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COMMON FIXED POINTS FOR TWO IMPLICIT CONTRACTIVE MAPPINGS OF INTEGRAL TYPE ON ORDERED METRIC SPACES

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Abstract. In this paper, we introduce three real functional classes $\mathscr{C}^*$ and Φ_c^* and Φ_u^*, and discuss the existence problems of common fixed points for two mappings of integral type with implicit contractive conditions determined by $\mathscr{C}^*$ and Φ_c^* and Φ_u^* on non-complete ordered metric spaces and give more general results.

Keywords: class $\mathscr{C}^*$; class Φ_c^*; class Φ_u^*; implicit; common fixed point.

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1. Introduction and Preliminaries

Throughout this paper, we assume that $\mathbb{R}^+ = [0, +\infty)$ and

$\Phi = \{\phi : \phi : \mathbb{R}^+ \to \mathbb{R}^+$satisfying that ϕ is Lebesgue integral, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \phi(t)dt > 0$ for each $\varepsilon > 0\}$

The following results is the famous Banach's contraction principle:

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Theorem 1.1([1]) Let f be a self mapping on a complete metric space (X,d) satisfying

$$d(fx,fy) \le cd(x,y),\ \forall\, x,y \in X, \tag{1.1}$$

where $c \in [0,1)$ is a constant. Then f has a unique fixed point $\hat{x} \in X$ such that $\lim_{n\to\infty} f^n x = \hat{x}$ for each $x \in X$.

It is known that the Banach contraction principle has a lot of generalizations and various applications in many different directions, see, for examples, [2-13] and the references cited therein.

Especially, Branciari[14] gave an integral version of Theorem 1.1 as follows:

Theorem 1.2([14]) Let f be a self-mapping on a complete metric space (X,d) satisfying

$$\int_0^{d(fx,fy)} \phi(t)dt \le c \int_0^{d(x,y)} \phi(t)dt,\ \forall\, x,y \in X, \tag{1.2}$$

where $c \in (0,1)$ is a constant and $\phi \in \Phi$. Then f has a unique fixed point $\hat{x} \in X$ such that $\lim_{n\to\infty} f^n x = \hat{x}$ for each $x \in X$.

In 2011, Liu and Li[15] modified the method in [13] to generalized the Branciari's fixed point theorem with replacing the contraction constant c in (1.2) by contraction functions α and β and established the following fixed point theorem:

Theorem 1.3([15]) Let f be a self-mapping on a complete metric space (X,d) satisfying

$$\int_0^{d(fx,fy)} \phi(t)dt \le \alpha(d(x,y)) \int_0^{d(x,fx)} \phi(t)dt + \beta(d(x,y)) \int_0^{d(y,fy)} \phi(t)dt, \forall\, x,y \in X, \tag{1.3}$$

where $\phi \in \Phi$ and $\alpha,\beta : \mathbb{R}^+ \to [0,1)$ are two functions with

$$\alpha(t)+\beta(t) < 1, \forall\, t \in \mathbb{R}^+;\ \limsup_{s\to 0^+} \beta(s) < 1;\ \limsup_{s\to t^+} \frac{\alpha(s)}{1-\beta(s)} < 1,\ \forall\, t > 0.$$

Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

In [16], Jin and Piao discussed the following existence problems of unique common fixed points for two mappings of integral type with variable coefficient in metric spaces which generalize and improve Theorem 1.3.

Theorem 1.4([16]) Let (X,d) be a complete metric space, $f,g:X\to X$ two mappings. If for each $x,y\in X$,

$$\int_0^{d(fx,gy)}\phi(t)dt\leq\alpha(d(x,y))\int_0^{d(x,y)}\phi(t)dt+\beta(d(x,y))\int_0^{d(x,fx)}\phi(t)dt+\gamma(d(x,y))\int_0^{d(y,gy)}\phi(t)dt, \tag{1.4}$$

where $\phi\in\Phi$ and $\alpha,\beta,\gamma:\mathbb{R}^+\to[0,1)$ are three functions satisfying the following conditions

$$\alpha(t)+\beta(t)+\gamma(t)<1,\forall t\in\mathbb{R}^+,\ \max\left\{\limsup_{s\to0^+}\beta(s),\limsup_{s\to0^+}\gamma(s)\right\}<1, \tag{1.5}$$

$$\max\left\{\limsup_{s\to t^+}\alpha(t),\limsup_{s\to t^+}\frac{\alpha(t)+\gamma(t)}{1-\beta(t)},\limsup_{s\to t^+}\frac{\alpha(t)+\beta(t)}{1-\gamma(t)}\right\}<1,\forall t\in\mathbb{R}^+. \tag{1.6}$$

Then f and g have a unique common fixed point u, and the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_{2n+1}=fx_{2n}$ and $x_{2n+2}=gx_{2n+1}$ for any $x_0\in X$ converges to u.

Here we will introduce three classes $\{\mathscr{C}^*,\Phi_u^*,\Phi_u^*\}$ of 1-or 2-dimensional functions and establish the implicit contractive conditions determined by $\{\mathscr{C}^*,\Phi_u^*,\Phi_u^*\}$ about two mappings, and then discuss the existence problems of common fixed points for two self-mappings of integral type with the new implicit limitation in a non-complete ordered metric space and give some more general results. The obtained results further generalize and improve the corresponding conclusions in the literature, especially the results in [14-16].

To do this, we first introduce the definitions of classes of $\mathscr{C}^*$ and Φ_c^* and Φ_u^*.

Definition 1.5 $F\in\mathscr{C}^*\Longleftrightarrow F:[0,\infty)^2\to\mathbb{R}$ is a continuous and non-decreasing function satisfying following axioms:

(1)$F(s,t)\leq s$;

(2) $F(s,t)=s$ implies that either $s=0$ or $t=0$; for all $s,t\in[0,\infty)$.

Note for some F we have that $F(0,0)=0$.

Example 1.6 The following functions $F:[0,\infty)^2\to\mathbb{R}$ are elements of $\mathscr{C}^*$:

(1) $F(s,t)=s-t$, $F(s,t)=s\Rightarrow t=0$;

(2) $F(s,t)=ms$, $0<m<1$, $F(s,t)=s\Rightarrow s=0$;

(3) $F(s,t)=s\beta(s)$, where $\beta:\mathbb{R}^+\to\mathbb{R}^+$ is continuous and non-decreasing;

(4) $F(s,t)=\frac{s}{(1+s)^r}$, where $r\in(0,1)$;

(5) $F(s,t)=sh(s,t)$, where $h:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$is a continuous and non-decreasing function such that $h(t,s)<1$ for all $t,s>0$;

Definition 1.7 Let Φ_c^* be a set of functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

ψ_1: ψ is continuous and strictly increasing

ψ_2: $\psi(t) = 0 \Longleftrightarrow t = 0$.

Definition 1.8 Let Φ_u^* be a set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

φ_1: φ is continuous and non-decreasing;

φ_2: $\varphi(t) > 0$ if $t > 0$ and $\varphi(0) \geq 0$.

Remark 1.9 If the non-decreasing condition in Definition 1.5 and 1.8 is removed, then $\mathscr{C}^*$ and Φ_u^* are $\mathscr{C}$ and Φ_u in [17-18] respectively. Obviously, $f \in \Phi_c^* \Longrightarrow f \in \Phi_u^*$, but the inverse does not hold.

Lemma 1.10([19]) Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)},x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1},x_{n(k)}) < \varepsilon$ and the following result holds

$$\lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)}) = \varepsilon.$$

Remark 1.11 Under the conditions of Lemma 1.9, We easily obtian the following result:

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)},x_{n(k)-1}) = \varepsilon.$$

2. Common Fixed Points

The following two Lemmas are well known results.

Lemma 2.1([15]) Let $\phi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)dt = \int_0^a \phi(t)dt.$$

Lemma 2.2([15]) Let $\phi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)dt = 0 \Longleftrightarrow \lim_{n\to\infty} r_n = 0.$$

Definition 2.3 $\phi \in \Phi$ is called to be strictly increasing about integral type if for any $x, y \in [0,\infty)$ with $x < y$,

$$\int_0^x \phi(t)dt < \int_0^y \phi(t)dt.$$

Example 2.4 Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\phi(t) = \frac{1}{1+t}$ for each $t \in \mathbb{R}^+$. Then obviously for $0 \le x < y$,

$$\int_0^x \frac{1}{1+t} dt = \ln^{(1+x)} < \ln^{(1+y)} = \int_0^y \frac{1}{1+t} dt.$$

Hence $\phi(t) = \frac{1}{1+t}$ is a strictly increasing function about integral type.

Now, we give the first main result about common fixed point problems for two implicit contractive mappings with integral type on a non-complete metric space.

Theorem 2.5 Let (X,d) be a metric space, $f,g : X \to X$ two mappings. Suppose that for each $x,y \in X$,

$$\begin{aligned}
&\psi\Big(\int_0^{d(fx,gy)} \phi(t)dt\Big)\\
\le &F\Big(\psi\big(\alpha(d(x,y))\int_0^{d(x,y)} \phi(t)dt + \beta(d(x,y))\int_0^{d(x,fx)} \phi(t)dt + \gamma(d(x,y))\int_0^{d(y,gy)} \phi(t)dt\big),\\
&\varphi\big(\alpha(d(x,y))\int_0^{d(x,y)} \phi(t)dt + \beta(d(x,y))\int_0^{d(x,fx)} \phi(t)dt + \gamma(d(x,y))\int_0^{d(y,gy)} \phi(t)dt\big)\Big),
\end{aligned} \tag{2.1}$$

where $\phi \in \Phi$ is strictly increasing about integral type, $\varphi \in \Phi_u^*$, $\psi \in \Psi_c^*$, $F \in \mathscr{C}^*$ and $\alpha,\beta,\gamma :$ $\mathbb{R}^+ \to \mathbb{R}^+$ are three functions satisfying the following conditions

$$\alpha(t) + \beta(t) + \gamma(t) \le 1, \forall t \in \mathbb{R}^+. \tag{2.2}$$

If fX or gX is complete, then f and g have a common fixed point. Furthermore, if $\alpha(t) \ne 1$ for all $t > 0$, then f and g have a unique common fixed point

Proof. Take $x_0 \in X$. We construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying the following conditions $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Let $d_n = d(x_n, x_{n+1}), \forall n \in \mathbb{N} \cup \{0\}$.

For $n \in \mathbb{N} \cup \{0\}$, by (2.1) and $F \in \mathscr{C}^*$,

$$\begin{aligned}
&\psi\Big(\int_0^{d_{2n}} \phi(t)dt\Big) = \psi\Big(\int_0^{d(fx_{2n},gx_{2n-1})} \phi(t)dt\Big)\\
\le &F\Big(\psi(\alpha(d(x_{2n},x_{2n-1}))\int_0^{d(x_{2n},x_{2n-1})} \phi(t)dt + \beta(d(x_{2n},x_{2n-1}))\int_0^{d(x_{2n},fx_{2n})} \phi(t)dt
\end{aligned}$$

$$+\gamma(d(x_{2n},x_{2n-1}))\int_0^{d(x_{2n-1},gx_{2n-1})}\phi(t)dt),$$
$$\varphi(\alpha(d(x_{2n},x_{2n-1}))\int_0^{d(x_{2n},x_{2n-1})}\phi(t)dt+\beta(d(x_{2n},x_{2n-1}))\int_0^{d(x_{2n},fx_{2n})}\phi(t)dt$$
$$+\gamma(d(x_{2n},x_{2n-1}))\int_0^{d(x_{2n-1},gx_{2n-1})}\phi(t)dt)\Big)$$

$$\begin{aligned}
=&F\Big(\psi([\alpha(d(x_{2n},x_{2n-1}))+\gamma(d(x_{2n},x_{2n-1}))]\int_0^{d_{2n-1}}\phi(t)dt+\beta(d(x_{2n},x_{2n-1}))\int_0^{d_{2n}}\phi(t)dt),\\
&\varphi([(\alpha(d(x_{2n},x_{2n-1}))+\gamma(d(x_{2n},x_{2n-1}))]\int_0^{d_{2n-1}}\phi(t)dt+\beta(d(x_{2n},x_{2n-1}))\int_0^{d_{2n}}\phi(t)dt)\Big)\\
\leq&\psi([\alpha(d(x_{2n},x_{2n-1}))+\gamma(d(x_{2n},x_{2n-1}))]\int_0^{d_{2n-1}}\phi(t)dt+\beta(d(x_{2n},x_{2n-1}))\int_0^{d_{2n}}\phi(t)dt).
\end{aligned}\tag{2.3}$$

If there exists $n\in\mathbb{N}$ such that

$$\int_0^{d_{2n-1}}\phi(t)dt<\int_0^{d_{2n}}\phi(t)dt,$$

then by the strictly increasing condition of ψ and (2.2), we obtain from (2.3) that

$$\psi(\int_0^{d_{2n}}\phi(t)dt)<\psi(\int_0^{d_{2n}}\phi(t)dt).$$

This is a contradiction, hence we have

$$\int_0^{d_{2n}}\phi(t)dt\leq\int_0^{d_{2n-1}}\phi(t)dt,\ \forall\ n\in\mathbb{N}.\tag{2.4}$$

Notice that the following result is the one part of (2.3):

$$\begin{aligned}
&\psi\Big(\int_0^{d_{2n}}\phi(t)dt\Big)\\
\leq&F\Big(\psi([\alpha(d(x_{2n},x_{2n-1}))+\gamma(d(x_{2n},x_{2n-1}))]\int_0^{d_{2n-1}}\phi(t)dt+\beta(d(x_{2n},x_{2n-1}))\int_0^{d_{2n}}\phi(t)dt),\\
&\varphi([(\alpha(d(x_{2n},x_{2n-1}))+\gamma(d(x_{2n},x_{2n-1}))]\int_0^{d_{2n-1}}\phi(t)dt+\beta(d(x_{2n},x_{2n-1}))\int_0^{d_{2n}}\phi(t)dt)\Big).
\end{aligned}\tag{2.5}$$

Similarly, by (2.1) and $F \in \mathscr{C}^*$, we have

$$\begin{aligned}
&\psi(\int_0^{d_{2n+1}} \phi(t)dt) = \psi(\int_0^{d(fx_{2n},gx_{2n+1})} \phi(t)dt) \\
\leq &F\Big(\psi(\alpha(d(x_{2n},x_{2n+1}))\int_0^{d(x_{2n},x_{2n+1})} \phi(t)dt + \beta(d(x_{2n},x_{2n+1}))\int_0^{d(x_{2n},fx_{2n})} \phi(t)dt \\
&+\gamma(d(x_{2n},x_{2n+1}))\int_0^{d(x_{2n+1},gx_{2n+1})} \phi(t)dt), \\
&\varphi(\alpha(d(x_{2n},x_{2n+1}))\int_0^{d(x_{2n},x_{2n+1})} \phi(t)dt + \beta(d(x_{2n},x_{2n+1}))\int_0^{d(x_{2n},fx_{2n})} \phi(t)dt \\
&+\gamma(d(x_{2n},x_{2n+1}))\int_0^{d(x_{2n+1},gx_{2n+1})} \phi(t)dt)\Big) \\
\leq &\psi([\alpha(d(x_{2n},x_{2n+1}))+\beta(d(x_{2n},x_{2n+1}))]\int_0^{d_{2n}} \phi(t)dt + \gamma(d(x_{2n},x_{2n+1}))\int_0^{d_{2n+1}} \phi(t)dt)
\end{aligned}$$

and

$$\int_0^{d_{2n+1}} \phi(t)dt \leq \int_0^{d_{2n}} \phi(t)dt. \tag{2.6}$$

Combining (2.4) and (2.6), we have

$$\int_0^{d_{n+1}} \phi(t)dt \leq \int_0^{d_n} \phi(t)dt,\ \forall n = 0,1,2,\cdots. \tag{2.7}$$

Since ϕ is strictly increasing about integral type, so we obtain

$$d_{n+1} \leq d_n,\ \forall n = 0,1,2,\cdots. \tag{2.8}$$

Therefore there exists $u \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d_n = u$. By Lemma 2.1 and the properties of ϕ and ψ, from (2.5),

$$\begin{aligned}
&\psi\Big(\int_0^{u} \phi(t)dt\Big) = \lim_{n\to\infty}\psi\Big(\int_0^{d_{2n}} \phi(t)dt\Big) \\
\leq &F(\psi(\limsup_{n\to\infty}[\alpha(d(x_{2n},x_{2n-1}))+\gamma(d(x_{2n},x_{2n-1}))]\lim_{n\to\infty}\int_0^{d_{2n-1}} \phi(t)dt \\
&+\limsup_{n\to\infty}\beta(d(x_{2n},x_{2n-1})\lim_{n\to\infty}\int_0^{d_{2n}} \phi(t)dt), \\
&\varphi(\limsup_{n\to\infty}[\alpha(d(x_{2n},x_{2n-1}))+\gamma(d(x_{2n},x_{2n-1}))]\lim_{n\to\infty}\int_0^{d_{2n-1}} \phi(t)dt \\
&+\limsup_{n\to\infty}\beta(d(x_{2n},x_{2n-1})\lim_{n\to\infty}\int_0^{d_{2n}} \phi(t)dt)) \\
\leq &F(\psi(\int_0^{u} \phi(t)dt),\varphi(\int_0^{u} \phi(t)dt)),
\end{aligned}$$

hence $\psi(\int_0^u \phi(t)dt) = 0$ or $\varphi(\int_0^u \phi(t)dt) = 0$, so $\int_0^u \phi(t)dt = 0$. Therefore $u = 0$, that is, $\lim_{n\to\infty} d_n = 0$.

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence, then by Lemma 1.10 and Remark 1.11, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k, \forall k = 1, 2, \cdots$ such that

$$\varepsilon = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}). \tag{2.9}$$

We can assume that the parity of $n(k)$ and $m(k)$ are different for each k by $\lim_{n\to\infty} d_n = 0$.

Suppose that $m(k)$ is even and $n(k)$ is odd. Take $x = x_{m(k)}, y = x_{n(k)}$ in (2.1) and let $k \to \infty$, then using Lemma 2.1 and (2.9), we obtain that

$$\begin{aligned}
&\psi(\int_0^{\varepsilon} \phi(t)dt) = \lim_{k\to\infty} \psi(\int_0^{d(x_{m(k)+1}, x_{n(k)+1})} \phi(t)dt) = \lim_{k\to\infty} \psi(\int_0^{d(fx_{m(k)}, gx_{n(k)})} \phi(t)dt) \\
\leq &F(\psi(\int_0^{\varepsilon} \phi(t)dt), \phi(\int_0^{\varepsilon} \phi(t)dt)),
\end{aligned}$$

hence $\psi(\int_0^{\varepsilon} \phi(t)dt) = 0$ or $\varphi(\int_0^{\varepsilon} \phi(t)dt) = 0$ by $F \in \mathscr{C}^*$, thus $\int_0^{\varepsilon} \phi(t)dt = 0$. Therefore $\varepsilon = 0$, which is a contradiction. Similarly, we obtain the same contradiction for the case that $m(k)$ is odd and $n(k)$ is even. Hence $\{x_n\}$ is a Cauchy sequence.

Suppose that fX is complete. Since $x_{2n+1} = fx_{2n} \in fX$, so there exists $x^* \in fX$ such that $x_{2n+1} \to x^*$ as $n \to \infty$. Hence $d(x_{2n+2}, d) \leq d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x^*)$ implies that $x_{2n+1} \to x^*$ as $n \to \infty$, so we have $x_n \to x^*$ as $n \to \infty$. Similarly, we also obtain that there exists $y^* \in gX$ such that $x_n \to y^*$ as $n \to \infty$ for the case that gX is complete. Hence we may assume that $x_n \to x^* \in fX \cup gX$ as $n \to \infty$ in any case.

Using (2.1) and Lemma 2.1, we have

$$\begin{aligned}
&\psi(\int_0^{d(fx^*, x^*)} \phi(t)dt) = \lim_{n\to\infty} \psi(\int_0^{d(fx^*, gx_{2n+1})} \phi(t)dt) \\
\leq &F(\psi(\limsup_{n\to\infty} \Big[\alpha(d(x^*, x_{2n+1})) \int_0^{d(x^*, x_{2n+1})} \phi(t)dt\Big] \\
&+ \limsup_{n\to\infty} \Big[\beta(d(x^*, x_{2n+1})) \int_0^{d(x^*, fx^*)} \phi(t)dt\Big] \\
&+ \limsup_{n\to\infty} \Big[\gamma(d(x^*, x_{2n+1})) \int_0^{d(x_{2n+1}, gx_{2n+1})} \phi(t)dt\Big]),
\end{aligned}$$

$$
\begin{aligned}
&\varphi(\limsup_{n\to\infty}\Big[\alpha(d(x^*,x_{2n+1}))\int_0^{d(x^*,x_{2n+1})}\phi(t)dt\Big]\\
&+\limsup_{n\to\infty}\Big[\beta(d(x^*,x_{2n+1}))\int_0^{d(x^*,fx^*)}\phi(t)dt\Big]\\
&+\limsup_{n\to\infty}\Big[\gamma(d(x^*,x_{2n+1}))\int_0^{d(x_{2n+1},gx_{2n+1})}\phi(t)dt\Big])\\
&\le F(\psi(\int_0^{d(x^*,fx^*)}\phi(t)dt),\varphi(\int_0^{d(x^*,fx^*)}\phi(t)dt)),
\end{aligned}
$$

so $\psi(\int_0^{d(fx^*,x^*)}\phi(t)dt)=0$ or $\phi(\int_0^{d(fx^*,x^*)}\phi(t)dt)=0$, hence $\int_0^{d(fx^*,x^*)}\phi(t)dt=0$, therefore $fx^*=x^*$. Similarly, we have $gx^*=x^*$. Therefore x^* is a common fixed point of f and g.

If y^* is another common fixed point of f and g, then $d(x^*,y^*)\neq 0$, Using $\alpha(d(x^*,y^*))<1$ and (2.1), we obtain

$$
\begin{aligned}
&0<\psi(\int_0^{d(x^*,y^*)}\phi(t)dt)=\psi(\int_0^{d(fx^*,gy^*)}\phi(t)dt)\\
&\le F(\psi(\alpha(d(x^*,y^*)\int_0^{d(x^*,y^*)}\phi(t)dt),\phi(\alpha(d(x^*,y^*)\int_0^{d(x^*,y^*)}\phi(t)dt))\\
&\le\psi(\alpha(d(x^*,y^*)\int_0^{d(x^*,y^*)}\phi(t)dt)\\
&<\psi(\int_0^{d(x^*,y^*)}\phi(t)dt),
\end{aligned}
$$

which is a contradiction. Hence x^* is the unique common fixed point of f and g.

The following result is a more generalization of of Theorem 2.5.

Theorem 2.6 Let (X,d) be a metric space, $m,n\in\mathbb{N}$ and $f,g:X\to X$ two mappings. If f and g in all conditions in Theorem 2.1 are replaced by f^m and g^n respectively, then the same conclusion also holds.

Proof. Let $F=f^m$ and $G=g^n$, then F and G satisfy all of the conditions of Theorem 2.1, hence there exists an unique element $u\in X$ such that $f^m u=Fu=u=Gu=g^n u$.

Using (2.1), we obtain the next contradiction

$$
\begin{aligned}
&\psi(\int_0^{d(fu,u)}\phi(t)dt)=\psi(\int_0^{d(f^mfu,g^nu)}\phi(t)dt)=\psi(\int_0^{d(Ffu,Gu)}\phi(t)dt)\\
&\le F(\psi([\alpha(d(fu,u))\int_0^{d(fu,u)}\phi(t)dt+\beta(d(fu,u))\int_0^{d(fu,Ffu)}\phi(t)dt
\end{aligned}
$$

$$+\gamma(d(fu,u))\int_0^{d(u,Gu)}\phi(t)dt]),$$

$$\varphi([\alpha(d(fu,u))\int_0^{d(fu,u)}\phi(t)dt+\beta(d(fu,u))\int_0^{d(fu,Ffu)}\phi(t)dt$$

$$+\gamma(d(fu,u))\int_0^{d(u,Gu)}\phi(t)dt]))$$

$$\leq F(\psi(\int_0^{d(fu,u)}\phi(t)dt),\varphi(\int_0^{d(fu,u)}\phi(t)dt)),$$

hence $\int_0^{d(fu,u)}\phi(t)dt=0$ by the property of $\{\mathscr{C}^*,\Phi_u^*,\Phi_u^*\}$, i.e., $fu=u$. Similarly, $gu=u$, so u is the common fixed point of f and g. The uniqueness is obviously.

Next, we discuss the same problems as the above on ordered metric spaces.

Theorem 2.7 Let $(X,\preceq,d)$ be a ordered metric space, $f,g:X\to X$ two mappings. Suppose that The conditions (2.1) and (2.2) hold for each two comparable elements $x,y\in X$ and $t\geq 0$ respectively. Furthermore, if the following conditions hold

(i) for each $x\in X$, $x\preceq fx$ and $x\preceq gx$;

(ii) f and g are continuous;

(iii) $f(X)$ or $g(X)$ is complete.

Then f and g have a common fixed point $u\in fX\cup gX$.

Proof. Take any $x_0\in X$, then by (i), we have the next result:

$$x_0\preceq fx_0=:x_1,x_1\preceq gx_1=:x_2,x_2\preceq fx_2=:x_3,x_3\preceq gx_3=:x_4,\cdots.$$

Hence we obtain a sequence $\{x_n\}$ satisfying

$$x_{2n+1}=fx_{2n},\ x_{2n+2}=gx_{2n+1},\ x_n\preceq x_{n+1},n=0,1,2,\cdots. \tag{2.10}$$

(2.10) implies that x_n and x_m are comparable for all $m,n=0,1,2,\cdots$, hence modifying and repeating the process of the proof of Theorem 2.5, we know that $\{x_n\}$ is a Cauchy sequence and there exists $u\in fX\cup gX$ such that $x_n\to u$ as $n\to\infty$.

Using (ii), we have

$$u=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}fx_{2n}=f\lim_{n\to\infty}x_{2n}=fu,$$

$$u=\lim_{n\to\infty}x_{2n+2}=\lim_{n\to\infty}gx_{2n+1}=g\lim_{n\to\infty}x_{2n+1}=gu,$$

hence $Su=Tu=u$, i.e., u is a common fixed point of f and g.

The next result is the non-continuous version of Theorem 2.7.

Theorem 2.8 Let $(X, \preceq, d)$ be a ordered metric space, $f, g : X \to X$ two mappings. Suppose that The conditions (2.1) and (2.2) hold for each two comparable element $x, y \in X$ and $t \geq 0$ respectively. Furthermore, if the following conditions hold

(i) for each $x \in X$, $x \preceq fx$ and $x \preceq gx$;

(ii) If $\{x_n\}$ is a non-decreasing sequence and $\lim_{n\to\infty} x_n = u \in X$, then $x_n \preceq u$ for all $n \in \mathbb{N}$;

(iii) $f(X)$ or $g(X)$ is complete.

Then f and g have a common fixed point $u \in fX \cup gX$.

Proof. Just as the proof of Theorem 2.7, using (i) and (iii), we have that there exists a non-decreasing Cauchy sequence $\{x_n\}$ satisfying (2.10) such that $x_n \to u \in fX \cup gX$ as $n \to \infty$ and $x_n \preceq u$ for all $n \in \mathbb{N}$ by (ii). Hence by (2.1), we have

$$\begin{aligned}
&\psi(\int_0^{d(u,gu)} \varphi(t)dt) = \lim_{n\to\infty} \psi(\int_0^{d(fx_{2n},gu)} \varphi(t)dt) \\
\leq & F(\psi(\limsup_{n\to\infty}\{[\alpha(d(x_{2n},u))\int_0^{d(x_{2n},u)} \varphi(t)dt \\
&+\beta(d(x_{2n},u))\int_0^{d(x_{2n},fx_{2n})} \varphi(t)dt + \gamma(d(x_{2n},u))\int_0^{d(u,gu)} \varphi(t)dt]\}), \\
&\phi(\limsup_{n\to\infty}[\alpha(d(x_{2n},u))\int_0^{d(x_{2n},u)} \varphi(t)dt \\
&+\beta(d(x_{2n},u))\int_0^{d(x_{2n},fx_{2n})} \varphi(t)dt + \gamma(d(x_{2n},u))\int_0^{d(u,gu)} \varphi(t)dt]\})) \\
\leq & F(\psi(\int_0^{d(u,gu)} \varphi(t)dt), \phi(\int_0^{d(u,gu)} \varphi(t)dt)),
\end{aligned}$$

hence $\int_0^{d(u,gu)} \varphi(t)dt = 0$ by the property of $\{\mathscr{C}^*, \Phi_u^*, \Phi_u^*\}$, therefore $d(u, gu) = 0$, i.e., $gu = u$. Similarly, we also have $fu = u$, hence u is the common fixed point of f and g.

Let $C(f, g) = \{x \in X : fx = gx = x\}$ be the set of common fixed point of f and g.

Theorem 2.9 Suppose that all of the conditions of Theorem 2.7 or Theorem 2.8 are satisfied. Furthermore, if any two different elements $u, v \in C(f, g)$ are comparable and $\alpha(t) \neq 1$ for all $t > 0$, Then f and g have a unique common fixed point.

Proof. Obviously, there exists $u \in C(f, g)$ by Theorem 2.7 or Theorem 2.8. Suppose that v is also a common fixed point of f and g. If $u \neq v$, then $d(u, v) \neq 0$ and $\alpha(d(u, v)) < 1$.

Since u and v are comparable, using (2.1), we have

$$
\begin{aligned}
&0<\psi(\int_0^{d(u,v)}\varphi(t)dt=\psi(\int_0^{d(fu,gv)}\varphi(t)dt\\
&\leq F(\psi([\alpha(d(u,v))\int_0^{d(u,v)}\varphi(t)dt+\beta(d(u,v))\int_0^{d(u,fu)}\varphi(t)dt+\gamma(d(u,v))\int_0^{d(v,gv)}\varphi(t)dt]),\\
&\phi([\alpha(d(u,v))\int_0^{d(u,v)}\varphi(t)dt+\beta(d(u,v))\int_0^{d(u,fu)}\varphi(t)dt+\gamma(d(u,v))\int_0^{d(v,gv)}\varphi(t)dt]))\\
&\leq\psi(\alpha(d(u,v))\int_0^{d(u,v)}\varphi(t)dt)\\
&<\psi(\int_0^{d(u,v)}\varphi(t)dt),
\end{aligned}
$$

which is a contradiction. Hence $u=v$, so u is the unique common fixed point of f and g.

Remark 2.10 It is needed that the sum of the coefficient functions is less than 1 in Theorem 1.3 and 1.4, but the sum may equal to 1 in Theorem 2.5-Theorem 2.9.

Conflict of Interests

The authors declare that there is no conflict of interests.

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