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CONVEXITY FOR RATIO OF DIFFERENCE OF SOME SPECIAL MEANS IN TWO VARIABLES

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Abstract. In this paper, we study the Schur properties of convexities(concave) like Schur, Schur Geometric, Schur Harmonic convexities on the ratio of difference of means obtained by arithmetic mean, geometric mean, harmonic mean, contra harmonic mean, heron mean and root-square means. Also, established some inequalities related to these means.

Keywords: means; inequality; harmonic convexity.

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1. Introduction

In [19], Taneja has established chain of inequality for the binary means as follows.

$$H(a,b) \leq G(a,b) \leq N_1(a,b) \leq H_e(a,b) \leq N_2(a,b) \leq A(a,b) \leq S(a,b) \tag{1.1}$$

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For the real numbers $a,\ b>0$, $A(a,b)=\frac{a+b}{2}$, $G(a,b)=\sqrt{ab}$, $H(a,b)=\frac{2ab}{a+b}$, $H_e(a,b)=\frac{a+\sqrt{ab}+b}{3}$, $S(a,b)=\sqrt{\frac{a^2+b^2}{2}}$, $C(a,b)=\frac{a^2+b^2}{a+b}$ are respectively called arithmetic mean, geometric mean, harmonic mean, heron mean, root square mean and contra-harmonic mean.
$N_1(a,b)=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2$ $N_2(a,b)=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right)$ are the subsidiary means discussed in [19]

Many ratio of difference of well known means are studied in [13]

$$M_{SA}(a,b)=S(a,b)-A(a,b) \tag{1.2}$$

$$M_{SN_2}(a,b)=S(a,b)-N_2(a,b) \tag{1.3}$$

$$M_{SN_3}(a,b)=S(a,b)-N_3(a,b) \tag{1.4}$$

$$M_{SH_e}(a,b)=S(a,b)-H_e(a,b) \tag{1.5}$$

$$M_{SN_1}(a,b)=S(a,b)-N_1(a,b) \tag{1.6}$$

$$M_{SG}(a,b)=S(a,b)-G(a,b) \tag{1.7}$$

$$M_{SH}(a,b)=S(a,b)-H(a,b) \tag{1.8}$$

$$M_{AN_2}(a,b)=A(a,b)-N_2(a,b) \tag{1.9}$$

$$M_{AG}(a,b)=A(a,b)-G(a,b) \tag{1.10}$$

$$M_{AH}(a,b)=A(a,b)-H(a,b) \tag{1.11}$$

$$M_{N_2N_1}(a,b)=N_2(a,b)-N_1(a,b) \tag{1.12}$$

$$M_{N_2G}(a,b)=N_2(a,b)-G(a,b) \tag{1.13}$$

The above difference of means are nonnegative and convex in $R_+^2\to(0,\infty)\times(0,\infty)$.

In this paper, we discussed convexities(concavity) related to ratio of difference of dual means.

2. Preliminaries

This section deals with prerequisites essential to development of our work.

Lemma 2.1. *In* [14] *Jamal Rooin and Mehdi Hassni introduced the homogeneous functions $f(x)$ and $g(x)$, where*

$$f(x) = \frac{a^x - b^x}{c^x - d^x} \quad and \quad g(x) = ln\frac{a^x - b^x}{c^x - d^x}, \qquad for \quad x \in (-\infty, \infty) \tag{2.1}$$

(1) convex, if $ad - bc \geq 0$

(2) concave, if $ad - bc \leq 0$ and

(3) equalities holds, if $ad - bc = 0$. for $a > b \geq c > d > 0$.

The Schur convex function was introduced by I. Schur, In 1923 and it has many important applications in analytic inequalities. In 2003, X.M. Zhang propose the concept of Schur-geometrically convex function which is an extension of Schur-convexity function ([5]-[18]).

Definition 2.1. [13], [20] *Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n) \in R^n$*

(1) x is majorized by y, (in symbol $x \prec y$). If $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ and $\sum_{i=1}^{n} x_{[i]} \leq \sum_{i=1}^{n} y_{[i]}$, where $x_{[1]} \geq, ..., \geq x_{[n]}$ and $y_{[1]} \geq, ..., \geq y_{[n]}$ are rearrangements of x and y in descending order.

(2) $x \geq y$ means $x_i \geq y_i$ for all $i = 1, 2.....n$. Let $\Omega \in R^n (n \geq 2)$. The function $\varphi : \Omega \to R$ is said to be decreasing if and only if $-\varphi$ is increasing.

(3) $\Omega \subseteq R^n$ is called a convex set if $(\alpha x_1 + \beta y_1, ..., \alpha x_n + \beta y_n)$ for every x and $y \in \Omega$ where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

(4) Let $\Omega \subseteq R^n$ the function $\varphi : \Omega \to R$ be said to be a Schur convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \leq \varphi(y)$ then φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is Schur convex.

Definition 2.2. *Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n) \in R_+^n$.*
Let $\Omega \subseteq R^n$ is called harmonically convex set if $(x_1^{\alpha} y_1^{\beta}, ..., x_1^{\alpha} y_1^{\beta}) \in \Omega$ for all x and $y \in \Omega$ where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. Let $\Omega \subseteq R_+^n$, the function $\varphi : \Omega \to R_+$ is said to be Schur harmonically convex function on Ω if $(lnx_1, ..., ln_n) \prec (lny_1, ..., lny_n)$ on Ω implies $\varphi(x) \leq \varphi(y)$ then φ is

said to be a Schur harmonically concave function on Ω *if and only if* $-\varphi$ *is Schur harmonically convex.*

Definition 2.3. [11],[20] *Let* $\Omega \subseteq R^n$ *is called symmetric set if* $x \in \Omega$ *implies* $Px \in \Omega$ *for every* $n \times n$ *permutation matrix* P *the function* $\varphi : \Omega \to R$ *is called symmetric if for every permutation matrix* P, $\varphi(Px) = \varphi(x)$ *for all* $x \in \Omega$.

Definition 2.4. *Let* $\Omega \subseteq R^n$ $\varphi : \Omega \to R$ *is called symmetric and convex function. Then* φ *is Schur convex on* Ω .

Lemma 2.2. *Let* $\Omega \subseteq R^n$ *be symmetric with non empty interior convex set and let* $\varphi : \Omega \to R_+$ *be continuous on* Ω *and differentiable in* Ω^0.*If* φ *is symmetric on* Ω *and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0(\leq 0). \tag{2.2}$$

holds for any $x = (x_1, x_2, ..., x_n) \in \Omega^0$, *then* φ *is a Schur convex (Schur concave) function.*

Lemma 2.3. *Let* $\Omega \subseteq R^n$ *be symmetric with non empty interior geometrically convex set and let* $\varphi : \Omega \to R_+$ *be continuous on* Ω *and differentiable in* Ω^0. *If* φ *is symmetric on* Ω *and*

$$(lnx_1 - lnx_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \geq 0(\leq 0). \tag{2.3}$$

holds for any $x = (x_1, x_2, ..., x_n) \in \Omega^0$, *then* φ *is a Schur-geometrically convex (Schur-geometrically concave) function.*

Lemma 2.4. *Let* $\Omega \subseteq R^n$ *be symmetric with non empty interior Harmonically convex set and let* $\varphi : \Omega \to R_+$ *be continuous on* Ω *and differentiable in* Ω^0. *If* φ *is symmetric on* Ω *and*

$$(x_1 - x_2)(x_1^2\frac{\partial \varphi}{\partial x_1} - x_2^2\frac{\partial \varphi}{\partial x_2}) \geq 0(\leq 0) \tag{2.4}$$

holds for any $x = (x_1, x_2, ..., x_n) \in \Omega^0$, *then* φ *is a Schur-harmonically convex (Schur-harmonically concave) function.*

3. Main results

We set of this section with the difference of means defined in the equations 1.2 to 1.13. The difference between the means are as follows.

$$M_{SN_2}(a,b) - M_{SA}(a,b) = A(a,b) - N_2(a,b)$$

$$M_{SH_e}(a,b) - M_{SN_2}(a,b) = N_2(a,b) - H_e(a,b)$$

$$M_{SH_e}(a,b) - M_{SA}(a,b) = A(a,b) - H_e(a,b)$$

$$M_{SN_1}(a,b) - M_{SH_e}(a,b) = H_e(a,b) - N_1(a,b)$$

$$M_{SN_1}(a,b) - M_{SN_2}(a,b) = N_2(a,b) - N_1(a,b)$$

$$M_{N_1H}(a,b) - M_{GH}(a,b) = N_1(a,b) - G(a,b)$$

$$M_{SH}(a,b) - M_{SG}(a,b) = G(a,b) - H(a,b)$$

The above difference of the means are convex for all positive real value of $'t'$.

Now, we establish the ratio of difference of above means as follows:

$$\frac{M_{SN_2} - M_{SA}}{M_{SH_e} - M_{SN_2}} = \frac{A - N_2}{N_2 - H_e}$$

$$\frac{M_{SH_e} - M_{SA}}{M_{SN_1} - M_{SH_e}} = \frac{A - H_e}{H_e - N_1}$$

Theorem 3.1. *For $a > b > 0$, the ratio of difference of mean*

$$\frac{M_{SN_2} - M_{SA}}{M_{SH_e} - M_{SN_2}} = \frac{A - N_2}{N_2 - H_e}$$

is convex for all positive real values of 't'.

Proof. Let

$$f(t,1)=\frac{M_{SN_2}(t,1)-M_{SA}(t,1)}{M_{SH_e}(t,1)-M_{SN_2}(t,1)}=\frac{A(t,1)-N_2(t,1)}{N_2(t,1)-H_e(t,1)}$$

Using Lemma 2.1

$$f(t,1)=\ AH_e-N_2^2$$

$$f(t,1)=(\frac{t+1}{2})(\frac{t+1+\sqrt{t}}{3})-(\frac{\sqrt{t}+1}{2}\sqrt{\frac{t+1}{2}})^2$$

$$f(t,1)=\frac{t+1}{2}(\frac{t+1+\sqrt{t}}{3}-(\frac{\sqrt{t}+1}{2})^2)$$

$$f(t,1)=\frac{(t+1)(\sqrt{t}-1)^2}{24}\ \ t\geq 0$$

.

Hence, the ratio of difference of mean is convex for all positive real values of $'t'$. ☐

Theorem 3.2. *For $a>b>0$, the ratio of difference of mean*

$$\frac{M_{SH_e}-M_{SA}}{M_{SN_1}-M_{SH_e}}=\frac{A-H_e}{H_e-N_1} \tag{3.1}$$

is convex for all positive real values of $'t'$.

Proof. Let

$$f(t,1)=\frac{M_{SH_e}(t,1)-M_{SA}(t,1)}{M_{SN_1}(t,1)-M_{SH_e}(t,1)}=\frac{A(t,1)-H_e(t,1)}{H_e(t,1)-N_1(t,1)} \tag{3.2}$$

Using Lemma 2.1,

$$f(t,1)=\ AN_1-H_e^2$$

$$f(t,1)=(\frac{t+1}{2})(\frac{\sqrt{t}+1}{2})^2-(\frac{t+1+\sqrt{t}}{3})^2$$

$$f(t,1)=\frac{t^2-6t+1+2\sqrt{t}+2t\sqrt{t}}{72}\geq 0 forall \qquad t\geq 0$$

Hence it is convex for all positive real values of $'t'$. ☐

4. Schur convexity of difference of Means

In this section, the Schur convex, the Schur-geometric, the Schur-harmonic convexity for the difference of means are established.

Theorem 4.1. *The ratio of difference of mean*

$$\frac{M_{SN_2} - M_{SA}}{M_{SH_e} - M_{SN_2}}$$

is

1. *Schur convex*
2. *Schur geometrical convex*
3. *Schur harmonical convex, for all $a \geq b$.*

Proof. Let

$$f(a,b) = \frac{M_{SN_2}(a,b) - M_{SA}(a,b)}{M_{SH_e}(a,b) - M_{SN_2}(a,b)} = \frac{A(a,b) - N_2(a,b)}{N_2(a,b) - H_e(a,b)}$$

by Lemma 2.1,

$$f(a,b) = \; AH_e - N_2^2$$

$$f(a,b) = \frac{a+b}{2}\frac{a+b+\sqrt{ab}}{3} - (\frac{\sqrt{a}+\sqrt{b}}{2}\sqrt{\frac{a+b}{2}})^2$$

$$f(a,b) = \frac{(a+b)(\sqrt{a}-\sqrt{b})^2}{24}$$

$$f(a,b) = \frac{a^2+b^2+2ab-2a\sqrt{ab}-2b\sqrt{ab}}{24}$$

by finding the partial derivatives of $f(a,b)$ and simple manipulation we have,

$$\frac{\partial f}{\partial a} = \frac{2a+2b-3\sqrt{ab}-b(\sqrt{b}/\sqrt{a})}{24}$$

$$\frac{\partial f}{\partial b} = \frac{2b+2a-3\sqrt{ab}-a(\sqrt{a}/\sqrt{b})}{24}$$

Proof of (i):

$$Then\ s = (\frac{\partial f}{\partial a} - \frac{\partial f}{\partial b}) = \frac{1}{24}(a\frac{\sqrt{a}}{\sqrt{b}} - b\frac{\sqrt{b}}{\sqrt{a}})$$

$$Then\ s = (a-b)(\frac{\partial f}{\partial a} - \frac{\partial f}{\partial b}) = \frac{a-b}{24}(\frac{a^2-b^2}{\sqrt{ab}}) \geq 0\ for\ a \geq b.$$

This verifies the condition for Schur convexity.

Proof of (ii): we have

$$a\frac{\partial f}{\partial a} - b\frac{\partial f}{\partial b} = (\frac{a-b}{12})(a+b-\sqrt{ab})$$

$$Then\ s = (lna - lnb)\sum a\frac{\partial f}{\partial a} - \sum b\frac{\partial f}{\partial b} = (lna - lnb)(\frac{a-b}{12})(a+b-\sqrt{ab}) \geq 0\ for\ a \geq b.$$

This verifies the condition for Schur geometrically convex.

Proof of (iii): we have

$$a^2\frac{\partial f}{\partial a} - b^2\frac{\partial f}{\partial b} = (\frac{a-b}{24})(2a^2+2b^2+4ab-3\sqrt{ab}(a+b))$$

Then

$$s = (a-b)\sum a^2\frac{\partial f}{\partial a} - \sum b^2\frac{\partial f}{\partial b} = (a-b)(\frac{a-b}{24})(2a^2+2b^2+4ab-3\sqrt{ab}(a+b)) \geq 0\ for\ a \geq b.$$

This verifies the condition for Schur harmonic convex. □

Theorem 4.2. *The ratio of difference of mean*

$$\frac{M_{SH_e} - M_{SA}}{M_{SN_1} - M_{SH_e}}$$

is (i). Schur convex (ii). Schur geometrical convex (iii). Schur harmonically convex, for all $a \geq b$.

Proof.

$$let f(a,b) = \frac{M_{SH_e}(a,b) - M_{SA}(a,b)}{M_{SN_1}(a,b) - M_{SH_e}(a,b)} = \frac{A(a,b) - H_e(a,b)}{H_e(a,b) - N_1(a,b)}$$

Hence,

$$f(a,b) = AN_1 - H_e^2$$

$$f(a,b) = (\frac{a+b}{2})(\frac{\sqrt{a}+\sqrt{b}}{2})^2 - (\frac{a+b+\sqrt{ab}}{3})^2$$

$$f(a,b) = \frac{a^2+b^2-6ab+2a\sqrt{ab}+2b\sqrt{ab}}{72}$$

by finding the partial derivatives of $f(a,b)$ and simple manipulation gives

$$\frac{\partial f}{\partial a} = \frac{1}{72}(2a-6b+3\sqrt{ab}+b\sqrt{b}/\sqrt{a})$$

$$\frac{\partial f}{\partial b} = \frac{1}{72}(2b-6a+3\sqrt{ab}+a\sqrt{a}/\sqrt{b})$$

Proof of (i):

$$Then\ s = (a-b)\frac{\partial f}{\partial a} - \frac{\partial f}{\partial b} = \frac{(a-b)^2}{72\sqrt{ab}}(8\sqrt{ab}-(a+b)) \geq 0\ fora \geq b$$

This verifies the condition for Schur convex.

Proof of (ii):

Then

$$a\frac{\partial f}{\partial a} - b\frac{\partial f}{\partial b} = \frac{2(a-b)}{72}(a+b+\sqrt{ab})$$

$$Then\ s = (\ lna - lnb)(a\frac{\partial f}{\partial a}) - (b\frac{\partial f}{\partial b}) = (\ lna - lnb)\frac{2(a-b)}{72}(a+b+\sqrt{ab}) \geq 0\ for\ a \geq b.$$

This verifies the condition for Schur geometrically convex.

Proof of (iii):

Then

$$a^2\frac{\partial f}{\partial a} - b^2\frac{\partial f}{\partial b} = \frac{(a-b)}{72}(2(a^2+b^2+ab)+3\sqrt{ab}(a+b)-6ab)$$

$$Then\ s = (a-b)a^2\frac{\partial f}{\partial a} - b^2\frac{\partial f}{\partial b} = \frac{(a-b)^2}{72}(2(a^2+b^2+ab)+3\sqrt{ab}(a+b)-6ab) \geq 0\ for\ a \geq b.$$

This verifies the condition for Schur harmonic convex. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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