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STABILITY OF ITERATIVE ALGORITHM FOR A COUNTABLE FAMILY OF QUASI-CONTRACTIVE OPERATORS

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Abstract. In this paper we establish the stability of an iterative algorithm for a countable family of quasi-contractive operators in an arbitrary Banach space in a more general form.

Keywords: fixed point; quasi-contraction; iteration procedure, stability.

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1. Preliminaries

Let X be a Banach space. T a selfmap of X. Let $y_{n+1} = f(T, y_n)$ be some iteration procedure. Suppose that $F(T)$, the fixed point set of T, is nonempty and that x_n converges to a point $p \in F(T)$. Let $\{x_n\} \subset X$ be bounded, and define $\varepsilon_n = \|y_{n+1}, f(T, y_n)\|$. If $\lim \varepsilon_n = 0$ implies that $\lim y_n = p$, then the iteration procedure $y_{n+1} = f(T, y_n)$ is said to be T-stable or stable with respect to T.

Harder and Hicks [4] showed how such sequences $\{x_n\}$ could arise in practice and the importance of investigating the stability of various iteration procedures for certain classes of nonlinear mappings. It was remarked by Massa (Math. Reviews 90a(1990), no. 54109a, 54H25) that the

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discussion about stability is very rich in examples. In [5], some applications of stability results to first order differential equations are discussed. Stability results for several iteration procedures for certain classes of nonlinear mappings have been established in recent papers by many authors (see, for example, A. M. Harder [4] [5], A. M. Harder and T. L. Hicks[6], M. O. Osilike [7] [8] [9] [10] [11] [12], and B.E.Rhoades [13]).

Ravi P. Agarwal, Yeol Je Cho, Jun Li, and Nan Jing Huang [3] prove a stability result of an Ishikawa type iteration procedure for a couple of quasi-contractive mappings in q-uniformly smooth Banach spaces.

We shall prove a stability result of an iteration procedure for a countable family of quasi-contractions in an arbitrary Banach space.

Let X be a Banach space and $\{T_k\}_{k=1}^{\infty}$ a countable family of selfmap of X, satisfying there exist $0 \leq h < 1$ such that

$$\|T_i x - T_j y\| \leq h \max\{\|x-y\|, \|x - T_i x\|, \|x - T_j y\|, \|y - T_j y\|, \|y - T_i x\|\}$$

for all i, j. Then we call $\{T_k\}_{k=1}^{\infty}$ a countable family of quasi-contractions.

Define the iteration procedure for $\{T_k\}_{k=1}^{\infty}$ as follows:

$$y_n = \frac{1}{n-1} \sum_{k=1}^{n-1} T_k y_{n-1}$$

Let $F(\{T_k\}_{k=1}^{\infty})$ denote the common fixed point set of $\{T_k\}_{k=1}^{\infty}$. Suppose $\{T_k\}_{k=1}^{\infty}$ is nonempty and y_n converges to a point $p \in F(\{T_k\}_{k=1}^{\infty})$. Let $\{x_n\} \subset X$ be bounded, and define $\varepsilon_n = \|x_n - \frac{1}{n-1}\sum_{k=1}^{n-1} T_k x_{n-1}\|$. If $\lim \varepsilon_n = 0$ implies that $\lim x_n = p$, then the iteration procedure $y_n = \frac{1}{n-1}\sum_{k=1}^{n-1} T_k y_{n-1}$ is said to be stable with respect to $\{T_k\}_{k=1}^{\infty}$.

We shall prove $y_n = \frac{1}{n-1}\sum_{k=1}^{n-1} T_k y_{n-1}$ is stable with respect to $\{T_k\}_{k=1}^{\infty}$ in a more general form.

We need the following lemmas in order to prove our main theorem.

Lemma 1. [2] *Let $\{x_n\}, \{\varepsilon_n\}$ be nonnegative sequences satisfying $x_{n+1} \leq h x_n + \varepsilon_n$ for all $n \in \mathbb{N}, 0 \leq h < 1, \lim \varepsilon_n = 0$. Then $\lim x_n = 0$.*

Lemma 2. *Let $\{T_k\}_{k=1}^{\infty}$ be a countable family of quasi-contractions, then*

$$\|T_i x - p\| \leq \frac{h}{1-h} \|x - p\|$$

Proof.

$$\begin{aligned}\|T_ix-p\| &\le h\max\{\|x-p\|,\|x-T_ix\|,\|x-T_ip\|,\|p-T_ix\|,\|p-T_ip\|\}\\ &\le h\max\{\|x-p\|,\|x-T_ix\|,\|x-p\|,\|p-T_ix\|,0\}\\ &= h\max\{\|x-p\|,\|x-T_ix\|,\|x-p\|,\|p-T_ix\|\}\end{aligned}$$

Hence $\|T_ix-p\|\le h\|x-p\|$ or $\|T_ix-p\|\le h\|x-T_ix\|$ or $\|T_ix-p\|\le h\|p-T_ix\|$.

If $\|T_ix-p\|\le h\|x-p\|$, it is clear

$$\|T_ix-p\|\le h\|x-p\|\le \frac{h}{1-h}\|x-p\|.$$

If $\|T_ix-p\|\le h\|p-T_ix\|$, then

$$\|T_ix-p\|=0\le \frac{h}{1-h}\|x-p\|$$

If $\|T_ix-p\|\le h\|x-T_ix\|$, then

$$\|x-T_ix\|\le\|T_ix-p\|+\|x-p\|\le h\|x-T_ix\|+\|x-p\|.$$

Hence, $\|T_ix-p\|\le\frac{h}{1-h}\|x-p\|$. □

Lemma 3. *Let $\{T_k\}_{k=1}^{\infty}$ be a countable family of quasi-contractions, then $\{T_k\}_{k=1}^{\infty}$ have a unique common fixed point.*

Proof. For each i, it follows from Ciric [1] that T_i has a unique fixed point p_i. It is sufficient to prove $p_i=p_j$, for all i,j.

$$\begin{aligned}\|p_i-p_j\| &= \|T_ip_i-T_jp_j\|\\ &\le h\max\{\|p_i-p_j\|,0,\|p_i-p_j\|,0,\|p_j-p_i\|\}=h\|p_i-p_j\|\end{aligned}$$

So, $\|p_i-p_j\|=0$ and $p_i=p_j$. The proof is completed. □

2. Main results

Theorem 1. *Let X be a Banach space, $\{T_k\}_{k=1}^{\infty}$ a countable family of quasi-contractions. Let $\{x_n\} \subset X$ be bounded, and define $\varepsilon_n = \|x_n - \frac{1}{n-1}\sum_{k=1}^{n-1} T_k x_{n-1}\|$. Assume $\lim \varepsilon_n = 0$. Define p_n to be the diameter of $\{x_k : k \geq n\} \bigcup \{T_l x_k : k \geq n, l \geq 1\}$; i.e., $p_n = \delta(\{x_k : k \geq n\} \bigcup \{T_l x_k : k \geq n, l \geq 1\})$. Then $\lim p_n = 0$.*

Proof. First, we show p_n is bounded. Since $\{x_n\}$ is bounded, there exist $M > 0$ such that $\|x_n\| \leq M$. It follows from Lemma 2 that $|T_l x_k - p\| \leq \frac{h}{1-h}\|x_k - p\|$, for all l, k. Thus, $|T_l x_k\| \leq \frac{h}{1-h}(\|x_k\| + \|p\|) + \|p\| \leq \frac{h}{1-h}(M + \|p\|) + \|p\|$, for all l, k. So it is easy to see p_n is bounded and $p_n \leq p_{n-1}$.

Define $\delta_n = \sup\{\varepsilon_k\}_{k=n}^{\infty}$, then $\lim \delta_n = 0$ since $\lim \varepsilon_n = 0$. Next, we show $p_n \leq hp_{n-2} + 2\delta_n$. $\forall i, j \geq n$,

$$\|T_k x_i - T_l x_j\| \leq h \max\{\|x_i - y_j\|, \|x_i - T_l x_j\|, \|x_i - T_k x_i\|, \|x_j - T_k x_i\|, \|x_j - T_l x_j\|\} \leq hp_n$$

$$\begin{aligned}
\|x_i - T_l x_j\| &\leq \|x_i - \frac{1}{i-1}\sum_{k=1}^{i-1} T_k x_{i-1}\| + \|\frac{1}{i-1}\sum_{k=1}^{i-1} T_k x_{i-1} - T_l x_j\| \\
&\leq \varepsilon_i + \|\frac{1}{i-1}\sum_{k=1}^{i-1}(T_k x_{i-1} - T_l x_j)\| \\
&\leq \varepsilon_i + \frac{1}{i-1}\sum_{k=1}^{i-1}\|T_k x_{i-1} - T_l x_j\| \\
&\leq \varepsilon_i + \frac{1}{i-1}\sum_{k=1}^{i-1} hp_{n-1} \\
&= \varepsilon_i + hp_{n-1} \\
&\leq \delta_n + hp_{n-1}
\end{aligned}$$

$$\|x_i - x_j\| \leq \|x_i - \frac{1}{i-1}\sum_{k=1}^{i-1} T_k x_{i-1}\| + \|\frac{1}{i-1}\sum_{k=1}^{i-1} T_k x_{i-1} - x_j\|$$

$$\le \varepsilon_i + \|\frac{1}{i-1}\sum_{k=1}^{i-1}(T_k x_{i-1} - x_j)\|$$

$$\le \varepsilon_i + \frac{1}{i-1}\sum_{k=1}^{i-1}\|T_k x_{i-1} - x_j\|$$

$$\le \varepsilon_i + \frac{1}{i-1}\sum_{k=1}^{i-1}(\delta_n + hp_{n-2})$$

$$= \varepsilon_i + \delta_n + hp_{n-2}$$

$$\le 2\delta_n + hp_{n-2}$$

So, $\delta(\{x_k : k \ge n\} \bigcup \{T_l x_k : k \ge n, l \ge 1\}) \le 2\delta_n + hp_{n-2}$, i.e., $p_n \le hp_{n-2} + 2\delta_n$.

Let $a_k = p_{2k}$, then $p_{2k} \le hp_{2k-2} + 2\delta_{2k}$ implies $a_k \le ha_{k-1} + 2\delta_k$. It follows from Lemma 1 that $\lim a_k = 0$, i.e., $\lim p_{2k} = 0$.

Let $b_k = p_{2k+1}$, then $p_{2k+1} \le hp_{2k-1} + 2\delta_{2k+1}$ implies $b_k \le hb_{k-1} + 2\delta_k$. It follows from Lemma 1 that $\lim b_k = 0$, i.e., $\lim p_{2k+1} = 0$.

Hence $\lim p_n = 0$. □

As a corollary, we establish the stability of the iteration procedure $y_n = \frac{1}{n-1}\sum_{k=1}^{n-1} T_k y_{n-1}$ with respect to a countable family of quasi-contractions $\{T_k\}_{k=1}^{\infty}$.

Theorem 2. *Let X be a Banach space, $\{T_k\}_{k=1}^{\infty}$ a countable family of quasi-contractions. Then the iteration procedure for $\{T_k\}_{k=1}^{\infty}$ defined as*

$$y_n = \frac{1}{n-1}\sum_{k=1}^{n-1} T_k y_{n-1}$$

is stable with respect to $\{T_k\}_{k=1}^{\infty}$.

Proof.

$$\|x_n - p\| \le \|x_n - \frac{1}{n-1}\sum_{k=1}^{n-1} T_k x_{n-1}\| + \|\frac{1}{n-1}\sum_{k=1}^{n-1} T_k x_{n-1} - p\|$$

$$\le \varepsilon_n + \frac{1}{n-1}\sum_{k=1}^{n-1}\|T_k x_{n-1} - Tp\|$$

But

$$\|T_kx_{n-1}-T_kp\| \le h\max\{\|x_{n-1}-p\|,\|x_{n-1}-T_kx_{n-1}\|,\|T_kx_{n-1}-p\|\}$$

So,

$$\begin{aligned}\|T_kx_{n-1}-T_kp\| &\le h\max\{\|x_{n-1}-p\|,\|x_{n-1}-T_kx_{n-1}\|\}\\ &\le h(\|x_{n-1}-p\|+\|x_{n-1}-T_kx_{n-1}\|)\\ &\le h\|x_{n-1}-p\|+hp_{n-1})\end{aligned}$$

substituting it into the first inequality, we obtain

$$\begin{aligned}\|x_n-p\| &\le \varepsilon_n+\frac{1}{n-1}\sum_{k=1}^{n-1}\|T_kx_{n-1}-T_kp\|\\ &\le \varepsilon_n+\frac{1}{n-1}\sum_{k=1}^{n-1}(h\|x_{n-1}-p\|+hp_{n-1})\\ &= \varepsilon_n+h\|x_{n-1}-p\|+hp_{n-1}\end{aligned}$$

Noting that $0\le h<1$, $\lim\varepsilon_n=0$ with addition $\lim p_n=0$, by Theorem 1. It follows from Lemma 1 that $\lim\|x_n-p\|=0$, i.e., $\lim x_n=p$. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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