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GENERALIZATION OF WEIGHTED OSTROWSKI INEQUALITY WITH APPLICATIONS IN NUMERICAL INTEGRATION

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Abstract. An integral inequality of Ostrowski type with weights is established for differentiable functions up to second order, whose second derivatives are bounded and first derivatives are absolutely continuous. The inequality of weighted integral is then functional to weighted composite quadrature rules.

Keywords: weighted Ostrowski's inequality; numerical integration.

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1. Introduction

It is a known that Ostrowski type inequalities can be used to estimate the absolute deviation from its integral mean. These inequalities can be used to provide explicit error bounds for numerical quadrature formulae. The weighted version of Ostrowski inequality was first presented in 1983 by J. E. Pečarić and B. Savić [10]. Keeping in view the importance of this inequality, for last few decades, the researchers are in continuous effort to obtain sharp bounds of Ostrowski's

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inequality in terms of weight.

We have introduced a weighted inequality which has its applications in Numerical quadrature rules and Probability theory. The probability density function, defined by us, has produced the results of Ostrowski's inequality in a more organized way.

In 1938, a Ukrainian Mathematician Alexandar Markowich Ostrowski discovered an inequality called Ostrowski inequality, which states that.

Lemma 1.1. [15] Let $\varphi : \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0 (I^0 is the interior of I) and let $a_0, b_1 \in \mathrm{I}^0$ with $a_0 < b_1$. If $\varphi' : (a_0, b_1) \to \mathbb{R}$ is bounded on (a_0, b_1), i.e.,

$$||\varphi'||_\infty = \sup_{t \in (a_0, b_1)} |\varphi'(t)| < \infty,$$

Then the inequality is

$$\left| \varphi(v) - \frac{1}{b_1 - a_0} \int_{a_0}^{b_1} \varphi(t) dt \right| \leq \left[\frac{1}{4} + \frac{(v - \frac{a_0+b_1}{2})^2}{(b_1 - a_0)^2} \right] (b_1 - a_0) ||\varphi'||_\infty$$

for all $v \in [a_0, b_1]$.The number $\frac{1}{4}$ is sharp which cannot be replaced by a smaller one.

G. V. Milovanović and J. E. Pečarić proved a generalised Ostrowski's inequality in 1976 for n-times differentiable functions [9]. In which he pointed out the speciality of twice differentiable functions [8, p. 470].

Lemma 1.2. [15] Let $\varphi : \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0 (I^0 is the interior of I) and let $a_0, b_1 \in \mathrm{I}^0$ with $a_0 < b_1$. If $\varphi' : (a_0, b_1) \to \mathbb{R}$ is bounded on (a_0, b_1), i.e.,

$$||\varphi'||_\infty = \sup_{t \in (a_0, b_1)} |\varphi'(t)| < \infty,$$

Then the inequality is

$$\left| \varphi(v) - \frac{1}{b_1 - a_0} \int_{a_0}^{b_1} \varphi(t) dt \right| \leq \left[\frac{1}{4} + \frac{(v - \frac{a_0+b_1}{2})^2}{(b_1 - a_0)^2} \right] (b_1 - a_0) ||\varphi'||_\infty$$

for all $v \in [a_0, b_1]$.The number $\frac{1}{4}$ is sharp which cannot be replaced by a smaller one.

Lemma 1.3. [15]*Let* $\varphi : [a_0, b_1] \to \mathbb{R}$ *be a twice differentiable function such that* $\varphi'' : (a_0, b_1) \to$

$\mathbb{R}$ *is bounded on* (a_0, b_1). *Then the following inequality holds for all* $v \in [a_0, b_1]$.

$$\begin{aligned}&\left|\frac{1}{2}\left[\varphi(v)+\frac{(v-a_0)\varphi(a_0)+(b_1-v)\varphi(b_1)}{b_1-a_0}\right]-\frac{1}{b_1-a_0}\int_{a_0}^{b_1}\varphi(t)dt\right|\\ \leq\ &\frac{\|\varphi''\|_\infty}{4}(b_1-a_0)^2\left[\frac{1}{12}+\frac{(v-\frac{a_0+b_1}{2})^2}{(b_1-a_0)^2}\right]\end{aligned}$$

In 1999, the following inequality is proved by Cerone et.al. in [1].

Lemma 1.4. [15]*Let the following assumption* (1.2) *be valid for all* $v \in [a_0, b_1]$

$$\begin{aligned}&\left|\varphi(v)-\left(v-\frac{a_0+b_1}{2}\right)\varphi'(v)-\frac{1}{b_1-a_0}\int_{a_0}^{b}{}_1\varphi(t)dt\right|\\ \leq\ &\left[\frac{(b_1-a_0)^2}{24}+\frac{1}{2}\left(v-\frac{a_0+b_1}{2}\right)^2\right]\|\varphi''\|_\infty\leq\frac{(b_1-a_0)^2}{6}\|\varphi''\|_\infty\end{aligned}$$

In the same year, the following result is stated by Dragomir and Barnett in [2].

Lemma 1.5. [15]*Let the following assumption* (1.2) *be valid for all* $v \in [a_0, b_1]$

$$\begin{aligned}&\left|\varphi(v)-\frac{\varphi(b_1)-\varphi(a_0)}{b_1-a_0}\left(v-\frac{a_0+b_1}{2}\right)-\frac{1}{b_1-a_0}\int_{a_0}^{b}{}_1\varphi(t)dt\right|\\ \leq\ &\frac{(b_1-a_0)^2}{2}\left\{\left[\left(\frac{v-\frac{a_0+b_1}{2}}{b_1-a_0}\right)^2+\frac{1}{4}\right]^2+\frac{1}{12}\right\}\|\varphi''\|_\infty\leq\frac{(b_1-a_0)^2}{6}\|\varphi''\|_\infty.\end{aligned}$$

An Ostrowski type integral inequality for functions whose second derivatives are bounded is established by P. Cerone, S. S. Dragomir and J. Roumeliotis in [1]. S. S. Dragomir and N. S. Barnett in [2] have also established a similar inequality. in [4], an Ostrowski type integral inequality is similar in sense as that of [1] or [2] pointed out by S. S. Dragomir and A. Sofo. The following integral inequality in [4] is proved by S. S. Dragomir and A. Sofo.

Lemma 1.6. [15]*Let the function* $\varphi : [a_0, b_1] \to \mathbb{R}$ *is considered having first derivative is absolutely continuous on* $[a_0, b_1]$ *and the second derivative* $\varphi'' \in L_\infty\,[a_0, b_1]$. *Then we get the following inequality for all* $v \in [a_0, b_1]$

$$\begin{aligned}&\left|\int_{a_0}^{b_1}\varphi(t)-\frac{(b_1-a_0)}{2}\left[\varphi(v)+\frac{\varphi(a_0)+\varphi(b_1)}{2}\right]+\frac{b_1-a_0}{2}\left(v-\frac{a_0+b_1}{2}\right)\varphi'(v)\right|\\ &\leq\|\varphi''\|_\infty\left(\frac{1}{3}\left|v-\frac{a_0+b_1}{2}\right|^3+\frac{(b_1-a_0)^3}{48}\right)\qquad(1)\end{aligned}$$

In present paper, we give weighted version of Ostrowski-type inequality (1) using different kernel and will also state its application in Numerical quadrature rules, probability theory and special means. This paper is arranged in the following approach. The first part is based on preliminaries whereas in second part we are using weights in inequality (1) from which we get main result, which is in fact probability density functions. In the last part we would state about the application in terms of composite quadrature rules.

3. Main results

We use here a weighted form of integral inequality which is proved by S. S. Dragomir and A. Sofo in [4], and also we are using it in numerical integration.

Theorem 3.1.

$$\begin{aligned} &\Bigg|\int_{a_0}^{b_1}\varphi(t)w(t)dt-\frac{1}{2}\Bigg[\varphi(v)+\frac{b_1-a_0}{2}\left[\varphi(a_0)w(a_0)+\varphi(b_1)w(b_1)\right] \\ &-\int_{a_0}^{b_1}\varphi(t)\left(t-\frac{a_0+b_1}{2}\right)w'(t)dt-\int_{a_0}^{b_1}w(t)\left(v-\frac{a_0+b_1}{2}\right)\varphi'(v)dt\Bigg]\Bigg| \\ \leq\;& \|\varphi''\|_\infty\left[\int_{a_0}^{b_1}w(u)du\left(\frac{v^2}{4}-\frac{a_0+b_1}{4}v\right)+\int_{a_0}^{b_1}\left(\frac{a_0+b_1}{4}t-\frac{t^2}{4}\right)w(t)dt\right] \end{aligned} \tag{2}$$

where $w:[a_0,b_1]\to[0,\infty)$ is a PDF.

Proof. Let us begin with the following integral identity [8] (see also [6]).

$$\psi(v)=\int_{a_0}^{b_1}w(t)\psi(t)dt+\int_{a_0}^{b_1}P_w(v,t)\psi'(t)dt \tag{3}$$

for all $v\in[a_0,b_1]$, given that f is absolutely continuous on $[a_0,b_1]$ and the weighted kernel $P_w:[a_0,b_1]\times[a_0,b_1]\to\mathbb{R}$ is given by:

$$P_w(v,t)=\begin{cases}\displaystyle\int_{a_0}^{t}w(u)du, & \text{if}\quad t\in[a_0,v],\\ \displaystyle\int_{\frac{a_0+b_1}{2}}^{t}w(u)du, & \text{if}\quad t\in(v,a_0+b_1-v],\\ \displaystyle\int_{b_1}^{t}w(u)du, & \text{if}\quad t\in(a_0+b_1-v,b_1],\end{cases}$$

where $t \in [a_0, b_1]$. Let us consider, $\psi(v) = \left(v - \frac{a_0+b_1}{2}\right)\varphi'(v)$. Then (3) implies

$$\begin{aligned}\left(v - \frac{a_0+b_1}{2}\right)\varphi'(v) = \int_{a_0}^{b_1} w(t)\left(t - \frac{a_0+b_1}{2}\right)\varphi'(t)dt + \int_{a_0}^{b_1} P_w(v,t)\big[\varphi'(t) \\ + \left(t - \frac{a_0+b_1}{2}\right)\varphi''(t)\big]\,dt.\end{aligned} \tag{4}$$

Using integration by parts, we get

$$\begin{aligned}&\int_{a_0}^{b_1}\left(t - \frac{a_0+b_1}{2}\right)w(t)\varphi'(t)dt = \frac{b_1-a_0}{2}\left[\varphi(a_0)w(a_0) + \varphi(b_1)w(b_1)\right] \\ &-\int_{a_0}^{b_1}\varphi(t)w(t)dt - \int_{a_0}^{b_1}\varphi(t)\left(t - \frac{a_0+b_1}{2}\right)w'(t)dt,\end{aligned} \tag{5}$$

also

$$\int_{a_0}^{b_1} P_w(v,t)\varphi'(t)dt = \varphi(v) - \int_{a_0}^{b_1}\varphi(t)w(t)dt. \tag{6}$$

Now using equations (5) and (6) in (4), we get

$$\begin{aligned}&\left(v - \frac{a_0+b_1}{2}\right)\varphi'(v) = \frac{b_1-a_0}{2}\left[\varphi(a_0)w(a_0) + \varphi(b_1)w(b_1)\right] \\ &+\varphi(v) - 2\int_{a_0}^{b_1}\varphi(t)w(t)dt - \int_{a_0}^{b_1}\varphi(t)\left(t - \frac{a_0+b_1}{2}\right)w'(t)dt \\ &+\int_{a_0}^{b_1} P_w(v,t)\left(t - \frac{a_0+b_1}{2}\right)\varphi''(t)dt.\end{aligned}$$

or

$$\begin{aligned}&\int_{a_0}^{b_1}\varphi(t)w(t)dt = \frac{b_1-a_0}{4}\left[\varphi(a_0)w(a_0) + \varphi(b_1)w(b_1)\right] + \frac{1}{2}\varphi(v) \\ &-\frac{1}{2}\left(v - \frac{a_0+b_1}{2}\right)\varphi'(v) - \frac{1}{2}\int_{a_0}^{b_1}\varphi(t)\left(t - \frac{a_0+b_1}{2}\right)w'(t)dt \\ &+\frac{1}{2}\int_{a_0}^{b_1} P_w(v,t)\left(t - \frac{a_0+b_1}{2}\right)\varphi''(t)dt,\end{aligned}$$

for all $v \in [a_0, b_1]$,

which gives us

$$\begin{aligned}
&\left| \int_{a_0}^{b_1} \varphi(t)w(t)dt - \frac{1}{2}\left[\varphi(v) + \frac{b_1-a_0}{2}\left[\varphi(a_0)w(a_0)+\varphi(b_1)w(b_1)\right]\right.\right. \\
&\left.\left. - \int_{a_0}^{b_1}\varphi(t)\left(t-\frac{a_0+b_1}{2}\right)w'(t)dt - \left(v-\frac{a_0+b_1}{2}\right)\varphi'(v)\right]\right| \\
= &\left|\frac{1}{2}\int_{a_0}^{b_1} P_w(v,t)\left(t-\frac{a_0+b_1}{2}\right)\varphi''(t)dt\right| \\
\leq &\frac{1}{2}\int_{a_0}^{b_1} |P_w(v,t)|\left|t-\frac{a_0+b_1}{2}\right|\left|\varphi''(t)\right|dt
\end{aligned}$$

Then we have

$$\begin{aligned}
&\int_{a_0}^{b_1} |P_w(v,t)|\left|t-\frac{a_0+b_1}{2}\right|\left|\varphi''(t)\right|dt \\
&\leq \|\varphi''\|_\infty \int_{a_0}^{b_1} |P_w(v,t)|\left|t-\frac{a_0+b_1}{2}\right|dt, \qquad (7)
\end{aligned}$$

Also

$$I = \int_{a_0}^{b_1} |P_w(v,t)|\left|t-\frac{a_0+b_1}{2}\right|dt$$

or

$$\begin{aligned}
I = &\int_{a_0}^{v}\left|\int_{a_0}^{t} w(u)du\right|\left|t-\frac{a_0+b_1}{2}\right|dt + \int_{v}^{a_0+b_1-v}\left|\int_{\frac{a_0+b_1}{2}}^{t} w(u)du\right| \\
&\times\left|t-\frac{a_0+b_1}{2}\right|dt + \int_{a_0+b_1-v}^{b_1}\left|\int_{b_1}^{t} w(u)du\right|\left|t-\frac{a_0+b_1}{2}\right|dt.
\end{aligned}$$

(8)

Now, we have a case:

for $v \in \left[a_0, \frac{a_0+b_1}{2}\right]$, we find

$$\begin{aligned}
I = &\int_{a_0}^{v}\left(\int_{a_0}^{t} w(u)du\right)\left(\frac{a_0+b_1}{2}-t\right)dt + \int_{v}^{\frac{a_0+b_1}{2}}\left(\int_{t}^{b_1} w(u)du\right) \\
&\times\left(\frac{a_0+b_1}{2}-t\right)dt + \int_{\frac{a_0+b_1}{2}}^{b_1}\left(\int_{t}^{b_1} w(u)du\right)\left(t-\frac{a_0+b_1}{2}\right)dt.
\end{aligned}$$

After solving the equations,we find

$$\begin{aligned}
I = & \int_{a_0}^{v} w(u)du\left(\frac{v^2}{2}-\frac{a_0+b_1}{2}v\right)-\int_{a_0}^{v}\left(\frac{t^2}{2}-\frac{a_0+b_1}{2}t\right)w(t)dt \\
& +\int_{v}^{\frac{a_0+b_1}{2}} w(u)du\left(\frac{v^2}{2}-\frac{a_0+b_1}{2}v\right)+\int_{v}^{\frac{a_0+b_1}{2}}\left(\frac{a_0+b_1}{2}t-\frac{t^2}{2}\right)w(t)dt \\
& -\int_{\frac{a_0+b_1}{2}}^{a_0+b_1-v} w(u)du\left(\frac{a_0+b_1}{2}v-\frac{v^2}{2}\right)+\int_{\frac{a_0+b_1}{2}}^{a_0+b_1-v}\left(\frac{a_0+b_1}{2}-\frac{t^2}{2}\right)w(t)dt \\
& -\int_{a_0+b_1-v}^{b_1} w(u)du\left(\frac{a_0+b_1}{2}v-\frac{v^2}{2}\right)+\int_{a_0+b_1-v}^{b_1}\left(\frac{a_0+b_1}{2}t-\frac{t^2}{2}\right)w(t)dt \\
= & \int_{a_0}^{v} w(u)du\left(\frac{v^2}{2}-\frac{a_0+b_1}{2}v\right)+\int_{a_0}^{v}\left(\frac{a_0+b_1}{2}t-\frac{t^2}{2}\right)w(t)dt \\
& +\int_{v}^{\frac{a_0+b_1}{2}} w(u)du\left(\frac{v^2}{2}-\frac{a_0+b_1}{2}v\right)+\int_{v}^{\frac{a_0+b_1}{2}}\left(\frac{a_0+b_1}{2}t-\frac{t^2}{2}\right)w(t)dt \\
& +\int_{\frac{a_0+b_1}{2}}^{a_0+b_1-v} w(u)du\left(\frac{v^2}{2}-\frac{a_0+b_1}{2}v\right)+\int_{\frac{a_0+b_1}{2}}^{a_0+b_1-v}\left(\frac{a_0+b_1}{2}-\frac{t^2}{2}\right)w(t)dt \\
& +\int_{a_0+b_1-v}^{b_1} w(u)du\left(\frac{v^2}{2}-\frac{a_0+b_1}{2}v\right)+\int_{a_0+b_1-v}^{b_1}\left(\frac{a_0+b_1}{2}t-\frac{t^2}{2}\right)w(t)dt
\end{aligned}$$

(9)

and finally

$$I=\int_{a_0}^{b_1} w(u)du\left(\frac{v^2}{2}-\frac{a_0+b_1}{2}v\right)+\int_{a_0}^{b_1}\left(\frac{a_0+b_1}{2}t-\frac{t^2}{2}\right)w(t)dt \tag{10}$$

Using equations (7), (8), and (9),we find

$$\begin{aligned}
& \left|\int_{a_0}^{b_1}\varphi(t)w(t)dt-\frac{1}{2}\left[\varphi(v)+\frac{b_1-a_0}{2}\left[\varphi(a_0)w(a_0)+\varphi(b_1)w(b_1)\right]\right.\right. \\
& \left.\left.-\int_{a_0}^{b_1}\varphi(t)\left(t-\frac{a_0+b_1}{2}\right)w'(t)dt-\left(v-\frac{a_0+b_1}{2}\right)\varphi'(v)\right]\right| \\
\leq & \frac{\|\varphi''\|_\infty}{2}\left[\int_{a_0}^{b_1} w(u)du\left(\frac{v^2}{2}-\frac{a_0+b_1}{2}v\right)+\int_{a_0}^{b_1}\left(\frac{a_0+b_1}{2}t-\frac{t^2}{2}\right)w(t)dt\right] \\
= & \|\varphi''\|_\infty\left[\int_{a_0}^{b_1} w(u)du\left(\frac{v^2}{4}-\frac{a_0+b_1}{4}v\right)+\int_{a_0}^{b_1}\left(\frac{a_0+b_1}{4}t-\frac{t^2}{4}\right)w(t)dt\right]
\end{aligned}$$

which is our required result.

remarks 3.2. *If we put* $w(t) \equiv \frac{1}{b_1 - a_0}$ *in (2), then we will get*

$$\left| \int_{a_0}^{b_1} \varphi(t) - \frac{(b_1 - a_0)}{2} \left[\varphi(v) + \frac{\varphi(a_0) + \varphi(b_1)}{2} \right] + \frac{b_1 - a_0}{2} \left(v - \frac{a_0 + b_1}{2} \right) \varphi'(v) \right|$$
$$\leq \|\varphi''\|_\infty \frac{(b_1 - a_0)}{4} \left(x - \frac{a_0 + b_1}{2} \right)^2 + \frac{(a_0 - b_1)^3}{48} \tag{11}$$

.

remarks 3.3. *If we put the midpoint* $v = \frac{a_0 + b_1}{2}$ *in (2), then we will get*

$$\left| \int_{a_0}^{b_1} \varphi(t) w(t) dt - \frac{1}{2} \left[\varphi \left(\frac{a_0 + b_1}{2} \right) + \frac{b_1 - a_0}{2} \left(\varphi(a_0) w(a_0) + \varphi(b_1) w(b_1) \right) \right. \right.$$
$$\left. \left. - \int_{a_0}^{b_1} \varphi(t) \left(t - \frac{a_0 + b_1}{2} \right) w'(t) dt \right] \right|$$
$$\leq \quad \|\varphi''\|_\infty \left[\int_{a_0}^{b_1} \left(\frac{a_0 + b_1}{4} t - \frac{t^2}{4} \right) w(t) dt - \int_{a_0}^{b_1} w(u) du \left(\frac{a_0 + b_1}{4} \right)^2 \right] \tag{12}$$

.

remarks 3.4. *If we put* $w(t) \equiv \frac{1}{b_1 - a_0}$ *in (12), then we will get*

$$\left| \int_{a_0}^{b_1} \varphi(t) dt - \frac{(b_1 - a_0)}{2} \left[\varphi \left(\frac{a_0 + b_1}{2} \right) + \frac{b_1 - a_0}{2} \left(\varphi(a_0) w(a_0) + \varphi(b_1) w(b_1) \right) \right. \right.$$
$$\left. \left. - (b_1 - a_0) \int_{a_0}^{b_1} \varphi(t) \left(t - \frac{a_0 + b_1}{2} \right) w'(t) dt \right] \right|$$
$$\leq \quad \|\varphi''\|_\infty \left[\int_{a_0}^{b_1} \left(\frac{a_0 + b_1}{4} t - \frac{t^2}{4} \right) dt - \int_{a_0}^{b_1} du \left(\frac{a_0 + b_1}{4} \right)^2 \right] \tag{13}$$

.

remarks 3.5. *If we examine an approximation for the end point*

$v = a_0$ *in (2), then we will get*

$$\left| \int_{a_0}^{b_1} \varphi(t) w(t) dt - \frac{1}{2} [\varphi(a_0) + \frac{b_1 - a_0}{2} \left(\varphi(a_0) w(a_0) + \varphi(b_1) w(b_1) \right) \right.$$
$$\left. - \int_{a_0}^{b_1} \varphi(t) \left(t - \frac{a_0 + b_1}{2} \right) w'(t) dt + (b_1 - a_0) \int_{a_0}^{b_1} w(t) \varphi'(a_0) dt] \right|$$
$$\leq \quad \|\varphi''\|_\infty \left[\int_{a_0}^{b_1} \left(\frac{a_0 + b_1}{4} t - \frac{t^2}{4} \right) w(t) dt - \int_{a_0}^{b_1} w(u) du \left(\frac{a_0 b_1}{4} \right) \right]$$

.

remarks 3.6. *If we examine an approximation for the end point*
$v = b_1$ in (2), we will get

$$\begin{aligned} & \left| \int_{a_0}^{b_1} \varphi(t)w(t)dt - \frac{1}{2}\left[\varphi(b_1) + \frac{b_1 - a_0}{2}\ (\varphi(a_0)w(a_0) + \varphi(b_1)w(b_1)) \right.\right. \\ & \left.\left. - \int_{a_0}^{b_1} \varphi(t)\left(t - \frac{a_0+b_1}{2}\right) w'(t)dt\ - \int_{a_0}^{b_1} w(t)(b_1 - a_0)\varphi'(b_1)dt \right]\right| \\ \leq\ & \|\varphi''\|_\infty \left[\int_{a_0}^{b_1}\left(\frac{a_0+b_1}{4}t\ - \frac{t^2}{4}\right) w(t)dt - \int_{a_0}^{b_1} w(u)du\left(\frac{a_0 b_1}{4}\right)\right]. \end{aligned}$$

3.Application in Numerical Integration

Theorem 3.1. *Let $I_n : a_0 = v_0 < v_1 < \ldots < v_{n-1} < v_n = b_1$ be a partition of interval $[a_0, b_1]$, $\Delta\zeta = v_{i+1} - v_i,$, $a_{0i} \leq \xi_i \leq b_{1i}$, where $a_{0i} = v_i$, $b_{1i} = v_{i+1}$, $i = 0, \ldots, n-1$ then*

$$\int_{v_i}^{v_{i+1}} \varphi(t)w(t)dt = S(\varphi, \varphi', I_n, v, w) + R(\varphi, \varphi', I_n, v, w)$$

where

$$\begin{aligned} & |S(\varphi, \varphi', I_n, v, w)| \\ =\ & \frac{1}{2}\sum_{i=0}^{n-1}\left[\varphi(\xi_i) + \frac{\Delta\zeta}{2}\left[\varphi(v_i)w(v_i) + \varphi(v_{i+1})w(v_{i+1})\right]\right. \\ & - \int_{v_i}^{v_{i+1}}\left(\xi - \frac{v_i + v_{i+1}}{2}\right)\varphi'(\xi_i)dt \\ & \left. - \int_{v_i}^{v_{i+1}} \varphi(t)\left(t - \frac{v_i+v_{i+1}}{2}\right) w'(t)dt\right]\Delta\zeta \end{aligned}$$

and

$$\begin{aligned} & \left|R(\varphi, \varphi', I_n, v, w)\right| \leq \|\varphi''\|_\infty \sum_{i=0}^{n-1}\left[\left|\int_{v_i}^{v_{i+1}}\left(\frac{(v_i+v_{i+1})t}{4} - \frac{t^2}{4}\right) w(t)dt\right.\right. \\ & \left.\left. + \int_{v_i}^{v_{i+1}}\left(\frac{\xi^2}{4} - \frac{(v_i+v_{i+1})\xi}{4}\right)\right|\right]. \end{aligned} \tag{14}$$

.

Proof. Applying inequality (2) on $\xi_i \in [x_i, x_{i+1}]$ and summing over i from to $n-1$ and using triangular inequality we get (14) $\square$

remarks 3.6. *If we put $w(t) \equiv \frac{1}{b_1 - a_0}$ in (2) then we will get*

$$
\begin{aligned}
&|S(\varphi,\varphi',I_n,v,w)| = \\
&\frac{(v_{i+1}-v_i)}{2}\sum_{i=0}^{n-1}\left[\varphi(v)+\frac{\varphi(v_i)+\varphi(v_{i+1})}{2}+\frac{v_{i+1}-v_i}{2}\left(v-\frac{v_i+v_{i+1}}{2}\right)\varphi'(v)\right]
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
&\left|R(\varphi,\varphi',I_n,v,w)\right| \le \left\|\varphi''\right\|_\infty \\
&\frac{(v_{i+1}-v_i)}{4}\sum_{i=0}^{n-1}\left(\xi_i-\frac{v_i+v_{i+1}}{2}\right)^2+\frac{(v_i-v_{i+1})^3}{48}
\end{aligned} \tag{16}
$$

.

for all $\xi_i \in [v_i, v_{i+1}]$

remarks 3.7. *for $\xi_i = \frac{v_i+v_{i+1}}{2} (i = 0,\cdots n-1)$, in (2) then the following quadrature rule is:*

$$
\begin{aligned}
S(\varphi,\varphi',I_n,v,w) &= \frac{1}{2}\sum_{i=0}^{n-1}\left[\varphi(\xi_i)+\frac{\Delta\zeta}{2}\left[\varphi(v_i)w(v_i)+\varphi(v_{i+1})w(v_{i+1})\right]\right. \\
&\left.-\int_{v_i}^{v_{i+1}}\varphi(t)\left(t-\frac{v_i+v_{i+1}}{2}\right)w'(t)dt\right]\Delta\zeta
\end{aligned} \tag{17}
$$

and

$$
\begin{aligned}
&\left|R(\varphi,\varphi',I_n,v,w)\right| \le \left\|\varphi''\right\|_\infty\left[\int_{v_i}^{v_{i+1}}w(t)dt\left(\frac{(v_i+v_{i+1})t}{2}-\frac{t^2}{2}\right)\right. \\
&\left.+\int_{v_i}^{v_{i+1}}\left(\frac{v_i+v_{i+1}}{4}\right)^2\right]
\end{aligned} \tag{18}
$$

.

remarks 3.8. *If we put $w(t) \equiv \frac{1}{v_{i+1}-v_i}$ in (12) then we will get*

$$
\begin{aligned}
&|S(\varphi,\varphi',I_n,v,w)| = \\
&\frac{(v_{i+1}-v_i)}{2}\sum_{i=0}^{n-1}\left[\varphi\left(\frac{v_i+v_{i+1}}{2}\right)+\frac{b_1-v_i}{2}\left(\varphi(v_i)w(v_i)+\varphi(v_{i+1})w(v_{i+1})\right)\right. \\
&\left.-(v_{i+1}-v_i)\int_{v_i}^{v_{i+1}}\varphi(t)\left(t-\frac{v_i+v_{i+1}}{2}\right)w'(t)dt\right]
\end{aligned} \tag{19}
$$

and

$$\left|R(\varphi,\varphi',I_n,v,w)\right|$$
$$\leq \left\|\varphi''\right\|_\infty \sum_{i=0}^{n-1}\left[\int_{v_i}^{v_{i+1}}\left(\frac{v_i+v_{i+1}}{4}t-\frac{t^2}{4}\right)dt-\int_{v_i}^{v_{i+1}}du\left(\frac{v_i+v_{i+1}}{4}\right)^2\right]$$

(20)

.

remarks 3.9. *If we examine an approximation for the end point*

$v=v_i$ *in (2), we will get*

$$|S(\varphi,\varphi',I_n,v,w)|=\frac{1}{2}\sum_{i=0}^{n-1}\left[\varphi(v_i)+\frac{v_{i+1}-v_i}{2}\left(\varphi(v_i)w(v_i)+\varphi(v_{i+1})w(v_{i+1})\right)\right.$$
$$\left.-\int_{v_i}^{v_{i+1}}\varphi(t)\left(t-\frac{v_i+v_{i+1}}{2}\right)w'(t)dt+(v_{i+1}-v_i)\int_{v_i}^{v_{i+1}}w(t)\varphi'(v_i)dt\right] \tag{21}$$

and

$$\left|R(\varphi,\varphi',I_n,v,w)\right|\leq$$
$$\sum_{i=0}^{n-1}\left\|\varphi''\right\|_\infty\left[\int_{v_i}^{v_{i+1}}\left(\frac{v_i+v_{i+1}}{4}t-\frac{t^2}{4}\right)w(t)dt-\int_{v_i}^{v_{i+1}}w(u)du\left(\frac{v_iv_{i+1}}{4}\right)\right]$$

.

remarks 3.10. *If we examine an approximation for the end point*

$v=b_1$ *in (2), we will get:*

$$|S(\varphi,\varphi',I_n,v,w)|=$$
$$\frac{1}{2}\sum_{i=0}^{n-1}\left[\varphi(v_{i+1})+\frac{v_{i+1}-v_i}{2}\left(\varphi(v_i)w(v_i)+\varphi(v_{i+1})w(v_{i+1})\right)\right.$$
$$\left.-\int_{v_i}^{v_{i+1}}\varphi(t)\left(t-\frac{v_i+v_{i+1}}{2}\right)w'(t)dt-\int_{v_i}^{v_{i+1}}w(t)(v_{i+1}-v_i)\varphi'(v_{i+1})dt\right]$$

(22)

and

$$\left|R(\varphi,\varphi',I_n,v,w)\right|\leq\left\|\varphi''\right\|_\infty$$
$$\sum_{i=0}^{n-1}\left[\int_{v_i}^{b_1}\left(\frac{v_i+v_{i+1}}{4}t\;-\frac{t^2}{4}\right)w(t)dt-\int_{v_i}^{v_{i+1}}w(u)du\left(\frac{v_iv_{i+1}}{4}\right)\right] \tag{23}$$

.

4.Application for Probability Density Function

Let X be a continuous random variable having the probability density function $\Psi : [a_0, b_1] \to \mathbb{R}_+$ and the cumulative distribution function $\Psi : [a_0, b_1] \to [0,1], i.e,$

$$\Psi(\xi) = \int_{a_0}^{\xi} \psi(t)dt, \quad \xi \in [\alpha, \beta] \subset [a_0, b_1],$$

is the expectation of the random variable X on the interval $[a_0; b_1]$ and weighted expectation would be

$$E_w(X) = \int_{a_0}^{b_1} t\psi(t)w(t)dt,$$

$$t - \xi + \frac{b_1 - a_0}{2} \leq \xi \leq t - \xi - \frac{b_1 - a_0}{2}$$

is the expected of the random variable X on the interval $[a_0, b_1]$. so, we get the following theorem.

Theorem 4. *Let the suppositions of Theorem* 2.1 *be valid if probability density function belongs to* $L_2[a_0, b_1]$ *space, then we get the following inequality*

$$\begin{aligned}
&\Bigg| b_1 w(b_1) - E_w(X) - \int_{a_0}^{b_1} t\Psi(t)w'(t)dt \\
&-\frac{1}{2}\left[\Psi(v) + \frac{b_1 - a_0}{2}\left[\Psi(a_0)w(a_0)\right]\Psi(b_1)w(b_1) - \int_{a_0}^{b_1} \Psi(t)\left(t - \frac{a_0 + b_1}{2}\right)\right. \\
&\left.\times w'(t)dt - \int_{a_0}^{b_1} w(t)\left(v - \frac{a_0 + b_1}{2}\right)\Psi'(v)dt\right]\Bigg| \\
\leq\ & \left\|\Psi''\right\|_\infty \left[\left(\frac{v^2}{4} - \frac{a_0 + b_1}{4}v\right) + \int_{a_0}^{b_1}\left(\frac{a_0 + b_1}{4}t - \frac{t^2}{4}\right)w(t)dt\right]
\end{aligned}$$

(24)

for all $\xi \in [a_0, b_1]$.

Proof. Put $\psi = \Psi$ in (2) we get (24), by using this identities

$$\int_{a_0}^{b_1} \Psi(t)w(t)dt = b_1 w(b_1) - E_w(X) - \int_{a_0}^{b_1} t\Psi(t)w'(t)dt$$

□

where $w : [a_0, b_1] \to [0, \infty)$ is a probability density function

Proof. Put $\psi = \Psi$ in (2) we get (24), by using this identities

$$\int_{a_0}^{b_1} \Psi(t)w(t)dt = b_1 w(b_1) - E_w(X) - \int_{a_0}^{b_1} t\Psi(t)w'(t)dt$$

□

where $w : [a_0, b_1] \to [0, \infty)$ is a probability density function.

Conclusion

Weighted Peano Kernels are used for Ostrowski type inequalities, depending on the second derivatives, are stated in this paper. In the research paper [5] and [11], Weighted Ostrowski type inequalities for the second derivative of the functions are addressed.A generalization and extension of the inequalities is presented in [5] and [11]. we have presented a generalization (2) of the inequality (1) found in [5] for twice differentiable function whose first derivatives are absolutely continuous and 2^{nd} derivative belong to $L_\infty - (a_0, b_1)$ by presenting a parameter $\lambda \in [0, 1]$. This generalization also results in finding a inequality for a specific value of λ as stated in remarks . The inequality thus found has a better bound than the inequalities presented in [5] and [11]. Remarks also shows that the perturbed trapezoid inequality that can be found from (2) is better than the perturbed inequalities presented in [5] and [11] of perturbed trapezoid type. The inequality is then functional for a partition of the interval $[a_0, b_1]$ to find some composite quadrature rules and also functional to special means.

Conflict of Interests

The authors declare that there is no conflict of interests.

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