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ON HERMITE-HADAMARD TYPE INEQUALITIES FOR GENERALIZED (s,m,φ)-PREINVEX FUNCTIONS VIA k-FRACTIONAL INTEGRALS

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Abstract. In the present paper, the notion of generalized (s,m,φ)-preinvex function is applied for establish some new Hermite-Hadamard type inequalities by using new identity for k-fractional Riemann-Liouville integrals. At the end, some applications to special means are given. These results provide new estimates on these Hermite-Hadamard type inequalities via k-fractional Riemann-Liouville integrals.

Keywords: Hermite-Hadamard type inequality; Hölder's inequality; Minkowski's inequality; power mean inequality; Riemann-Liouville fractional integral.

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1. Introduction and Preliminaries

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbb{R}=(-\infty,+\infty)$ and I° to denote the interior of I. For any subset $K\subseteq\mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ=[0,+\infty)$. The set of integrable functions on the interval $[a,b]$ is denoted by $L_1[a,b]$.

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The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The following definition will be used in the sequel.

Definition 1.2. The hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$$

for $c > b > 0$ and $|z| < 1$, where $\beta(x,y)$ is the Euler beta function for all $x, y > 0$.

In recent years, various generalizations, extensions and variants of such inequalities have been obtained (see [7], [8], [9]). For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [14]) and the references cited therein, also (see [6]) and the references cited therein.

Fractional calculus (see [14]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.3. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha)=\int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$.
In the case of $\alpha=1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [11], [14]) and the references cited therein.

Definition 1.4. (see [2]) If $k>0$, then k-Gamma function Γ_k is defined as

$$\Gamma_k(\alpha)=\lim_{n\longrightarrow\infty}\frac{n!k^n(nk)^{\frac{\alpha}{k}}-1}{(\alpha)_{n,k}}.$$

If $\mathrm{Re}(\alpha)>0$ then k-Gamma function in integral form is defined as

$$\Gamma_k(\alpha)=\int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt,$$

with the property that

$$\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha).$$

Definition 1.5. (see [3]) Let $f\in L_1[a,b]$. Then k-fractional integrals of order $\alpha,k>0$ with $a\geq 0$ are defined as

$$I^{\alpha,k}_{a+}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x(x-t)^{\frac{\alpha}{k}-1}f(t)dt,\quad x>a$$

and

$$I^{\alpha,k}_{b-}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b(t-x)^{\frac{\alpha}{k}-1}f(t)dt,\quad b>x.$$

For $k=1$, k-fractional integrals give Riemann-Liouville integrals.

Now, let us recall some definitions of various convex functions.

Definition 1.6. (see [5]) A function $f:\mathbb{R}_\circ\longrightarrow\mathbb{R}$ is said to be s-convex in the second sense, if

$$f(\lambda x+(1-\lambda)y)\leq\lambda^s f(x)+(1-\lambda)^s f(y) \tag{1.2}$$

for all $x,y\in\mathbb{R}_\circ$, $\lambda\in[0,1]$ and $s\in(0,1]$.

It is clear that a 1-convex function must be convex on $\mathbb{R}_\circ$ as usual. The s-convex functions in the second sense have been investigated in (see [5]).

Definition 1.7. (see [10]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please see (see [10], [12]) and the references therein.

Definition 1.8. (see [13]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f(x + t\eta(y,x)) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The aim of this paper is to establish some generalizations of Hermite-Hadamard type inequalities using new identity given in Section 2 2 for generalized (s,m,φ)-preinvex functions via k-fractional Riemann-Liouville integrals. In Section 33, some applications to special means are given. These results provide new estimates on these types.

2. Main results

Definition 2.1. (see [4]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$.

Definition 2.3. (see [1]) Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ and $\varphi : I \longrightarrow K$ a continuous increasing function. For $f : K \longrightarrow \mathbb{R}$ and any fixed $s, m \in (0,1]$, if

$$f(m\varphi(x) + \lambda\eta(\varphi(y), \varphi(x), m)) \leq m(1-\lambda)^s f(\varphi(x)) + \lambda^s f(\varphi(y)) \tag{2.1}$$

is valid for all $x, y \in I, \lambda \in [0,1]$, then we say that $f(x)$ is a generalized (s,m,φ)-preinvex function with respect to η.

Throughout this paper we denote

$$K_{\alpha,k}(\eta,\varphi,\lambda,r,m,a,b)$$
$$= \frac{\left(1+\frac{\alpha}{k}(1-\lambda)\right) f\left(m\varphi(b)+\frac{\eta(\varphi(a),\varphi(b),m)}{r+1}\right)+\left(1-\frac{\alpha}{k}(1-\lambda)\right) f(m\varphi(b))}{2}$$
$$-\frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{2\eta(\varphi(a),\varphi(b),m)^{\frac{\alpha}{k}}}$$
$$\times\left[I^{\alpha,k}_{(m\varphi(b))+} f\left(m\varphi(b)+\frac{\eta(\varphi(a),\varphi(b),m)}{r+1}\right)+I^{\alpha,k}_{\left(m\varphi(b)+\frac{\eta(\varphi(a),\varphi(b),m)}{r+1}\right)-} f(m\varphi(b))\right].$$

In this section, in order to prove our main results regarding some Hermite-Hadamard type inequalities for generalized (s,m,φ)-preinvex function via k-fractional Riemann-Liouville integrals, we need the following new lemma:

Lemma 2.4. *Let $\varphi : I \longrightarrow K$ be a continuous increasing function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m, \lambda \in (0,1]$, $r \in [0,1]$ and let $\varphi(a), \varphi(b) \in K, a < b$ with $\eta(\varphi(a),\varphi(b),m) \neq 0$. Assume that $f : K \longrightarrow \mathbb{R}$ is a differentiable function on K°, $f' \in L_1[m\varphi(b), m\varphi(b)+\eta(\varphi(a),\varphi(b),m)]$. Then for $\alpha, k > 0$, the following identity for k-fractional Riemann-Liouville integrals holds:*

$$K_{\alpha,k}(\eta,\varphi,\lambda,r,m,a,b) = \frac{\eta(\varphi(a),\varphi(b),m)}{2(r+1)}$$

$$\times\int_0^1 \left(t^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)-(1-t)^{\frac{\alpha}{k}}\right) f'\left(m\varphi(b)+\frac{t}{r+1}\eta(\varphi(a),\varphi(b),m)\right) dt. \tag{2.2}$$

Proof. A simple proof of the equality (2.2) can be done by performing two integration by parts in the integrals from the right side and changing the variables. The details are left to the interested reader. □

Now we turn our attention to establish new inequalities of Hermite-Hadamard type for generalized (s,m,φ)-preinvex functions via k-fractional Riemann-Liouville integrals. Using Lemma 2.4, the following results can be obtained for the corresponding version for power of the absolute value of the first derivative.

Theorem 2.5. *Let $\varphi : I \longrightarrow A$ be a continuous increasing function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s,m \in (0,1], r \in [0,1]$ and let $\varphi(a), \varphi(b) \in A$, $a < b$ with $\eta(\varphi(a), \varphi(b), m) \neq 0$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s,m,φ)-preinvex function on $[m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$, then for any $\alpha, \lambda \in (0,1]$ and $k > 0$, the following inequality for k-fractional Riemann-Liouville integrals holds:*

$$|K_{\alpha,k}(\eta,\varphi,\lambda,r,m,a,b)| \leq \frac{1}{(r+1)^{1+\frac{s}{q}}} \frac{1}{(s+1)^{\frac{1}{q}}} \frac{|\eta(\varphi(a),\varphi(b),m)|}{2}$$

$$\times \left[\frac{\alpha}{k}(1-\lambda) + 2\left(\frac{k}{k+p\alpha}\right)^{\frac{1}{p}}\right] \left[|f'(\varphi(a))|^q + m\left((r+1)^{s+1} - r^{s+1}\right)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}. \tag{2.3}$$

Proof. Suppose that $q > 1$. Using Lemma 2.4, the fact that $|f'|^q$ is a generalized (s,m,φ)-preinvex function, property of the modulus, Hölder's inequality and Minkowski's inequality, we have

$$|K_{\alpha,k}(\eta,\varphi,\lambda,r,m,a,b)|$$

$$\leq \frac{|\eta(\varphi(a),\varphi(b),m)|}{2(r+1)} \left[\int_0^1 \left(t^{\frac{\alpha}{k}} + \frac{\alpha}{k}(1-\lambda)\right) f'\left|\left(m\varphi(b) + \frac{t}{r+1}\eta(\varphi(a),\varphi(b),m)\right)\right| dt\right.$$

$$\left. + \int_0^1 (1-t)^{\frac{\alpha}{k}} f'\left|\left(m\varphi(b) + \frac{t}{r+1}\eta(\varphi(a),\varphi(b),m)\right)\right| dt\right]$$

$$\leq \frac{|\eta(\varphi(a),\varphi(b),m)|}{2(r+1)} \left[\left(\int_0^1 \left(t^{\frac{\alpha}{k}} + \frac{\alpha}{k}(1-\lambda)\right)^p dt\right)^{\frac{1}{p}}\right.$$

$$\times \left(\int_0^1 f'\left|\left(m\varphi(b) + \frac{t}{r+1}\eta(\varphi(a),\varphi(b),m)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\left. + \left(\int_0^1 (1-t)^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}} \left(\int_0^1 f'\left|\left(m\varphi(b) + \frac{t}{r+1}\eta(\varphi(a),\varphi(b),m)\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{|\eta(\varphi(a),\varphi(b),m)|}{2(r+1)} \left[\left(\int_0^1 t^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}} + \left(\int_0^1 \left(\frac{\alpha}{k}(1-\lambda)\right)^p dt\right)^{\frac{1}{p}} + \left(\int_0^1 (1-t)^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}}\right]$$

$$\times \left(\int_0^1 \left(m\left(1 - \frac{t}{r+1}\right)^s |f'(\varphi(b))|^q + \left(\frac{t}{r+1}\right)^s |f'(\varphi(a))|^q\right) dt\right)^{\frac{1}{q}}$$

$$= \frac{1}{(r+1)^{1+\frac{s}{q}}} \frac{1}{(s+1)^{\frac{1}{q}}} \frac{|\eta(\varphi(a),\varphi(b),m)|}{2}$$

$$\times\left[\frac{\alpha}{k}(1-\lambda)+2\left(\frac{k}{k+p\alpha}\right)^{\frac{1}{p}}\right]\left[|f'(\varphi(a))|^q+m\left((r+1)^{s+1}-r^{s+1}\right)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}.$$

The proof of Theorem 2.5 is completed. □

Corollary 2.6. *Under the conditions of Theorem 2.5, if we choose $r=0$, $m=\lambda=k=1$ and $\eta(\varphi(a),\varphi(b),m)=\varphi(a)-\varphi(b)$, then we get the following generalized Hermite-Hadamard type inequality for fractional integrals*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(a)-\varphi(b))^{\alpha}}\left[J^{\alpha}_{\varphi(b)+}f(\varphi(a))+J^{\alpha}_{\varphi(a)-}f(\varphi(b))\right]\right|$$

$$\leq(\varphi(b)-\varphi(a))\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[|f'(\varphi(a))|^q+|f'(\varphi(b))|^q\right]^{\frac{1}{q}}.$$

Theorem 2.7. *Let $\varphi: I\longrightarrow A$ be a continuous increasing function. Suppose $A\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta: A\times A\times(0,1]\longrightarrow\mathbb{R}$ for any fixed $s,m\in(0,1], r\in[0,1]$ and let $\varphi(a),\varphi(b)\in A$, $a<b$ with $\eta(\varphi(a),\varphi(b),m)\neq 0$. Assume that $f: A\longrightarrow\mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s,m,φ)-preinvex function on $[m\varphi(b),m\varphi(b)+\eta(\varphi(a),\varphi(b),m)]$, $q\geq 1$, then for any $\alpha,k>0$ and $\lambda\in(0,1]$, the following inequality for k-fractional Riemann-Liouville integrals holds:*

$$|K_{\alpha,k}(\eta,\varphi,\lambda,r,m,a,b)|$$

$$\leq\frac{|\eta(\varphi(a),\varphi(b),m)|}{2(r+1)^{1+\frac{s}{q}}}\times\left\{\left(\frac{k}{k+\alpha}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\left[\left(\frac{\alpha(1-\lambda)}{k(s+1)}+\frac{k}{k(s+1)+\alpha}\right)|f'(\varphi(a))|^q\right.\right.$$

$$\left.+\left(m\left(\frac{k}{k+\alpha}\right)(r+1)^s\cdot{}_2F_1\left(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{r+1}\right)+\frac{m\alpha(1-\lambda)}{k}\left(\frac{(r+1)^{s+1}-r^{s+1}}{s+1}\right)\right)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}$$

(2.4)

$$\left.+\left(\frac{k}{k+\alpha}\right)^{1-\frac{1}{q}}\left[\beta\left(s+1,\frac{\alpha}{k}+1\right)|f'(\varphi(a))|^q+m\left(\frac{k}{k+\alpha}\right)(r+1)^s\cdot{}_2F_1\left(-s,1;\frac{\alpha}{k}+2;\frac{1}{r+1}\right)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right\}.$$

Proof. Suppose that $q\geq 1$. Using Lemma 2.4, the fact that $|f'|^q$ is a generalized (s,m,φ)-preinvex function, property of the modulus and the well-known power mean inequality, we have

$$|K_{\alpha,k}(\eta,\varphi,\lambda,r,m,a,b)|$$

$$\leq\frac{|\eta(\varphi(a),\varphi(b),m)|}{2(r+1)}\left[\int_0^1\left(t^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right)f'\left|\left(m\varphi(b)+\frac{t}{r+1}\eta(\varphi(a),\varphi(b),m)\right)\right|dt\right.$$

$$\left.+\int_0^1(1-t)^{\frac{\alpha}{k}}f'\left|\left(m\varphi(b)+\frac{t}{r+1}\eta(\varphi(a),\varphi(b),m)\right)\right|dt\right]$$

$$\leq \frac{|\eta(\varphi(a),\varphi(b),m)|}{2(r+1)}\Bigg[\left(\int_0^1\left(t^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right)dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left(t^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right)f'\left|\left(m\varphi(b)+\frac{t}{r+1}\eta(\varphi(a),\varphi(b),m)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1(1-t)^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\frac{\alpha}{k}}f'\left|\left(m\varphi(b)+\frac{t}{r+1}\eta(\varphi(a),\varphi(b),m)\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{|\eta(\varphi(a),\varphi(b),m)|}{2(r+1)}\Bigg[\left(\int_0^1\left(t^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right)dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left(t^{\frac{\alpha}{k}}+\frac{\alpha}{k}(1-\lambda)\right)\left(m\left(1-\frac{t}{r+1}\right)^s|f'(\varphi(b))|^q+\left(\frac{t}{r+1}\right)^s|f'(\varphi(a))|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1(1-t)^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\frac{\alpha}{k}}\left(m\left(1-\frac{t}{r+1}\right)^s|f'(\varphi(b))|^q+\left(\frac{t}{r+1}\right)^s|f'(\varphi(a))|^q\right)dt\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{|\eta(\varphi(a),\varphi(b),m)|}{2(r+1)^{1+\frac{s}{q}}}\times\Bigg\{\left(\frac{k}{k+\alpha}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\Bigg[\left(\frac{\alpha(1-\lambda)}{k(s+1)}+\frac{k}{k(s+1)+\alpha}\right)|f'(\varphi(a))|^q$$

$$+\left(m\left(\frac{k}{k+\alpha}\right)(r+1)^s\cdot{}_2F_1\left(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{r+1}\right)+\frac{m\alpha(1-\lambda)}{k}\left(\frac{(r+1)^{s+1}-r^{s+1}}{s+1}\right)\right)|f'(\varphi(b))|^q\Bigg]^{\frac{1}{q}}$$

$$+\left(\frac{k}{k+\alpha}\right)^{1-\frac{1}{q}}\left[\beta\left(s+1,\frac{\alpha}{k}+1\right)|f'(\varphi(a))|^q+m\left(\frac{k}{k+\alpha}\right)(r+1)^s\cdot{}_2F_1\left(-s,1;\frac{\alpha}{k}+2;\frac{1}{r+1}\right)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\Bigg\}.$$

The proof of Theorem 2.7 is completed. □

Corollary 2.8. *Under the conditions of Theorem 2.7, if we choose $r=0$, $m=\lambda=k=1$ and $\eta(\varphi(a),\varphi(b),m)=\varphi(a)-\varphi(b)$, then we get the following generalized Hermite-Hadamard type inequality for fractional integrals*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(a)-\varphi(b))^\alpha}\left[J^\alpha_{\varphi(b)+}f(\varphi(a))+J^\alpha_{\varphi(a)-}f(\varphi(b))\right]\right|$$

$$\leq\frac{(\varphi(b)-\varphi(a))}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg\{\left[\frac{|f'(\varphi(a))|^q}{\alpha+s+1}+\frac{{}_2F_1(-s,\alpha+1;\alpha+2;1)}{\alpha+1}|f'(\varphi(b))|^q\right]^{\frac{1}{q}}$$

$$+\left[\beta(s+1,\alpha+1)|f'(\varphi(a))|^q+\frac{{}_2F_1(-s,1;\alpha+2;1)}{\alpha+1}|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\Bigg\}.$$

Remark 2.9. For $M > 0$ and $q \geq 1$, if $|f'|^q \leq M$, then by Theorem 2.5 and Theorem 2.7, we can get some special kinds of Hermite-Hadamard type inequalities via k-fractional Riemann-Liouville integrals. For $k = 1$, we obtain special kinds of Hermite-Hadamard type inequalities via Riemann-Liouville integrals. Also, for different choices of values λ, r, for example $\lambda = \frac{1}{2}, \frac{1}{3}$; $r = \frac{1}{2}, \frac{1}{3}, 1$ and function φ, by Theorem 2.5 and Theorem 2.7 we can get some interesting integral inequalities of these types.

3. Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3.1. (see [15]) A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

(1) Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,

(2) Symmetry: $M(x, y) = M(y, x)$,

(3) Reflexivity: $M(x, x) = x$,

(4) Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,

(5) Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for arbitrary positive real numbers α, β $(\alpha \neq \beta)$.

(1) The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$

(2) The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$

(3) The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

(4) The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \ \ r \geq 1.$$

(5) The identric mean:

$$I := I(\alpha,\beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|}; \ \ |\alpha| \neq |\beta|, \ \ \alpha\beta \neq 0.$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}; \ \ p \in \mathbb{R}\setminus\{-1,0\}, \ \ \alpha \neq \beta.$$

(8) The weighted p-power mean:

$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & ,\alpha_n \\ u_1, & u_2, & \cdots & ,u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0\,(i = 1,2,\ldots,n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(a),\varphi(b)) : [\varphi(a),\varphi(a) + \eta(\varphi(b),\varphi(a))] \times [\varphi(a),\varphi(a)+\eta(\varphi(b),\varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I \longrightarrow A$ be a continuous increasing function, therefore one can obtain various inequalities using the results of Section 22 for these means as follows:
Replace $\eta(\varphi(y),\varphi(x),m)$ with $\eta(\varphi(y),\varphi(x))$ and setting $\eta(\varphi(a),\varphi(b)) = M(\varphi(a),\varphi(b))$ for $m = 1$ in (2.3) and (2.4), one can obtain the following interesting inequalities involving means:

$$\Bigg|\frac{\left(1+\frac{\alpha}{k}(1-\lambda)\right) f\left(\varphi(b)+\frac{M(\varphi(a),\varphi(b))}{r+1}\right) + \left(1-\frac{\alpha}{k}(1-\lambda)\right) f(\varphi(b))}{2}$$

$$-\frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{2M(\varphi(a),\varphi(b))^{\frac{\alpha}{k}}}$$

$$\times\left[I^{\alpha,k}_{\varphi(b)+} f\left(\varphi(b)+\frac{M(\varphi(a),\varphi(b))}{r+1}\right) + I^{\alpha,k}_{\left(\varphi(b)+\frac{M(\varphi(a),\varphi(b))}{r+1}\right)-} f(\varphi(b))\right]\Bigg|$$

$$\leq \frac{1}{(r+1)^{1+\frac{s}{q}}}\frac{1}{(s+1)^{\frac{1}{q}}}\frac{M(\varphi(a),\varphi(b))}{2}$$

$$\times\left[\frac{\alpha}{k}(1-\lambda)+2\left(\frac{k}{k+p\alpha}\right)^{\frac{1}{p}}\right]\left[|f'(\varphi(a))|^q + \left((r+1)^{s+1}-r^{s+1}\right)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}, \tag{3.1}$$

$$\left|\frac{\left(1+\frac{\alpha}{k}(1-\lambda)\right) f\left(\varphi(b)+\frac{M(\varphi(a),\varphi(b))}{r+1}\right)+\left(1-\frac{\alpha}{k}(1-\lambda)\right) f(\varphi(b))}{2}\right.$$

$$-\frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{2M(\varphi(a),\varphi(b))^{\frac{\alpha}{k}}}$$

$$\left.\times\left[I^{\alpha,k}_{\varphi(b)+} f\left(\varphi(b)+\frac{M(\varphi(a),\varphi(b))}{r+1}\right)+I^{\alpha,k}_{\left(\varphi(b)+\frac{M(\varphi(a),\varphi(b))}{r+1}\right)-} f(\varphi(b))\right]\right|$$

$$\leq \frac{M(\varphi(a),\varphi(b))}{2(r+1)^{1+\frac{s}{q}}}\times\left\{\left(\frac{k}{k+\alpha}+\frac{\alpha}{k}(1-\lambda)\right)^{1-\frac{1}{q}}\left[\left(\frac{\alpha(1-\lambda)}{k(s+1)}+\frac{k}{k(s+1)+\alpha}\right)|f'(\varphi(a))|^q\right.\right.$$

$$\left.+\left(\left(\frac{k}{k+\alpha}\right)(r+1)^s\cdot {}_2F_1\left(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{r+1}\right)+\frac{\alpha(1-\lambda)}{k}\left(\frac{(r+1)^{s+1}-r^{s+1}}{s+1}\right)\right)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}$$

(3.2)

$$\left.+\left(\frac{k}{k+\alpha}\right)^{1-\frac{1}{q}}\left[\beta\left(s+1,\frac{\alpha}{k}+1\right)|f'(\varphi(a))|^q+\left(\frac{k}{k+\alpha}\right)(r+1)^s\cdot {}_2F_1\left(-s,1;\frac{\alpha}{k}+2;\frac{1}{r+1}\right)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}\right\}.$$

Letting $M(\varphi(a),\varphi(b))=A,G,H,P_r,I,L,L_p,M_p$ in (3.1) and (3.2), we get inequalities involving means for a particular choices of a differentiable generalized $(s,1,\varphi)$-preinvex function f. The details are left to the interested reader.

Conflict of Interests

The authors declare that there is no conflict of interests.

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