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ON SOME EXPLICIT BOUNDS ON CERTAIN RETARDED NONLINEAR INTEGRAL INEQUALITIES WITH APPLICATIONS

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Abstract. In this article, we investigate some new nonlinear retarded integral inequalities of Gronwall type. By some techniques, we generalize the results established by Pachpatte in [5] to nonlinear retarded integral inequalities. We also introduce some applications to illustrate the benefit of some of our results.

Keywords: Integral inequality; Analysis technique; Estimation; Retarded integral and differential equation.

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1. Introduction

Retarded Integral inequalities which give us explicit bounds on unknown functions provide a very important device in the study of many qualitative properties of solutions of nonlinear retarded differential equations. Gronwall in [1] proved the famous inequality (see Theorem 1.1). Integral inequalities of Gronwall type are important handy tools in the study of existence, uniqueness and stability of solutions of differential equations. Gronwall inequality has been

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extended to the more general cases including the generalized Gronwall type inequalities and other related inequalities (see [2, 3, 4, 7, 8, 9, 10, 11, 12]).

Theorem 1.1. *Let u be a continuous function defined on the interval* $J=[\alpha,\alpha+h]$ *and*

$$0 \le u(t) \le \int_{\alpha}^{t}[bu(s)+a]ds, \qquad \forall t \in J, \tag{1.1}$$

where a, b are nonnegative constants. Then

$$0 \le u(t) \le ah\exp[bh], \qquad \forall t \in J. \tag{1.2}$$

We introduce the following inequality that used in the proof of our results:

Theorem 1.2. [6] If $x \ge 0$, $y \ge 0$ and $\frac{1}{p}+\frac{1}{q}$ with $p \ge 1$, then

$$x^{\frac{1}{p}}y^{\frac{1}{q}} \le \frac{x}{p}+\frac{y}{q}. \tag{1.3}$$

However, many problems in our real life that have in the past, usually been modeled by initial value problems. In some situations, we need to study some delay nonlinear integral inequalities, so, the inequalities proved in [5] are not directly applicable in the study of delay nonlinear differential equations. It is desirable to investigate some new delay inequalities of the above type, which can be used effectively in the study of certain of delay nonlinear differential equations.

2. Main results

Outset, we locate the set of real numbers by $\mathbb{R}$ and $\mathbb{R}_+=[0,\infty), I=(0,\infty)$ are subsets of $\mathbb{R}$.

Theorem 2.1. *Let* u,a,b,g,h,f *be real-valued nonnegative continuous function defined on* $\mathbb{R}_+$ *and* $\alpha(t)$ *be differentiable, with* $\alpha(t) \le t$, $\alpha(0)=0$ *and* $p \ne 0$, $p \ge q \ge 0$ *be constant. If*

$$u^p(t) \le a(t)+b(t)\int_0^{\alpha(t)}\left[g(s)u^p(s)+h(s)u^q(s)+f(s)\right]ds, \tag{2.1}$$

for all $t \in \mathbb{R}_+$*, then*

$$\begin{aligned} u(t) \leq & \left\{ a(t)+b(t)\exp\left[\int_0^{\alpha(t)} b(r)\left(g(r)+\frac{q}{p}h(r)\right)dr\right] \right. \\ & \times \int_0^{\alpha(t)} \left[g(s)a(s)+h(s)\left(\frac{p-q}{p}+\frac{q}{p}a(s)\right)+f(s)\right] \\ & \left. \times \exp\left[-\int_0^{s} b(r)\left(g(r)+\frac{q}{p}h(r)\right)dr\right]ds \right\}^{1/p}, \end{aligned} \tag{2.2}$$

for all $t \in \mathbb{R}_+$.

Proof. We take the function $z(t)$ by:

$$z(t) = \int_0^{\alpha(t)} \left[g(s)u^p(s)+h(s)u^q(s)+f(s)\right]ds, \qquad \forall t \in \mathbb{R}_+. \tag{2.3}$$

That $z(t) \geq 0$ nondecreasing on $\mathbb{R}_+$ with $z(0) = 0$. Then we can write (2.1) as the following:

$$u^p(t) \leq a(t)+b(t)z(t), \qquad \forall t \in \mathbb{R}_+. \tag{2.4}$$

From (2.4) and using Theorem 1.2, we have,

$$\begin{aligned} u^q(t) & \leq \left(a(t)+b(t)z(t)\right)^{1/\frac{p}{q}} \left(1\right)^{1/\frac{p}{(p-q)}} \\ & \leq \frac{p-q}{p}+\frac{q}{p}a(t)+\frac{q}{p}b(t)z(t), \qquad \forall t \in \mathbb{R}_+. \end{aligned} \tag{2.5}$$

By differentiating (2.3) and using (2.4) and (2.5), we get,

$$\begin{aligned} z'(t) & \leq b(\alpha(t))\left(g(\alpha(t))+\frac{q}{p}h(\alpha(t))\right)z(\alpha(t))\alpha'(t) \\ & \quad +\left[g(\alpha(t))a(\alpha(t))+h(\alpha(t))\left(\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))\right)+f(\alpha(t))\right]\alpha'(t) \\ & \leq b(\alpha(t))\left(g(\alpha(t))+\frac{q}{p}h(\alpha(t))\right)z(t)\alpha'(t) \\ & \quad +\left[g(\alpha)(t)a(\alpha(t))+h(\alpha(t))\left(\frac{p-q}{p}+\frac{q}{p}a(t)\right)+f(\alpha(t))\right]\alpha'(t), \end{aligned} \tag{2.6}$$

for all $t \in \mathbb{R}_+$. The inequality (2.6) gives us the following estimation,

$$\begin{aligned} z(t) \leq\ & \exp\left[\int_0^{\alpha(t)} b(r)\left(g(r)+\frac{q}{p}h(r)\right)dr\right] \\ & \times \int_0^{\alpha(t)} \left[g(s)a(s)+h(s)\left(\frac{p-q}{p}+\frac{q}{p}a(s)\right)+f(s)\right] \\ & \times \exp\left[-\int_0^{s} b(r)\left(g(r)+\frac{q}{p}h(r)\right)dr\right]ds, \end{aligned} \tag{2.7}$$

for all $t \in \mathbb{R}_+$. We get the required inequality (2.2) from (2.7) and (2.4). The proof is completed.

Remark 2.1. If $\alpha(t)=t$, $q=1$ and $f(t)=0$, then Theorem 2.1 reduces to Theorem 1 part (a_1) in [5].

Theorem 2.2. *Let u,a,b,g,h,f be real-valued nonnegative continuous function defined on $\mathbb{R}_+$ and $c(t)$ be a real-valued positive continuous and nondecreasing function defined on $\mathbb{R}_+$ and $\alpha(t)$ be differentiable, with $\alpha(t)\leq t$, $\alpha(0)=0$ and $p\neq 0$, $p\geq q\geq 0$ be constant. If*

$$u^p(t) \leq c^p(t)+b(t)\int_0^{\alpha(t)}\left[g(s)u^p(s)+h(s)u^q(s)+f(s)\right]ds, \tag{2.8}$$

for all $t \in \mathbb{R}_+$, then

$$\begin{aligned} u(t) \leq\ & c(t)\Bigg\{1+b(t)\exp\left[\int_0^{\alpha(t)} b(r)\left(g(r)+\frac{q}{p}h(r)c^{q-p}(r)\right)dr\right] \\ & \times \int_0^{\alpha(t)}\left[g(s)+h(s)c^{q-p}(s)+c^{-p}(s)f(s)\right] \\ & \times \exp\left[-\int_0^{\alpha(t)} b(r)\left(g(r)+\frac{q}{p}h(r)c^{q-p}(r)\right)dr\right]ds\Bigg\}^{1/p}, \end{aligned} \tag{2.9}$$

for all $t \in \mathbb{R}_+$.

Proof. Since $c(t)$ is a positive continuous and nondecreasing function on $\mathbb{R}_+$ from (2.8), we observe that,

$$\left(\frac{u(t)}{c(t)}\right)^p \leq 1+b(t)\int_0^{\alpha(t)}\left[g(s)\left(\frac{u(t)}{c(t)}\right)^p + h(s)c^{q-p}(s)\left(\frac{u(t)}{c(t)}\right)^q + c^{-p}(s)f(s)\right]ds, \tag{2.10}$$

for all $t \in \mathbb{R}_+$. Now an application of the inequality given in theorem 2.1, yields the desired result in (2.9). This completes the proof.

Remark 2.2. If $\alpha(t)=t$, $q=1$ and $f(t)=0$, then Theorem 2.2 reduces to Theorem 1 part (a_2) in [5].

Theorem 2.3. *Let u,a,b,g,h be real-valued nonnegative continuous function defined on $\mathbb{R}_+$ and $k(t,s)$ and its partial derivative $\frac{\partial}{\partial t}k(t,s)$ be real-valued nonnegative continuous function for $0 \le s \le t < \infty$ and $\alpha(t)$ be differentiable, with $\alpha(t) \le t$, $\alpha(0) = 0$ and $p \neq 0$, $p \ge q \ge 0$ be constant. If*

$$u^p(t) \le a(t) + b(t)\int_0^{\alpha(t)} k(t,s)\Big[g(s)u^p(s) + h(s)u^q(s)\Big]ds, \tag{2.11}$$

for all $t \in \mathbb{R}_+$, then

$$u(t) \le \left\{a(t) + b(t)\exp\left[\int_0^{\alpha(t)} A(r)dr\right]\int_0^{\alpha(t)} B(s)\exp\left[-\int_0^{s} A(r)dr\right]ds\right\}^{\frac{1}{p}}, \tag{2.12}$$

for all $t \in \mathbb{R}_+$, where

$$\begin{aligned} A(t) &= k(t,\alpha(t))b(\alpha(t))\left[g(\alpha(t)) + \frac{q}{p}h(\alpha(t))\right]\alpha'(t) \\ &\quad + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)b(s)\left(g(s) + \frac{q}{p}h(s)\right)ds, \end{aligned} \tag{2.13}$$

for all $t \in \mathbb{R}_+$.

$$\begin{aligned} B(t) &= k(t,\alpha(t))\left[g(\alpha(t))a(\alpha(t)) + h(\alpha(t))\left(\frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\right)\right]\alpha'(t) \\ &\quad + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)\left(g(s)a(s) + h(s)\left(\frac{p-q}{p} + \frac{q}{p}a(s)\right)\right)ds, \end{aligned} \tag{2.14}$$

for all $t \in \mathbb{R}_+$.

proof. Define a function $z(t)$ by,

$$z(t) = \int_0^{\alpha(t)} k(t,s)\Big[g(s)u^p(s) + h(s)u^q(s)\Big]ds, \forall t \in \mathbb{R}_+. \tag{2.15}$$

Which is a nonnegative and nondecreasing function on $\mathbb{R}_+$ with $z(0) = 0$. Then as in the proof of Theorem 2.1 from (2.3) we see that the inequalities (1.3) and (2.4) hold. Differentiating (2.15) and using (1.3) and (2.4), we get,

$$\begin{aligned} z'(t) &= k(t,\alpha(t))\Big[g(\alpha(t))u^p(\alpha(t)) + h(\alpha(t))u^q(\alpha(t))\Big]\alpha'(t) \\ &\quad + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)\Big[g(s)u^p(s) + h(s)u^q(s)\Big]ds. \end{aligned} \tag{2.16}$$

For all $t \in \mathbb{R}_+$. From Theorem 2.1 and using (2.4) and (2.5) in (2.16),

$$\begin{aligned} z'(t) \leq\ & k(t,\alpha(t))\Big[g(\alpha(t))\Big(a(\alpha(t))+b(\alpha(t))z(\alpha(t))\Big) \\ & +h(\alpha(t))\Big(\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))+\frac{q}{p}b(\alpha(t))z(\alpha(t))\Big)\Big]\alpha'(t) \\ & +\int_0^{\alpha(t)}\frac{\partial}{\partial t}k(t,s)\Big[g(s)(a(s)+b(s)z(s)) \\ & +h(s)\Big(\frac{p-q}{p}+\frac{q}{p}a(s)+\frac{q}{p}b(s)z(s)\Big)\Big]ds, \end{aligned} \tag{2.17}$$

for all $t \in \mathbb{R}_+$. Using the fact that $z(\alpha) \leq z(t)$, from (2.17), we get,

$$\begin{aligned} z'(t) \leq\ & \Big[k(t,\alpha(t))\Big[b(\alpha(t))(g(\alpha(t))+\frac{q}{p}h(\alpha(t)))\Big]\alpha'(t) \\ & +\int_0^{\alpha(t)}\frac{\partial}{\partial t}k(t,s)b(s)\Big(g(s)+\frac{q}{p}h(s)\Big)ds\Big]z(t) \\ & +k(t,\alpha(t))\Big[g(\alpha(t))a(\alpha(t))+h(\alpha(t))(\frac{p-q}{p}+\frac{q}{p}a(\alpha(t)))\Big]\alpha'(t) \\ & +\int_0^{\alpha(t)}\frac{\partial}{\partial t}k(t,s)\Big[g(s)a(s)+h(s)(\frac{p-q}{p}+\frac{q}{p}a(s))\Big]ds, \\ =\ & A(t)z(t)+B(t), \end{aligned} \tag{2.18}$$

for all $t \in \mathbb{R}_+$. The inequality (2.18) implies the estimate,

$$z(t) \leq \exp\Big[\int_0^{\alpha(t)}A(r)dr\Big]\int_0^{\alpha(t)}B(s)\exp\Big[-\int_0^{s}A(r)dr\Big]ds. \tag{2.19}$$

For all $t \in \mathbb{R}_+$. Using (2.19) in $u^p(t) \leq a(t)+b(t)z(t)$, we get the required inequality in (2.12). This completes the proof.

Remark 2.3. If $\alpha(t) = t$ and $q = 1$, then Theorem 2.3 reduces to Theorem 1 part (a_3) in [5].

Theorem 2.4. *Let u, a, b, g, h be real-valued nonnegative continuous function defined on $\mathbb{R}_+$ and $c(t)$ be a real-valued positive continuous and nondecreasing function defined on $\mathbb{R}_+$, $k(t,s)$ and its partial derivative $\frac{\partial}{\partial t}k(t,s)$ be real-valued nonnegative continuous function for $0 \leq s \leq t < \infty$ and $\alpha(t)$ be differentiable, with $\alpha(t) \leq t$, $\alpha(0) = 0$ and $p \neq 0$, $p \geq q \geq 0$ be constant. If*

$$u^p(t) \leq c^p(t)+b(t)\int_0^{\alpha(t)}k(t,s)\Big[g(s)u^p(s)+h(s)u^q(s)\Big]ds, \tag{2.20}$$

for all $t \in \mathbb{R}_+$, *then*

$$u(t) \le c(t)\left\{1 + b(t)\exp\left[\int_0^{\alpha(t)} A(r)dr\right]\int_0^{\alpha(t)} B(s)\exp\left[-\int_0^{s} A(r)dr\right]ds\right\}^{\frac{1}{p}}, \tag{2.21}$$

for all $t \in \mathbb{R}_+$, *where*

$$\begin{aligned} A(t) &= k(t,\alpha(t))b(\alpha(t))\left(g(\alpha(t)) + \frac{q}{p}h(\alpha(t))c^{q-p}(\alpha(t))\right)\alpha'(t) \\ &\quad + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)(b(s)\left(g(s) + \frac{q}{p}h(s)c^{q-p}(s)\right)ds, \end{aligned} \tag{2.22}$$

for all $t \in \mathbb{R}_+$.

$$\begin{aligned} B(t) &= k(t,\alpha(t))\left[g(\alpha(t)) + h(\alpha(t))c^{q-p}(\alpha(t)) + c^{-p}(\alpha(t))f(\alpha(t))\right]\alpha'(t) \\ &\quad + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)\left[g(s) + h(s)c^{q-p}(s)\right]ds, \end{aligned} \tag{2.23}$$

for all $t \in \mathbb{R}_+$.

Proof. Since $c(t)$ is a positive continuous and nondecreasing function on $\mathbb{R}_+$, from (2.20), we observe that,

$$\left(\frac{u(t)}{c(t)}\right)^p \le 1 + b(t)\int_0^{\alpha(t)} k(t,s)\left[g(s)\left(\frac{u(t)}{c(t)}\right)^p + h(s)c^{q-p}(s)\left(\frac{u(t)}{c(t)}\right)^q\right]ds,$$

for all $t \in \mathbb{R}_+$. Now an application of the inequality given in Theorem 2.3 yields the desired result in (2.21). This completes the proof.

Theorem 2.5. Let u, a, b, g, h, f be real-valued nonnegative continuous function defined on $\mathbb{R}_+$, $k(t,s)$ and its partial derivative $\frac{\partial}{\partial t}k(t,s)$ be real-valued nonnegative continuous function for $0 \le s \le t < \infty$ and $\alpha(t)$ be differentiable, with $\alpha(t) \le t$, $\alpha(0) = 0$ and $p \ne 0$, $p \ge q \ge 0$ be constant. If

$$u^p(t) \le a(t) + b(t)\int_0^{\alpha(t)} k(t,s)\left[g(s)u^p(s) + h(s)u^q(s) + f(s)\right]ds, \tag{2.24}$$

for all $t \in \mathbb{R}_+$, then

$$u(t) \le \left\{a(t) + b(t)\exp\left[\int_0^{\alpha(t)} A(r)dr\right]\int_0^{\alpha(t)} B(s)\exp\left[-\int_0^{s} A(r)dr\right]ds\right\}^{1/p}, \tag{2.25}$$

for all $t \in \mathbb{R}_+$, where

$$\begin{aligned} A(t) &= k(t,\alpha(t))b(\alpha(t))\left[g(\alpha(t))+\frac{q}{p}h(\alpha(t))\right]\alpha'(t) \\ &\quad +\int_0^{\alpha(t)}\frac{\partial}{\partial t}k(t,s)b(s)\left(g(s)+\frac{q}{p}h(s)\right)ds, \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} B(t) &= k(t,\alpha(t))\left[g(\alpha(t))a(\alpha(t))+h(\alpha(t))\left(\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))\right)+f(\alpha(t))\right]\alpha'(t) \\ &\quad +\int_0^{\alpha(t)}\frac{\partial}{\partial t}k(t,s)\left(g(s)a(s)+h(s)\left(\frac{p-q}{p}+\frac{q}{p}a(s)\right)+f(s)\right)ds, \end{aligned} \tag{2.27}$$

for all $t \in \mathbb{R}_+$.

Proof. Define a function $z(t)$ by,

$$z(t)=\int_0^{\alpha(t)}k(t,s)\left[g(s)u^p(s)+h(s)u^q(s)+f(s)\right]ds, \forall t\in\mathbb{R}_+. \tag{2.28}$$

Which is a nonnegative and nondecreasing function on $\mathbb{R}_+$ with $z(0)=0$. Then as in the proof of Theorem 2.1 from (2.3), we see that the inequalities (1.3) and (2.4) hold. Differentiating (2.28) and using (1.3) and (2.4), we get,

$$\begin{aligned} z'(t) &= k(t,\alpha(t))\left[g(\alpha(t))u^p(\alpha(t))+h(\alpha(t))u^q(\alpha(t))+f(\alpha(t))\right]\alpha'(t) \\ &\quad +\int_0^{\alpha(t)}\frac{\partial}{\partial t}k(t,s)\left[g(s)u^p(s)+h(s)u^q(s)+f(s)\right]ds. \end{aligned} \tag{2.29}$$

For all $t \in \mathbb{R}_+$. From Theorem 1.2 and using (2.4) and (2.5) in (2.29), we obtain,

$$\begin{aligned} z'(t) &\leq k(t,\alpha(t))\Bigg[g(\alpha(t))\Big(a(\alpha(t))+b(\alpha(t))z(\alpha(t))\Big) \\ &\quad +h(\alpha(t))\left(\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))+\frac{q}{p}b(\alpha(t))z(\alpha(t))\right)+f(\alpha(t))\Bigg]\alpha'(t) \\ &\quad +\int_0^{\alpha(t)}\frac{\partial}{\partial t}k(t,s)\Bigg[g(s)(a(s)+b(s)z(s)) \\ &\quad +h(s)\left(\frac{p-q}{p}+\frac{q}{p}a(s)+\frac{q}{p}b(s)z(s)\right)+f(s)\Bigg]ds, \end{aligned} \tag{2.30}$$

for all $t \in \mathbb{R}_+$. Using the fact that $z(\alpha(t)) \leq z(t)$, from (2.30), we get,

$$\begin{aligned}
z'(t) &\le \Big[k(t,\alpha(t))\Big[b(\alpha(t))(g(\alpha(t))+\frac{q}{p}h(\alpha(t)))\Big]\alpha'(t) \\
&\quad + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)b(s)\Big(g(s)+\frac{q}{p}h(s)\Big)ds\Big]z(t) \\
&\quad + k(t,\alpha(t))\Big[g(\alpha(t))a(\alpha(t))+h(\alpha(t))(\frac{p-q}{p}+\frac{q}{p}a(\alpha(t)))+f(\alpha(t))\Big]\alpha'(t) \\
&\quad + \int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)\Big[g(s)a(s)+h(s)(\frac{p-q}{p}+\frac{q}{p}a(s))+f(s)\Big]ds, \\
&= A(t)z(t)+B(t), \qquad (2.31)
\end{aligned}$$

for all $t \in \mathbb{R}_+$. The inequality (2.31) implies the estimate,

$$z(t) \le \exp\Big[\int_0^{\alpha(t)} A(r)dr\Big]\int_0^{\alpha(t)} B(s)\exp\Big[-\int_0^s A(r)dr\Big]ds. \qquad (2.32)$$

For all $t \in \mathbb{R}_+$. Using (2.32) in $u^p(t) \le a(t)+b(t)z(t)$, we get the required inequality in (2.25). This completes the proof.

Remark 2.4. If $\alpha(t)=t$, $q=1$ and $f(t)=0$, then Theorem 2.5 reduces to Theorem 1 part (a_3) in [5].

Remark 2.5. If $f(t)=0$, then Theorem 2.5 reduces to Theorem 2.3.

Theorem 2.6. *Let u,a,b,g,h,f be real-valued nonnegative continuous function defined on $\mathbb{R}_+$ and $c(t)$ be a real-valued positive continuous and nondecreasing function defined on $\mathbb{R}_+$, $k(t,s)$ and its partial derivative $\frac{\partial}{\partial t}k(t,s)$ be real-valued nonnegative continuous function for $0 \le s \le t < \infty$ and $\alpha(t)$ be differentiable, with $\alpha(t) \le t$, $\alpha(0)=0$ and $p \ne 0$, $p \ge q \ge 0$ be constant. If*

$$u^p(t) \le c^p(t)+b(t)\int_0^{\alpha(t)} k(t,s)\Big[g(s)u^p(s)+h(s)u^q(s)+f(s)\Big]ds, \qquad (2.33)$$

for all $t \in \mathbb{R}_+$, then

$$u(t) \le c(t)\Big\{1+b(t)\exp\Big[\int_0^{\alpha(t)} A(r)dr\Big]\int_0^{\alpha(t)} B(s)\exp\Big[-\int_0^s A(r)dr\Big]ds\Big\}^{\frac{1}{p}}, \qquad (2.34)$$

for all $t \in \mathbb{R}_+$, where

$$A(t) = k(t,\alpha(t))b(\alpha(t))\left(g(\alpha(t)) + \frac{q}{p}h(\alpha(t))c^{q-p}(\alpha(t))\right)\alpha'(t)$$
$$+\int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)(b(s)\left(g(s) + \frac{q}{p}h(s)c^{q-p}(s)\right)ds, \tag{2.35}$$

and

$$B(t) = k(t,\alpha(t))\left[g(\alpha(t)) + h(\alpha(t))c^{q-p}(\alpha(t)) + c^{-p}(\alpha(t))f(\alpha(t))\right]\alpha'(t)$$
$$+\int_0^{\alpha(t)} \frac{\partial}{\partial t}k(t,s)\left[g(s) + h(s)c^{q-p}(s) + c^{-p}(s)f(s)\right]ds, \tag{2.36}$$

for all $t \in \mathbb{R}_+$.

Proof. Since $c(t)$ is a positive continuous and nondecreasing function on $\mathbb{R}_+$, from (2.33) we observe that,

$$\left(\frac{u(t)}{c(t)}\right)^p \leq 1 + b(t)\int_0^{\alpha(t)} k(t,s)\left[g(s)\left(\frac{u(t)}{c(t)}\right)^p + h(s)c^{q-p}(s)\left(\frac{u(t)}{c(t)}\right)^q + c^{-p}(s)f(s)\right]ds,$$

for all $t \in \mathbb{R}_+$. Now an application of the inequality given in Theorem yields the desired result in (2.34). This completes the proof.

Remark 2.6. If $f(t) = 0$, then Theorem 2.6 reduces to Theorem 2.4.

Theorem 2.7. *Let u,a,b be real-valued nonnegative continuous function defined on $\mathbb{R}_+$ and $\alpha(t)$ differentiable, with $\alpha(0) = 0, \alpha(t) \leq t$ and $p \neq 0$, $p \geq q \geq 0$ be constant and $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that,*

$$0 \leq f(t,x) - f(t,y) \leq m(t,y)(x-y), \forall t \in \mathbb{R}_+, \tag{2.37}$$

and $x \geq 0$, $y \geq 0$, where $m : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

If

$$u^p(t) \leq a(t) + b(t)\int_o^{\alpha(t)} \left[f(s,u^q(s))\right]ds, \tag{2.38}$$

for all $t \in \mathbb{R}_+$, then

$$\begin{aligned} u(t) \leq & \Big\{a(t) + b(t)\exp\Big[\int_0^{\alpha(t)} m(r, \frac{p-q}{p} + \frac{q}{p}a(r))\frac{q}{p}b(r)dr\Big] \\ & \times \int_0^{\alpha(t)} \Big[f(s, \frac{p-q}{p} + \frac{q}{p}a(s))\Big] \\ & \times \exp\Big[-\int_0^s m(r, \frac{p-q}{p} + \frac{q}{p}a(r))\frac{q}{p}b(r)dr\Big]ds\Big\}^{1/p}, \end{aligned} \tag{2.39}$$

for all $t \in \mathbb{R}_+$.

Proof. Define a function $z(t)$ by,

$$z(t) = \int_0^{\alpha(t)} \Big(f(s, u^q(s))\Big)ds, \forall t \in \mathbb{R}_+. \tag{2.40}$$

Which is a nonnegative and nondecreasing function on $\mathbb{R}_+$ with $z(0) = 0$. Then as in the proof of Theorem 2.1 from (2.38) we see that the inequality (2.4) and (2.5) hold. from (2.40) and (2.5) and the condition (2.37) it follows that,

$$\begin{aligned} z'(t) &= f\Big(\alpha(t), u^q(\alpha(t))\Big)\alpha'(t) \\ &\leq f\Big(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t)) + \frac{q}{p}b(\alpha(t))z(\alpha(t))\Big)\alpha'(t) \\ &\leq f\Big(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t)) + \frac{q}{p}b(\alpha(t))z(t)\Big)\alpha'(t) \\ &\quad - f\Big(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\Big)\alpha'(t) + f\Big(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\Big)\alpha'(t) \\ &\leq m\Big(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\Big)\frac{q}{p}b(\alpha(t))z(t)\alpha'(t) \\ &\quad + f\Big(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\Big)\alpha'(t). \end{aligned} \tag{2.41}$$

For all $t \in \mathbb{R}_+$. The inequality (2.41) implies the estimate,

$$\begin{aligned} z(t) \leq & \exp\Big[\int_0^{\alpha(t)} m\Big(r, \frac{p-q}{p} + \frac{q}{p}a(r)\Big)\frac{q}{p}b(r)dr\Big] \\ & \times \int_0^{\alpha(t)} \Big[f\Big(s, \frac{p-q}{p} + \frac{q}{p}a(s)\Big)\Big] \\ & \times \exp\Big[-\int_0^s m\Big(r, \frac{p-q}{p} + \frac{q}{p}a(r)\Big)\frac{q}{p}b(r)dr\Big]ds. \end{aligned} \tag{2.42}$$

For all $t \in \mathbb{R}_+$. From (2.41) and (2.4) the desired inequality in (2.39) follows. This completes the proof.

Remark 2.7. If $\alpha(t) = t$ and $q = 1$, then Theorem 2.7 reduces to Theorem 2 part (b_1) in [5].

Theorem 2.8. *Let u, a, b, h be real-valued nonnegative continuous function defined on $\mathbb{R}_+$ and $\alpha(t)$ differentiable, with $\alpha(0) = 0, \alpha(t) \leq t$ and $p \neq 0$, $p \geq q \geq 0$ be constant and $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that,*

$$0 \leq f(t,x) - f(t,y) \leq m(t,y)(x-y), \forall t \in \mathbb{R}_+, \tag{2.43}$$

and $x \geq 0$, $y \geq 0$, where $m : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.
If

$$u^p(t) \leq a(t) + b(t) \int_o^{\alpha(t)} \Big[f(s, u^q(s)) + h(s) \Big] ds, \tag{2.44}$$

for all $t \in \mathbb{R}_+$, then

$$\begin{aligned} u(t) \quad \leq \quad & \Bigg\{ a(t) + b(t) \exp\Big[\int_0^{\alpha(t)} m(r, \frac{p-q}{p} + \frac{q}{p} a(r)) \frac{q}{p} b(r) dr \Big] \\ & \times \int_0^{\alpha(t)} \Big[f(s, \frac{p-q}{p} + \frac{q}{p} a(s)) + h(s) \Big] \\ & \times \exp\Big[- \int_0^s m(r, \frac{p-q}{p} + \frac{q}{p} a(r)) \frac{q}{p} b(r) dr \Big] ds \Bigg\}^{\frac{1}{p}}, \end{aligned} \tag{2.45}$$

for all $t \in \mathbb{R}_+$.

Proof. Define a function $z(t)$ by,

$$z(t) = \int_0^{\alpha(t)} \Big(f(s, u^q(s)) + h(s) \Big) ds, \forall t \in \mathbb{R}_+. \tag{2.46}$$

Which is a nonnegative and nondecreasing function on $\mathbb{R}_+$ with $z(0) = 0$. Then as in the proof of Theorem 2.1 from (2.44) we see that the inequality (2.4) and (2.5) hold. from (2.46) and

(2.5) and the condition (2.43) it follows that,

$$
\begin{aligned}
z'(t) &= \left(f\left(\alpha(t),u^q(\alpha(t))\right)+h(\alpha(t))\right)\alpha'(t) \\
&\leq \left[f\left(\alpha(t),\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))+\frac{q}{p}b(\alpha(t))z(\alpha(t))\right)+h(\alpha(t))\right]\alpha'(t) \\
&\leq f\left(\alpha(t),\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))+\frac{q}{p}b(\alpha(t))z(t)\right)\alpha'(t)+h(\alpha(t))\alpha'(t) \\
&\quad -f\left(\alpha(t),\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))\right)\alpha'(t)+f\left(\alpha(t),\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))\right)\alpha'(t) \\
&\leq m\left(\alpha(t),\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))\right)\frac{q}{p}b(\alpha(t))z(t)\alpha'(t) \\
&\quad +f\left(\alpha(t),\frac{p-q}{p}+\frac{q}{p}a(\alpha(t))\right)\alpha'(t)+h(\alpha(t))\alpha'(t). \qquad (2.47)
\end{aligned}
$$

For all $t \in \mathbb{R}_+$. The inequality (2.47) implies the estimate,

$$
\begin{aligned}
z(t) &\leq \exp\left[\int_0^{\alpha(t)} m\left(r,\frac{p-q}{p}+\frac{q}{p}a(r)\right)\frac{q}{p}b(r)dr\right] \\
&\quad \times \int_0^{\alpha(t)}\left[f\left(s,\frac{p-q}{p}+\frac{q}{p}a(s)\right)+h(s)\right] \\
&\quad \times \exp\left[-\int_0^{s} m\left(r,\frac{p-q}{p}+\frac{q}{p}a(r)\right)\frac{q}{p}b(r)dr\right]ds. \qquad (2.48)
\end{aligned}
$$

For all $t \in \mathbb{R}_+$. From (2.47) and (2.4) the desired inequality in (2.45) follows. This completes the proof.

Remark 2.8. If $\alpha(t) = t$, $q = 1$ and $h(t) = 0$, then Theorem 2.8 reduces to Theorem 2 part (b_1) in [5].

Remark 2.9. If $h(t) = 0$, then Theorem 2.8 reduces to Theorem 2.7.

Theorem 2.9. *Let u,a,b be real-valued nonnegative continuous function defined on $\mathbb{R}_+$ and $\alpha(t)$ differentiable, with $\alpha(t) \leq t$, $\alpha(0) = 0$, and $p \neq 0$, $p \geq q \geq 0$ be constant and $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function and $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function with $\phi(0) = 0$ such that,*

$$0 \leq f(t,x) - f(t,y) \leq m(t,y)\phi^{-1}(x-y), \qquad (2.49)$$

for all $t \in \mathbb{R}_+$, and $x \geq y \geq 0$, where $m : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and ϕ^{-1} is the inverse function of ϕ and,

$$\phi^{-1}(xy) \leq \phi^{-1}(x)\phi^{-1}(y), \tag{2.50}$$

for all $x, y \in \mathbb{R}_+$. If

$$u^p(t) \leq a(t) + b(t)\phi \int_o^{\alpha(t)} \Big[f(s, u^q(s)) \Big] ds, \tag{2.51}$$

for all $t \in \mathbb{R}_+$, then

$$\begin{aligned} u(t) \leq & \Bigg\{ a(t) + b(t)\phi \Bigg[\exp\Bigg[\int_0^{\alpha(t)} m(r, \frac{p-q}{p} + \frac{q}{p}a(r))\phi^{-1}(\frac{q}{p}b(r))dr \Bigg] \\ & \times \int_0^{\alpha(t)} \Bigg[f(s, \frac{p-q}{p} + \frac{q}{p}a(s)) \Bigg] \\ & \times \exp\Bigg[-\int_0^s m(r, \frac{p-q}{p} + \frac{q}{p}a(r))\phi^{-1}(\frac{q}{p}b(r))dr \Bigg] ds \Bigg] \Bigg\}^{1/p}, \end{aligned} \tag{2.52}$$

for all $t \in \mathbb{R}_+$.

Proof. Defining a function $z(t)$ by (2.40) and following the arguments as in the proof of theorem 2.1 we see that corresponding to the inequalities (2.4) and (2.5), we get,

$$u^p(t) \leq a(t) + b(t)\phi(z(t)), \tag{2.53}$$

and

$$u^q(t) \leq \frac{p-q}{p} + \frac{q}{p}a(t) + \frac{q}{p}b(t)\phi(z(t)), \qquad \forall t \in \mathbb{R}_+. \tag{2.54}$$

From (2.40) and (2.54) it follows that,

$$\begin{aligned} z'(t) &= f\Big(\alpha(t), u^q(\alpha(t))\Big)\alpha'(t) \\ &\leq f\Big(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t)) + \frac{q}{p}b(\alpha(t))\phi(z(\alpha(t)))\Big)\alpha'(t) \\ &\leq f\Big(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t)) + \frac{q}{p}b(\alpha(t))\phi(z(t))\Big)\alpha'(t) \\ & \quad - f\Big(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\Big)\alpha'(t) + f\Big(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\Big)\alpha'(t). \end{aligned} \tag{2.55}$$

For all $t \in \mathbb{R}_+$. Using the condition (2.49) in (2.55), we obtain,

$$
\begin{aligned}
z'(t) &\le m\left(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\right)\phi^{-1}\left(\frac{q}{p}b(\alpha(t))\phi(z(t))\right)\alpha'(t) \\
&\quad + f\left(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\right)\alpha'(t). \qquad (2.56)
\end{aligned}
$$

For all $t \in \mathbb{R}_+$. Using the condition (2.50) in (2.56), we get,

$$
\begin{aligned}
z'(t) &\le m\left(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\right)\phi^{-1}\left(\frac{q}{p}b(\alpha(t))\right)z(t)\alpha'(t) \\
&\quad + f\left(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\right)\alpha'(t). \qquad (2.57)
\end{aligned}
$$

For all $t \in \mathbb{R}_+$. The inequality (2.57) implies the estimate,

$$
\begin{aligned}
z(t) &\le \exp\left[\int_0^{\alpha(t)} m\left(r, \frac{p-q}{p} + \frac{q}{p}a(r)\right)\phi^{-1}\left(\frac{q}{p}b(r)\right)dr\right] \\
&\quad \times \int_0^{\alpha(t)} \left[f\left(s, \frac{p-q}{p} + \frac{q}{p}a(s)\right)\right] \\
&\quad \times \exp\left[-\int_0^{s} m\left(r, \frac{p-q}{p} + \frac{q}{p}a(r)\right)\phi^{-1}\left(\frac{q}{p}b(r)\right)dr\right]ds. \qquad (2.58)
\end{aligned}
$$

For all $t \in \mathbb{R}_+$. The required inequality (2.52) follows from (2.53) and (2.58). This completes the proof.

Remark 2.10. If $\alpha(t) = t$ and $q = 1$, then Theorem 2.9 reduces to Theorem 2 part (b_2) in [5].

Theorem 2.10. *Let u, a, b, h be real-valued nonnegative continuous function defined on $\mathbb{R}_+$ and $\alpha(t)$ differentiable, with $\alpha(t) \le t$, $\alpha(0) = 0$, and $p \ne 0$, $p \ge q \ge 0$ be constant and $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function and $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function with $\phi(0) = 0$ such that,*

$$0 \le f(t,x) - f(t,y) \le m(t,y)\phi^{-1}(x-y), \qquad (2.59)$$

for all $t \in \mathbb{R}_+$, and $x \ge y \ge 0$, where $m : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and ϕ^{-1} is the inverse function of ϕ and,

$$\phi^{-1}(xy) \le \phi^{-1}(x)\phi^{-1}(y), \qquad (2.60)$$

for all $x, y \in \mathbb{R}_+$. If

$$u^p(t) \le a(t) + b(t)\phi \int_o^{\alpha(t)} \Big[f(s, u^q(s)) + h(s) \Big] ds, \tag{2.61}$$

for all $t \in \mathbb{R}_+$, then

$$\begin{aligned} u(t) \quad \le \quad & \Bigg\{ a(t) + b(t)\phi \Bigg[\exp\Bigg[\int_0^{\alpha(t)} m(r, \frac{p-q}{p} + \frac{q}{p} a(r)) \phi^{-1}(\frac{q}{p} b(r)) dr \Bigg] \\ & \times \int_0^{\alpha(t)} \Bigg[f(s, \frac{p-q}{p} + \frac{q}{p} a(s)) + h(s) \Bigg] \\ & \times \exp\Bigg[- \int_0^s m(r, \frac{p-q}{p} + \frac{q}{p} a(r)) \phi^{-1}(\frac{q}{p} b(r)) dr \Bigg] ds \Bigg] \Bigg\}^{1/p}, \end{aligned} \tag{2.62}$$

for all $t \in \mathbb{R}_+$.

Proof. Defining a function $z(t)$ by (2.46) and following the arguments as in the proof of theorem 2.1 we see that corresponding to the inequalities (2.4) and (2.5), we get,

$$u^p(t) \le a(t) + b(t)\phi(z(t)), \tag{2.63}$$

and

$$u^q(t) \le \frac{p-q}{p} + \frac{q}{p} a(t) + \frac{q}{p} b(t)\phi(z(t)), \forall t \in \mathbb{R}_+. \tag{2.64}$$

From (2.46) and (2.64) it follows that,

$$\begin{aligned} z'(t) \quad = \quad & \Bigg(f\Big(\alpha(t), u^q(\alpha(t)) \Big) + h(\alpha(t)) \Bigg) \alpha'(t) \\ \le \quad & \Bigg[f\Bigg(\alpha(t), \frac{p-q}{p} + \frac{q}{p} a(\alpha(t)) + \frac{q}{p} b(\alpha(t)) \phi(z(\alpha(t))) \Bigg) + h(\alpha(t)) \Bigg] \alpha'(t) \\ \le \quad & f\Bigg(\alpha(t), \frac{p-q}{p} + \frac{q}{p} a(\alpha(t)) + \frac{q}{p} b(\alpha(t)) \phi(z(t)) \Bigg) \alpha'(t) + h(\alpha(t)) \alpha'(t) \\ & - f\Bigg(\alpha(t), \frac{p-q}{p} + \frac{q}{p} a(\alpha(t)) \Bigg) \alpha'(t) + f\Bigg(\alpha(t), \frac{p-q}{p} + \frac{q}{p} a(\alpha(t)) \Bigg) \alpha'(t). \end{aligned} \tag{2.65}$$

For all $t \in \mathbb{R}_+$. Using the condition (2.59) in (2.65), we obtain,

$$\begin{aligned} z'(t) \quad \le \quad & m\Bigg(\alpha(t), \frac{p-q}{p} + \frac{q}{p} a(\alpha(t)) \Bigg) \phi^{-1}\Bigg(\frac{q}{p} b(\alpha(t)) \phi(z(t)) \Bigg) \alpha'(t) \\ & + f\Bigg(\alpha(t), \frac{p-q}{p} + \frac{q}{p} a(\alpha(t)) \Bigg) \alpha'(t) + h(\alpha(t)) \alpha'(t). \end{aligned} \tag{2.66}$$

For all $t \in \mathbb{R}_+$. Using the condition (2.60) in (2.66) , we get,

$$
\begin{aligned}
z'(t) \quad \le \quad & m\left(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\right)\phi^{-1}\left(\frac{q}{p}b(\alpha(t))\right)z(t)\alpha'(t) \\
& + f\left(\alpha(t), \frac{p-q}{p} + \frac{q}{p}a(\alpha(t))\right)\alpha'(t) + h(\alpha(t))\alpha'(t).
\end{aligned} \tag{2.67}
$$

For all $t \in \mathbb{R}_+$. The inequality (2.67) implies the estimate,

$$
\begin{aligned}
z(t) \quad \le \quad & \exp\left[\int_0^{\alpha(t)} m\left(r, \frac{p-q}{p} + \frac{q}{p}a(r)\right)\phi^{-1}\left(\frac{q}{p}b(r)\right)dr\right] \\
& \times \int_0^{\alpha(t)}\left[f\left(s, \frac{p-q}{p} + \frac{q}{p}a(s)\right) + h(s)\right] \\
& \times \exp\left[-\int_0^{s} m\left(r, \frac{p-q}{p} + \frac{q}{p}a(r)\right)\phi^{-1}\left(\frac{q}{p}b(r)\right)dr\right]ds.
\end{aligned} \tag{2.68}
$$

For all $t \in \mathbb{R}_+$. The required inequality (2.62) follows from (2.63) and (2.68). This completes the proof.

Remark 2.11. If $\alpha(t) = t$, $q = 1$ and $h(t) = 0$, then 10.2 reduces to Theorem 2 part (b_2) in [5].

Remark 2.12. If $h(t) = 0$, then Theorem 2.10 reduces to Theorem 2.9.

3. Application

In this section we offer some applications of the inequalities presented in our Theorems 2.1 in order to illustrate the benefits of our results

Example 3.1.

consider the following retarded differential equation,

$$
\begin{cases}
pu^{p-1}(t)\frac{du(t)}{dt} = M(t, u^p(\alpha(t)), H(t, u^q(\alpha(t)))), \forall t \in \mathbb{R}_+, \\
\\
u(0) = u_0,
\end{cases} \tag{3.1}
$$

where M and H are real-valued nonnegative continuous function defined on $[\mathbb{R}^3{}_+, \mathbb{R}]$, $|u_0| > 0$ is a constant, satisfy the following condition:

$$
|H(t, u)| \le h(t)|u| \qquad \forall t \in \mathbb{R}_+, \tag{3.2}
$$

$$|M(t,u,w)| \leq g(t)|u| + w \qquad \forall t \in \mathbb{R}_+, \tag{3.3}$$

where $u(t), g(t), h(t), (\alpha(t))$, p and q as defined in theorem 2.1. Integrate both sides of the equation (3.1) from 0 to t, we have the equivalent the following integral equation:

$$u^p(t) = u^p{}_0 + \int_0^t M(s, u^p(\alpha(s)), H(s, u^q(\alpha(s))))ds, \tag{3.4}$$

for all $t \in \mathbb{R}_+$, using the condition (3.2) and (3.3), from (3.4), we get,

$$\begin{aligned} |u(t)|^p &\leq |u_0|^p + \int_0^t \Big\{ g(s)u^p(\alpha(s)) + h(s)u^q(\alpha(s)) \Big\} ds \\ &\leq |u_0|^p + \int_0^{\alpha(t)} \Big\{ \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} u^p(s) + \frac{h(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} u^q(s) \Big\} ds \end{aligned}$$

for all $t \in \mathbb{R}_+$. Now a suitable application of theorem 2.1 with $b(t) = 1$ and $f(t) = 0$ yields.

$$\begin{aligned} |u(t)| \leq &\Bigg\{ |u_0|^p + \exp\left[\int_0^{\alpha(t)} \left(\frac{g(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + \frac{qh(\alpha^{-1}(r))}{p\alpha'(\alpha^{-1}(r))} \right) dr \right] \\ &\times \int_0^{\alpha(t)} \left[\frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} |u_0|^p + \frac{h(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \left(\frac{p-q}{p} + \frac{q|u_0|^p}{p} \right) \right] \\ &\times \exp\left[-\int_0^s \left(\frac{g(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} + \frac{qh(\alpha^{-1}(r))}{p\alpha'(\alpha^{-1}(r))} \right) dr \right] ds \Bigg\}^{1/p} \end{aligned} \tag{3.5}$$

for all $t \in \mathbb{R}_+$. Thus, the estimation in (3.5), implies the boundedness of the solution u(t) of (3.1).

Conflict of Interests

The authors declare that there is no conflict of interests.

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