{\frac{x_i+x_{i+1}}{2}}\left((x_i+x_{i+1})t-t^2\right)w(t)dt\right|+\left(\int_{\alpha_i}^{\frac{x_i+x_{i+1}}{2}} w(u)du\right)\frac{(x_i+x_{i+1})^2}{8} \qquad (3.2)\\
&-\left(\int_{x_i}^{\alpha_i} w(u)du+\int_{\beta_i}^{x_{i+1}} w(u)du\right)\frac{x_ix_{i+1}}{4}-2\int_{\alpha_i}^{\beta_i}\left(\frac{(x_i+x_{i+1})t}{4}-\frac{t^2}{4}\right)w(t)dt\\
&\left.+\int_{x_i}^{x_{i+1}}\left(\frac{(x_i+x_{i+1})t}{4}-\frac{t^2}{4}\right)w(t)dt\right]
\end{aligned}
$$

Proof. Applying inequality (2.1) on $\xi_i\in[\alpha_i,\beta_i]=\left[x_i+h\frac{\Delta x_i}{2},x_{i+1}-h\frac{\Delta x_i}{2}\right]$ and summing over i from 0 to $n-1$ and using triangular inequality, we get (3.1) and (3.2).

Special Case 1. If we simply put $w(t)\equiv\frac{1}{x_{i+1}-x_i}$ in (3.1) and (3.2), then we will get $Theorem$ 2 of [11].

Remark 3.2. If we put $h=0$, then $\alpha_i=x_i$ and $\beta_i=x_{i+1}$ in (3.1) and (3.2) we get the inequality

$$
\begin{aligned}
&S(g,g',I_n,\xi_i,h,w)=\frac{1}{2}\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}} w(t)dt g(\xi_i)\right.\\
&+\frac{\Delta x_i}{2}\left(g(x_i)w(x_i)+g(x_{i+1})w(x_{i+1})\right)-\int_{x_i}^{x_{i+1}} w(t)dt\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)g'(\xi_i) \qquad (3.3)\\
&\left.-\int_{x_i}^{x_{i+1}} g(t)\left(t-\frac{x_i+x_{i+1}}{2}\right)w'(t)dt\right]\Delta x_i
\end{aligned}
$$

and

$$\begin{aligned}
&\left|R(g,g',I_n,\xi_i,h,w)\right| \\
&\leq \|g''\|_\infty \left[\left|\left(\int_{x_i}^{\xi_i} w(u)du + \int_{x_{i+1}}^{\xi_i} w(u)du\right)\left(\frac{(x_i+x_{i+1})\xi_i}{4} - \frac{{\xi_i}^2}{4}\right)\right.\right. \\
&\left.+\frac{1}{2}\int_{\xi_i}^{\frac{x_i+x_{i+1}}{2}} \left((x_i+x_{i+1})t - t^2\right) w(t)dt\right| \\
&\left.+\left(\int_{x_i}^{\frac{x_i+x_{i+1}}{2}} w(u)du\right)\frac{(x_i+x_{i+1})^2}{8} - \int_{x_i}^{x_{i+1}}\left(\frac{(x_i+x_{i+1})t}{4} - \frac{t^2}{4}\right)w(t)dt\right].
\end{aligned} \tag{3.4}$$

Corollary 3.3. *For $\xi_i = \frac{x_i+x_{i+1}}{2}$ in (3.1) and (3.2) for $i \in \{0,\ldots,n-1\}$, then we have the following quadrature rule*

$$\begin{aligned}
&S(g,g',I_n,h,w) \\
&= \frac{1}{2}\sum_{i=0}^{n-1}\left[\int_{\alpha_i}^{\beta_i} w(t)dt g(\frac{x_i+x_{i+1}}{2}) + \frac{\Delta x_i}{2}\left(g(x_i)w(x_i) + g(x_{i+1})w(x_{i+1})\right)\right. \\
&+\int_{\beta_i}^{x_{i+1}} w(t)dt g(x_{i+1}) - \int_{\alpha_i}^{x_i} w(t)dt g(x_i) - \int_{x_i}^{x_{i+1}} g(t)\left(t - \frac{x_i+x_{i+1}}{2}\right)w'(t)dt \\
&\left.-\frac{\Delta x_i}{2}\left(\int_{\alpha_i}^{x_i} w(t)dt g'(x_i) + \int_{\beta_i}^{x_{i+1}} w(t)dt g'(x_{i+1})\right)\right]\Delta x_i
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\left|R(g,g',I_n,h,w)\right| \\
&\leq \|g''\|_\infty \left[\left|\left(\int_{\alpha_i}^{\frac{x_i+x_{i+1}}{2}} w(u)du + \int_{\beta_i}^{\frac{x_i+x_{i+1}}{2}} w(u)du\right)\left(\frac{(x_i+x_{i+1})^2}{16}\right)\right|\right. \\
&+\left(\int_{\alpha_i}^{\frac{x_i+x_{i+1}}{2}} w(u)du\right)\frac{(x_i+x_{i+1})^2}{8} - \left(\int_{x_i}^{\alpha_i} w(u)du + \int_{\beta_i}^{x_{i+1}} w(u)du\right)\frac{x_i x_{i+1}}{4} \\
&\left.-2\int_{\alpha_i}^{\beta_i}\left(\frac{(x_i+x_{i+1})t}{4} - \frac{t^2}{4}\right)w(t)dt + \int_{x_i}^{x_{i+1}}\left(\frac{(x_i+x_{i+1})t}{4} - \frac{t^2}{4}\right)w(t)dt\right].
\end{aligned} \tag{3.6}$$

Remark 3.4. If we choose $h = 0$, then $\alpha_i = x_i$ and $\beta_i = b$ in (3.5) and (3.6), ($i \in \{0,\ldots,n-1\}$), then

$$\begin{aligned}
S(g,I_n,w) &= \frac{1}{2}\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} g(t)w(t)dt - \frac{1}{2}\left[\int_{x_i}^{x_{i+1}} w(t)dt g(\frac{x_i+x_{i+1}}{2})\right.\right. \\
&\left.\left.+\frac{\Delta x_i}{2}\left(g(x_i)w(x_i) + g(x_{i+1})w(x_{i+1})\right) \; - \int_{x_i}^{x_{i+1}} g(t)\left(t - \frac{x_i+x_{i+1}}{2}\right)w'(t)dt\right]\right|\Delta x_i
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&|R(g,I_n,w)| \\
&\leq \|g''\|_\infty \left[\left| \left(\int_{x_i}^{\frac{x_i+x_{i+1}}{2}} w(u)du + \int_{x_{i+1}}^{\frac{x_i+x_{i+1}}{2}} w(u)du \right) \left(\frac{(x_i+x_{i+1})^2}{16} \right) \right| \right. \\
&\left. + \left(\int_{x_i}^{\frac{x_i+x_{i+1}}{2}} w(u)du \right) \frac{(x_i+x_{i+1})^2}{8} - \int_{x_i}^{x_{i+1}} \left(\frac{(x_i+x_{i+1})t}{4} - \frac{t^2}{4} \right) w(t)dt \right].
\end{aligned} \tag{3.8}$$

Remark 3.5. If we choose $h=1$, then α_i=β_i=$\frac{x_i+x_{i+1}}{2}$ in (3.5) and (3.6), $i \in \{0,\ldots,n-1\}$, we get

$$\begin{aligned}
&S(g,g',I_n,w) \\
&= \frac{1}{2}\sum_{i=0}^{n-1} \left[\frac{\Delta x_i}{2} \left(g(x_i)w(x_i) + g(x_{i+1})w(b) \right) + \int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} w(t)dtg(x_{i+1}) \right. \\
&- \int_{\frac{x_i+x_{i+1}}{2}}^{x_i} w(t)dtg(x_i) - \int_{x_i}^{x_{i+1}} g(t) \left(t - \frac{x_i+x_{i+1}}{2} \right) w'(t)dt \\
&\left. - \frac{\Delta x_i}{2} \left(\int_{\frac{x_i+x_{i+1}}{2}}^{x_i} w(t)dtg'(x_i) + \int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} w(t)dtg'(x_{i+1}) \right) \right] \Bigg| \Delta x_i
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\left| R(g,g',I_n,w) \right| \\
&\leq \|g''\|_\infty \left[- \int_{x_i}^{x_{i+1}} w(u)du \frac{x_i x_{i+1}}{4} + \int_{x_i}^{x_{i+1}} \left(\frac{(x_i+x_{i+1})t}{4} - \frac{t^2}{4} \right) w(t)dt \right].
\end{aligned} \tag{3.10}$$

Remark 3.6. If we put h=$\frac{3}{10}$ gives $\alpha_i = \frac{17x_i+3x_{i+1}}{20}$ and $\beta = \frac{3x_i+17x_{i+1}}{20}$ in (3.5) and (3.6) , $i \in \{0,\ldots,n-1\}$, we get the inequality

$$\begin{aligned}
&S(g,g',I_n,w) = \frac{1}{2}\sum_{i=0}^{n-1} \left[\int_{\frac{17x_i+3x_{i+1}}{20}}^{\frac{3x_i+17x_{i+1}}{20}} w(t)dtg(\frac{x_i+x_{i+1}}{2}) \right. \\
&+ \frac{\Delta x_i}{2} \left(g(x_i)w(x_i) + g(x_{i+1})w(x_{i+1}) \right) + \int_{\frac{3x_i+17x_{i+1}}{20}}^{x_{i+1}} w(t)dtg(x_{i+1}) \\
&- \int_{\frac{17x_i+3x_{i+1}}{20}}^{x_i} w(t)dtg(x_i) - \int_{x_i}^{x_{i+1}} g(t) \left(t - \frac{x_i+x_{i+1}}{2} \right) w'(t)dt \\
&\left. - \frac{\Delta x_i}{2} \left(\int_{\frac{17x_i+3x_{i+1}}{20}}^{x_i} w(t)dtg'(x_i) + \int_{\frac{3x_i+17x_{i+1}}{20}}^{x_{i+1}} w(t)dtg'(x_{i+1}) \right) \right] \Bigg| \Delta x_i
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\left|R(g,g',I_n,w)\right| \\
&\leq \|g''\|_\infty \left[\left| \left(\int_{\frac{17x_i+3x_{i+1}}{20}}^{\frac{x_i+x_{i+1}}{2}} w(u)du + \int_{\frac{3x_i+17x_{i+1}}{20}}^{\frac{x_i+x_{i+1}}{2}} w(u)du \right) \left(\frac{(x_i+x_{i+1})^2}{16} \right) \right| \right. \\
&+ \left(\int_{\frac{17x_i+3x_{i+1}}{20}}^{\frac{x_i+x_{i+1}}{2}} w(u)du \right) \frac{(x_i+x_{i+1})^2}{8} - \left(\int_{x_i}^{\frac{17x_i+3x_{i+1}}{20}} w(u)du \right. \qquad (3.12) \\
&\left. + \int_{\frac{3x_i+17x_{i+1}}{20}}^{x_{i+1}} w(u)du \right) \frac{x_i x_{i+1}}{4} - 2\int_{\frac{17x_i+3x_{i+1}}{20}}^{\frac{3x_i+17x_{i+1}}{20}} \left(\frac{(x_i+x_{i+1})t}{4} - \frac{t^2}{4} \right) w(t)dt \\
&\left. + \int_{x_i}^{x_{i+1}} \left(\frac{(x_i+x_{i+1})t}{4} - \frac{t^2}{4} \right) w(t)dt \right].
\end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] P. Cerone, S. S. Dragomir, J. Roumeliotis, An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications, East Asian Math. J., 15 (1) (1999), 1–9.

[2] S. S. Dragomir, A. Sofo, An integral inequality for twice dif ferentiable mappings and applications, Tamkang J. Math., 31 (4) (2000), 257–266.

[3] S. S. Dragomir, N. S. Barnett, An Ostrowski type inequality for mappings whose second derivatives are bounded and applications, J. Indian Math. Soc. (N.S.), 66 (1–4) (1999), 237–245.

[4] S. S. Dragomir, P. Cerone, J. Roumeliotis, A new generalization of Ostrowski's integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means, Appl. Math. Lett., 13 (2000), 19–25.

[5] Nazia Irshad and Asif R. Khan, On Weighted Ostrowski type Integral Inequality for Twice Differentiable Mappings, (Submitted).

[6] M. Matić, J. Pečarić, N. Ujević, Improvement and further generalization of inequalities of Ostrowski-Gruss type, Computers Math. Appl., 39 (2000), 161–175.

[7] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Inequalities for Functions and Their Integrals and Derivatives, Kluwer Academic, Dordrecht,1994.

[8] A. M. Ostrowski, *Ü*ber die absolutabweichung einer differentiebaren funktion von ihren integralmittelwert, Comment. Math. Helv., 10 (1938), 226–227.

[9] J. E. Pečarić and B. Savić, O novom postupku razvijanja funkcija u red i nekim primjenama, Zbor. Rad. AKoV (Beograd), 9 (1983), 171–202.

[10] N. Ujević, A generalization of Ostrowski's inequality and applications in numerical integration, Appl. Math. Lett., 17 (2004), 133–137.

[11] Fiza Zafar and Nazir Ahmad Mir, A Generalized Integral Inequality for Twice Differentiable Mappings, Kragujevac J. Math., 32 (2009), 81–96.