Available online at http://scik.org
Adv. Inequal. Appl. 2017, 2017:5
ISSN 2050-7461

STRONG CONVERGENCE OF AN ITERATIVE SCHEME FOR ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In 2009, Kumam [7] introduced a new iterative scheme for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for monotone, Lipschitz-continuous mappings and proved its strong convergence in a real Hilbert space. The aim of this paper is to prove a strong convergence result of this iterative scheme in the setting of Banach spaces involving an inverse strongly accretive operator under some conditions. As a special case, we shall prove that proposed iterative scheme converges strongly to minimum norm solution of some variational inequality problem.

Keywords: iterative scheme; accretive operators; Banach spaces.

2010 AMS Subject Classification: 47H06, 47J25.

1. Introduction.

Let E be any smooth Banach space with $||.||$and let E^* be its dual. Let C be a nonempty closed convex subset of E. Let $||.||$ denote the norm of E and E^*. We shall use the symbol $\to$ to denote the strong convergence.

Firstly, we give some definitions.

Definition 1.1 A Banach space E is called uniformly convex iff for any ε, $0 < \varepsilon \le 2$, the inequalities $\|x\| \le 1$, $\|y\| \le 1$and $\|x - y\| \ge \varepsilon$ imply there exists a $\delta > 0$ such that $\left\|\frac{x+y}{2}\right\| \le 1 - \delta$.

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Received October 23, 2016

Definition 1.2 Let E be any smooth Banach space and $\rho_E : R^+ \to R^+$ be any function. Then it is called modulus of smoothness of E if

$$\rho_E(t) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1; \|x\| = 1, \|y\| = t\right\}.$$

Definition 1.3 A Banach space E is called uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$$

Definition 1.4 Let q > 1. A Banach space E is called q-uniformly smooth if there exists a fixed constant c > 0 such that $\rho_E(t) = ct^q$ for all t > 0. See [5, 11], for more details. It is clear that if E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth.

Definition 1.5 Let J be any mapping from E into E^* satisfying the condition

$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\}$. Then J is called the normalized duality mapping of E.

Remark 1.6 It is known that $J_q(x) = \|x\|^{q-2} J(x)$ for all $x \in E$. If E is a Hilbert space, then J = I. The normalized duality mapping J satisfies the following properties:

1. If E is smooth, then J is single valued.
2. If E is reflexive, then J is surjective.
3. If E is strictly convex, then J is one-one and

   $\langle x - y, x^* - y^* \rangle > 0$ for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$.
4. If E is uniformly smooth, then J is uniformly norm to norm continuous on each bounded subset of E.
5. Also $q \langle y - x, j_x \rangle \leq \|y\|^q - \|x\|^q$ for all $x, y \in E$ and $j_x \in J_q(x)$.

Definition 1.7 Let C be a non-empty subset of a Banach space E. A mapping $T : C \to C$ is called nonexpansive [10] if

$\|Tx - Ty\| = \|x - y\| \quad \forall\ x, y \in C.$

Definition 1.8 A Banach space E is said to be smooth if the limit

$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for all $x, y \in U$, where $U = \{x \in E : \|x\| = 1\}$.

Let C be a nonempty closed convex subset of Banach space E. An operator $A : C \to E$ is called α-inverse strongly accretive if there exists a constant α > 0 such that

$\langle Ax - Ay, J(x - y) \rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$.

It is obvious from above equation that

$$\|Ax - Ay\| \leq \frac{1}{\alpha} \|x - y\|.$$

Let D be a subset of C and Q be a mapping from C to D. Then Q is said to be sunny if $Q(Qx + t(x - Qx)) = Qx$, whenever $Qx + t(x - Qx) \varepsilon C$ for $x \varepsilon C$ and $t \geq 0$. A mapping $Q : C \to C$ is called retraction if $Q^2 = Q$. If Q is any retraction, then $Qz = z$ for every $z \varepsilon R(Q)$, where R(Q) is the range set of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Let E be any smooth Banach space with $||.||$ and let E^* be its dual and $< x, f >$ denote the value of $f \epsilon E^*$ at $x \epsilon E$. Let C be a nonempty closed convex subset of E.

and let A be an accretive operator of C into E. The generalized variational inequality problem is to find an element $u \epsilon C$ such that

$$< Au, J(v - u) > \geq 0 \ \forall \ v \epsilon C, \qquad (1.1)$$

where J is the duality mapping of E into E^*.

This problem is connected with the fixed point problem for nonlinear mappings.

In order to find a solution of (1.1), Aoyama et al [4] gave the following result.

Theorem 1.1 [4] Let E be a uniformly convex and 2-uniformly smooth Banach space with best smooth constant K and C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A be α-inverse strongly accretive operator of C into E. Let $S(C, A) \neq \varphi$ and the sequence $\{x_n\}$ be generated by

$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n)$, $x_1 \epsilon C$, $n = 1, 2, 3, \ldots\ldots\ldots\ldots$,

where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in [0, 1]. and $\lambda_n \in [a, \alpha/K^2]$ for some $a > 0$ and let $\alpha_n \in [b, c]$, where $0 < b < c < 1$, then $\{x_n\}$converges weakly to some element z of S(C, A).

Motivated by this, Yao et al [12] introduced another iterative scheme and proved its strong convergence.

Theorem 1.2[12] Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping. Let $A: C \to E$ be an α-inverse strongly accretive operator such that $S(C, A) \neq \varphi$. Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [a, \alpha/K^2]$. For fixed $u \varepsilon E$ and given $x_0 \varepsilon C$ define the sequence $\{x_n\}$ by

$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q[\beta_n u + (1 - \beta_n)(y_n - \lambda_n A x_n)$, n = 0, 1, 2………… (1.2)

where Q is sunny nonexpansive retraction from E onto C. Suppose the following conditions are satisfied:

(i). $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1,$

(ii). $\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty,$

(iv). $\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0.$

Then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $Q^{'}u$, where $Q^{'}$ is a sunny nonexpansive retraction of E onto S(C, A).

In particular, if we take u = 0, then the sequence $\{x_n\}$ converges strongly to the minimum norm element in S(C, A).

In 2009, Kumam [7] gave the following result.

Theorem 1.3[7] Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to R$ satisfying the following conditions:

1. F(x, x) = 0 for all x εC
2. F is monotone
3. For each x, y, z εC, $\lim_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$
4. For each x εC, y → F(x y) is convex and lower semicontinuous.

Let A: C → H be a monotone and k-Lipschitz continuous and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \cap EP(F) \neq \varphi$. Suppose that the sequence $\{x_n\}$ be generated by $x_1 = u \in C$

$F(u_n, y) + \frac{1}{r_n} < y - u_n, u_n - x_n > \geq 0$, for all y ε C

$y_n = P_C(u_n - \lambda_n A u_n),$

$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(x_n - \lambda_n A y_n)$, n = 0, 1, 2………… (1.3)

Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [a, b]$ for some a, b ε(0, 1/k) and $\{r_n\} \subset (0, \infty)$.

Suppose the following conditions are satisfied:

(i). $\alpha_n + \beta_n + \gamma_n = 1,$

(ii). $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$

(iii). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$

(iv). $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

(iv). $\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0.$

Then the sequence $\{x_n\}$ converges strongly to $P_{F(S) \cap VI(A, C) \cap EP(F)}u$.

2. Preliminaries

In this section, we collect some lemmas and results, which will be used in the proof of our main result.

Lemma 2.1 [3] Let C be a nonempty closed convex subset of a smooth Banach space, D be a nonempty subset of C and Q be a retraction from C onto D. Then Q is sunny and nonexpansive iff

$< u - Qu, j(y - Qu) > \leq 0$ for all $u \in C$ and $y \in D$.

Lemma 2.2 [1] In a Banach E, the following inequality holds:

$\|x + y\|^2 \leq \|x\|^2 + 2 < y, j(x + y) >$, for all $x, y \in E$, where $j(x + y) \in J(x + y)$.

Lemma 2.3 [2] Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then for all $\lambda > 0$,

$S(C, A) = F(Q_C(I - \lambda A))$, where

$S(C, A) = \{ u \in C : < Au, J(v - u) > \geq 0$, for all $v \in C\}$.

Lemma 2.4 [**8**] Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in [0, 1] with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose

$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$ for all integers $n \geq 0$ and

$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$ Then $\lim_{n\to\infty} \|x_n - z_n\| = 0.$

Lemma 2.5 [9] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$s_{n+1} = (1-\alpha_n) s_n + \delta_n, \ \forall n \geq 0,$ where $\{\alpha_n\}$ is a sequence in (0, 1) and $\{\delta_n\}$ is a sequence such that

(i). $\sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii). $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$

Then $\lim_{n\to\infty} s_n = 0$.

3. Main Result.

In this section, we shall prove that the iterative scheme defined by Kumam et al.[7] converges strongly to a solution of variational inequality problem in the setting of uniformly convex and 2-uniformly smooth Banach space.

Theorem 3.1 Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping. Let A: C → E be an α-inverse strongly accretive operator such that $S(C, A) \neq \varphi$. Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [a, \alpha/K^2]$. Suppose that the sequence $\{x_n\}$ be generated by $x_1 \varepsilon C$

$y_n = Q(x_n - \lambda_n A x_n)$,

$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q(y_n - \lambda_n A y_n)$, n = 0, 1, 2………… (3.1)

Suppose the following conditions are satisfied:

(i). $\alpha_n + \beta_n + \gamma_n = 1$,

(ii). $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii). $0 < \liminf_{n\to\infty} \leq \limsup_{n\to\infty} < 1$,

(iv). $\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $Q'u$, where Q' is a sunny nonexpansive retraction of E onto S(C, A).

In particular, if we take u = 0, then $\{x_n\}$ converges strongly to the minimum norm element in S(C, A).

Proof. For all x, y εC, and $\lambda_n \varepsilon (0, \frac{\alpha}{K^2}]$, it is known that $I - \lambda_n A$ is nonexpansive.

Let p ε S(C, A). Then by lemma 3, $p = Q(p - \lambda_n A p)$.

Let $z_n = Q(y_n - \lambda_n A y_n)$.

Now, $\|y_n - p\| = \|Q(x_n - \lambda_n A x_n) - Q(p - \lambda_n A p)\|$

$\leq \|x_n - p\|$ (3.2)

And $\|z_n - p\| = \|Q(y_n - \lambda_n A y_n) - Q(p - \lambda_n A p)\|$

$\leq \|y_n - p\| \leq \|x_n - p\|$ (3.3)

Now, using (3.1), we have,

$\|x_{n+1} - p\| \le \|\alpha_n (u - p) + \beta_n (x_n - p) + \gamma_n (z_n - p)\|$

$\le \alpha_n \|u - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\|$

$= \alpha_n \|u - p\| + (1 - \alpha_n)\|x_n - p\|$

$\le \max. \{\|u - p\|, \|x_0 - p\|\}$,

which implies $\{x_n\}$ is bounded and hence using (3.2) and (3.3), $\{y_n\}$,$\{z_n\}$ and $\{Ax_n\}$ are also bounded.

Now,

$\|z_{n+1} - z_n\| = \| Q(y_{n+1} - \lambda_{n+1}A\, y_{n+1}) - Q(y_n - \lambda_n A\, y_n)\|$

$\le \| y_{n+1} - y_n \| + \| \lambda_{n+1}A\, y_{n+1} - \lambda_n A\, y_n\|$

$= \| y_{n+1} - y_n \| + \| \lambda_{n+1}A\, y_{n+1} - \lambda_n A\, y_{n+1} + \lambda_n A\, y_{n+1} - \lambda_n A\, y_n\|$

$\le \| y_{n+1} - y_n \| + | \lambda_{n+1} - \lambda_n|\| A\, y_{n+1}\| + | \lambda_n| \|Ay_{n+1} - A\, y_n\|$

$\le \| y_{n+1} - y_n \| + | \lambda_{n+1} - \lambda_n|\| A\, y_{n+1}\| + \frac{\lambda_n}{\alpha} \|y_{n+1} - y_n\|$ [∵ A is α- inverse strongly accretive]

$$= (1 + \frac{\lambda_n}{\alpha}) \| y_{n+1} - y_n \| + | \lambda_{n+1} - \lambda_n|\| A\, y_{n+1}\| \quad (3.4)$$

Also,

$\|y_{n+1} - y_n\|$

$= \| Q(x_{n+1} - \lambda_{n+1}A\, x_{n+1}) - Q(x_n - \lambda_n A\, x_n)\|$

$\le \| x_{n+1} - x_n \| + \| \lambda_{n+1}A\, x_{n+1} - \lambda_n A\, x_n\|$

$= \| x_{n+1} - x_n \| + \| \lambda_{n+1}A\, x_{n+1} - \lambda_n A\, x_{n+1} + \lambda_n A\, x_{n+1} - \lambda_n A\, x_n\|$

$\le \| x_{n+1} - x_n \| + | \lambda_{n+1} - \lambda_n|\| A\, x_{n+1}\| + | \lambda_n| \|Ax_{n+1} - A\, x_n\|$

$\le \| x_{n+1} - x_n \| + | \lambda_{n+1} - \lambda_n|\| A\, x_{n+1}\| + \frac{\lambda_n}{\alpha} \|x_{n+1} - x_n\|$ [∵ A is α- inverse strongly accretive]

$$= (1 + \frac{\lambda_n}{\alpha}) \| x_{n+1} - x_n \| + | \lambda_{n+1} - \lambda_n|\| A\, x_{n+1}\| \quad (3.5)$$

Let $x_{n+1} = \beta_n x_n + (1-\beta_n)\, t_n$

$$t_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} = \frac{\alpha_n u + \beta_n x_n + \gamma_n z_n - \beta_n x_n}{1-\beta_n}$$

$$= \frac{\alpha_n u + \gamma_n z_n}{1-\beta_n}.$$

Now,

$$t_{n+1} - t_n = \frac{\alpha_{n+1}u + \gamma_{n+1}z_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n z_n}{1-\beta_n}$$

$$= \frac{\alpha_{n+1}u + \gamma_{n+1}z_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}u + \gamma_{n+1}z_n}{1-\beta_{n+1}} - \frac{\alpha_{n+1}u + \gamma_{n+1}z_n}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n z_n}{1-\beta_n}$$

$$= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(z_{n+1} - z_n) + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)z_n$$

Using (3.4) and (3.5), we obtain,

$$\|t_{n+1} - t_n\| - \|x_{n+1} - x_n\|$$

$$\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|u\| + \left|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right|\|z_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(1 + \frac{\lambda_n}{\alpha}\right)\|y_{n+1} - y_n\|$$

$$+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1} - \lambda_n|\|Ay_{n+1}\| - \|x_{n+1} - x_n\|$$

$$\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\| + \|z_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(1 + \frac{\lambda_n}{\alpha}\right)\left[\left(1 + \frac{\lambda_n}{\alpha}\right)\|x_{n+1} - x_n\|\right.$$

$$\left. + |\lambda_{n+1} - \lambda_n|\|Ax_n\|\right]$$

$$+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1} - \lambda_n|\|Ay_{n+1}\| - \|x_{n+1} - x_n\|$$

$$= \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\| + \|z_n\|) + \left[\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(1 + \frac{\lambda_n}{\alpha}\right)^2 - 1\right]\|x_{n+1} - x_n\|$$

$$+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(1 + \frac{\lambda_n}{\alpha}\right)|\lambda_{n+1} - \lambda_n|\|Ax_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1} - \lambda_n|\|Ay_{n+1}\|$$

$$= \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\| + \|z_n\|) + \left[\frac{\gamma_{n+1}}{1-\beta_{n+1}} + \frac{\gamma_{n+1}.\frac{\lambda_n^2}{\alpha^2}}{1-\beta_{n+1}} + \frac{2\gamma_{n+1}.\frac{\lambda_n}{\alpha}}{1-\beta_{n+1}} - 1\right]\|x_{n+1} - x_n\|$$

$$+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(1 + \frac{\lambda_n}{\alpha}\right)|\lambda_{n+1} - \lambda_n|\|Ax_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1} - \lambda_n|\|Ay_{n+1}\|$$

$$= \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\| + \|z_n\|) + \left(\frac{\alpha_{n+1} + \gamma_{n+1}.\frac{\lambda_n^2}{\alpha^2} + 2\gamma_{n+1}.\frac{\lambda_n}{\alpha}}{1-\beta_{n+1}}\right)\|x_{n+1} - x_n\|$$

$$+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(1 + \frac{\lambda_n}{\alpha}\right)|\lambda_{n+1} - \lambda_n|\|Ax_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1} - \lambda_n|\|Ay_{n+1}\| \qquad (3.6)$$

Using (ii), (iii) and (iv) conditions in (3.6), we get,

$$\lim_{\sup n\to\infty} \left(\|t_{n+1} - t_n\| - \|x_{n+1} - x_n\| \right) \le 0 .$$

Using Lemma 4, we get,

$$\lim_{n\to\infty} \|t_n - x_n\| = 0 \qquad (3.7)$$

Consequently, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|t_n - x_n\| = 0$ (3.8)

Using this in (3.5), we get $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$ (3.9)

Using (3.9) in (3.4), we get $\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0$ (3.10)

Combining (3.9) and (3.10), we can write,

$$\lim_{\sup n\to\infty} \left(\|y_{n+1} - \mathrm{y}_n\| - \|x_{n+1} - x_n\| \right) \le 0$$

Using Lemma 4, we get,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0 \qquad (3.11)$$

Next, we shall prove that

$$\lim_{\sup n\to\infty} \left\langle u - Q'u,\ j(x_n - u) \right\rangle \le 0 , \qquad (3.12)$$

where Q' is sunny nonexpansive retraction of E onto S(C, A).

To prove it, we choose a subsequence $\{x_{n_i}\}$ of {xn} that weakly converge to z such that

$$\lim_{\sup n\to\infty} \left\langle u - Q'u,\ j(x_n - Q'u) \right\rangle = \lim_{\sup i\to\infty} \left\langle u - Q'u,\ j(x_{n_i} - Q'u) \right\rangle \qquad (3.13)$$

Firstly we show that z ε S(C, A). Since, $\lambda_n\ \varepsilon\, [a, \frac{\alpha}{K^2}]$ for some a >0, so $\{\lambda_{n_i}\}$ is bounded and so there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ that converges to $\lambda_0\ \varepsilon\, [a, \frac{\alpha}{K^2}]$. W. L. O. G., we can assume that $\lambda_{n_i} \to \lambda_0$. Since Q is nonexpansive, so

$$\left\|Q(x_{n_i} - \lambda_0 A x_{n_i}) - x_{n_i}\right\|$$

$$\le \left\|Q(x_{n_i} - \lambda_0 A x_{n_i}) - y_{n_i}\right\| + \left\|y_{n_i} - x_{n_i}\right\|$$

$$= \left\|Q(x_{n_i} - \lambda_0 A x_{n_i}) - Q(x_{n_i} - \lambda_{n_i} A x_{n_i})\right\| + \left\|y_{n_i} - x_{n_i}\right\|$$

$$\le \left|\lambda_0 - \lambda_{n_i}\right| \left\|A x_{n_i}\right\| + \left\|y_{n_i} - x_{n_i}\right\|$$

Using condition (iv) and (3.11) in this equation, we get,

$$\lim_{i\to\infty}\left\|Q(I-\lambda_0 A)x_{n_i}-x_{n_i}\right\|=0 \qquad (3.14)$$

By demiclosedness principle of nonexpansive mappings and (3.14), we obtain z ε F(Q(I – λ_0A)). Using Lemma 3, we have z ε S(C, A).

From equation (3.13) and Lemma 1, we have,

$$\lim_{\sup n\to\infty}\left\langle u-Q'u,\, j(x_n-Q'u)\right\rangle = \lim_{\sup i\to\infty}\left\langle u-Q'u,\, j(x_{n_i}-Q'u)\right\rangle$$

$$= \left\langle u-Q'u,\, j(z-Q'u)\right\rangle \le 0.$$

$$\lim_{\sup n\to\infty}\left\langle u-Q'u,\, j(x_n-Q'u)\right\rangle \le 0 \qquad (3.15)$$

Now,

$$\left\|x_{n+1}-Q'u\right\|^2$$

$$= \left\langle \alpha_n u+\beta_n x_n+\gamma_n z_n-Q'u,\, j(x_{n+1}-Q'u)\right\rangle$$

$$= \alpha_n\left\langle u-Q'u,\, j(x_{n+1}-Q'u)\right\rangle + \beta_n\left\langle x_n-Q'u,\, j(x_{n+1}-Q'u)\right\rangle + \gamma_n\left\langle z_n-Q'u,\, j(x_{n+1}-Q'u)\right\rangle$$

$$= \frac{\beta_n}{2}\left(\left\|x_n-Q'u\right\|^2+\left\|x_{n+1}-Q'u\right\|^2\right) + \frac{\gamma_n}{2}\left(\left\|z_n-Q'u\right\|^2+\left\|x_{n+1}-Q'u\right\|^2\right) + \alpha_n\left\langle u-Q'u,\, j(x_{n+1}-Q'u)\right\rangle$$

$$\le \frac{1}{2}[(1-\alpha_n)\left(\left\|x_n-Q'u\right\|^2+\left\|x_{n+1}-Q'u\right\|^2\right)] + \alpha_n\left\langle u-Q'u,\, j(x_{n+1}-Q'u)\right\rangle$$

$$\le \frac{1}{2}[(1-\alpha_n)\left(\left\|x_n-Q'u\right\|^2+\left\|x_{n+1}-Q'u\right\|^2\right)]$$

$$\Rightarrow \left\|x_{n+1}-Q'u\right\|^2 \le (1-\alpha_n)\left(\left\|x_n-Q'u\right\|^2\right) + \alpha_n\left\langle u-Q'u,\, j(x_{n+1}-Q'u)\right\rangle \qquad (3.16)$$

Using Lemma 5 and (3.15) in (3.16), we observe that $\{x_n\}$ converges strongly to Q'u.

In particular, if we take u = 0, then $\{x_n\}$ generated by (3.1) converges strongly to Q'u, which is the minimum norm element in S(C, A). Hence, the proof.

4. Application

In this section, we give an application of our main result.

Let C be a closed convex subset of a Hilbert space H. Then it is well known that if A is an α-strongly accretive and L-Lipschitz continuous operator of C into H and $\lambda\,\varepsilon\left(0, \frac{2\alpha}{L^2}\right)$, then the operator $P_C(I- \lambda A)$ is a contraction of C into itself.Now, we shall prove a strong convergence theorem for a strongly accretive operator.

Theorem 4.1 Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping. Let Q be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A: C → E be an α-inverse strongly accretive and L-Lipschitz continuous operator such that $S(C, A) \neq \varphi$. Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [a, \alpha/K^2L^2]$ for some $a > 0$.

Suppose the following conditions are satisfied:

(i). $\alpha_n + \beta_n + \gamma_n = 1$,

(ii). $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii). $0 < \liminf_{n\to\infty} \leq \limsup_{n\to\infty} < 1$,

(iv). $\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $Q'u$, where Q' is a sunny nonexpansive retraction of E onto S(C, A).

In particular, if we take u = 0, then $\{x_n\}$ converges strongly to the minimum norm element in S(C, A).

Proof. Since A: C → E be an α-inverse strongly accretive and L-Lipschitz continuous operator, so we have,

$< Ax – Ay, J(x - y) > \geq \alpha\|x - y\|^2 \geq \alpha/L^2\|Ax - Ay\|^2$, for all x, y εC.

$\Rightarrow$ A is α/L^2-inverse strongly accretive. Using theorem (3.1), we can obtain that $\{x_n\}$ generated by (3.1) converges strongly to $Q'u$.

Conflicts of Interests

The author declares that there is no conflict of interests

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