Available online at http://scik.org
Adv. Inequal. Appl. 2019, 2019:12
https://doi.org/10.28919/aia/4087
ISSN: 2050-7461

ON GENERALIZATION OF PACHPATTE TYPE DISCRETE INEQUALITIES AND APPLICATIONS

S.D. KENDRE[1,*], N.R. KALE[2]

[1]Department of Mathematics, Savitribai Phule Pune University, Pune-411007, India

[2]Department of Mathematics, Sant Rawool Maharaj Mahavidyalaya, Kudal, Sindhudurga-416520, India

Abstract. In this paper, we generalize some existing discrete Pachpatte type inequalities to more general situations. These inequalities in turn can be applied to study various qualitative as well as quantitative properties of solutions of various finite difference and sum-difference equations and its variants.

Keywords: nonlinear; difference inequalities; difference equations; sum-difference equation.

2010 AMS Subject Classification: 26D15, 65Q10, 39A70, 39A10.

1. Introduction

The role played by finite difference inequalities in the development of the theory of finite difference equations is well known. In recent years, the desire to widen the scope of applications of finite difference equations has resulted into the necessity of discovering various new finite difference inequalities in order to study the qualitative as well as quantitative behavior of solutions of such equations.

*Corresponding author
E-mail address: sdkendre@yahoo.com
Received April 06, 2019

During the past few years various mathematicians have discovered many fundamental and useful finite difference inequalities; see [1, 2, 3, 4, 5, 6, 7, 8, 9]. In [5, 6, 7], Pachpatte investigated a number of new finite difference inequalities which are useful in the study of new classes of difference and sum-difference eqautions. In present paper, we generalize some of the inequalities reported in [5, 6, 7] to obtain explicit bound on some of the nonlinear difference equations. Some applications are also given to convey the importance of our results.

In what follows, we let $N_0 = \{0,1,2,\dots\}$ and for any function $u(t),\ t \in N_0$, we define the operator Δ by $\Delta u(t) = u(t+1) - u(t)$. The class $C(\mathbb{R}_+,\mathbb{R}_+)$ consists of all real valued, nonnegative and continuous functions defined on $\mathbb{R}_+$. As usual, throughout the paper the empty sum and product are taken to be 0 and 1 respectively.

Before proceeding to the statement of our main result, we state some important lemmas and finite difference inequalities that will be used in further discussion.

Theorem 1.1. *(Pachpatte's Inequality* [5]*) Let $a(t)$ be nonnegative function defined on N_0 and $c \geq 1$ be a constant. Let $u(t) \geq 1$ be a function defined for $t \in N_0$. If*

$$u(t) \leq c + \sum_{s=0}^{t-1} a(s)u(s)\log u(s),\ t \in N_0,$$

then

$$u(t) \leq c^{\prod_{s=0}^{t-1}[1+a(s)]},\ \textit{for } t \in N_0.$$

Theorem 1.2. *(Hull and Luxemberg* [1]*) Let $u(t)$ and $b(t)$ be nonnegative functions defined for $t \in N_0$ and c be a nonnegative constant. Let $g(u)$ be a nondecreasing continuous function defined on $\mathbb{R}_+$ with $g(u) > 0$ for $u > 0$. If*

$$u(t) \leq c + \sum_{s=0}^{t-1} b(s)g(u(s)),\ t \in N_0,$$

then

$$u(t) \leq G^{-1}\left[G(c) + \sum_{s=0}^{t-1} b(s)\right],\ \textit{for } 0 \leq t \leq t_1,\ t, t_1 \in N_0,$$

where G^{-1} is the inverse of G defined as

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)},\ r > 0,\ r_0 \textit{ is arbitrary}$$

and $t_1 \in N_0$ *be chosen so that*

$$G(c)+\sum_{s=0}^{t-1} b(s) \in Dom(G^{-1}), \text{ for all } t \in N_0 \text{ such that } 0 \leq t \leq t_1.$$

Theorem 1.3. *(S. Sugiyama* [2]*) Let* $u(t)$ *and* $b(t)$ *be nonnegative functions defined on* N_0 *and* $c \geq 0$ *be a constant. If*

$$u(t) \leq c+\sum_{s=0}^{t-1} b(s)u(s),\ t \in N_0,$$

then

$$u(t) \leq c\prod_{s=0}^{t-1}[1+b(s)] \leq c\, exp\left(\sum_{s=0}^{t-1} b(s)\right), \text{ for } t \in N_0.$$

Lemma 1.4. *[Zhao* [10]*] Assume that* $p \geq 1, a \geq 0$*, then*

$$a^{\frac{1}{p}} \leq \frac{1}{p}k^{\frac{1-p}{p}}a+\frac{p-1}{p}k^{\frac{1}{p}}, \text{ for any } k > 0.$$

Lemma 1.5. *[Zhao* [10]*] Assume that* $a \geq 0,\ p \geq q \geq 0$ *and* $p \neq 0$*, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p}k^{\frac{q-p}{p}}a+\frac{p-q}{p}k^{\frac{q}{p}}, \text{ for any } k > 0.$$

2. Main results

In this section, we state and prove some new nonlinear difference inequalities of Pachpatte type and we obtain a bound on an unknown function, which can be used in the analysis of various problems in the theory of nonlinear difference and sum-difference equations.

Theorem 2.1. *Let* $u(t) \geq 1$ *be a function defined for* $t \in N_0$ *and* $c \geq 1$ *be a real constant. Let* $a(t)$ *be a nonnegative function defined for* $t \in N_0$*. Let* $\phi \in C(\mathbb{R}_+,\mathbb{R}_+)$ *be strictly increasing function with* $\phi(\infty)=\infty$ *and* $\psi,\chi \in C(\mathbb{R}_+,\mathbb{R}_+)$ *be nondecreasing such that* $\psi(u) \leq \phi(u), \chi(u) \leq \phi(u)$ *and* $\chi(1)=1$*. If*

$$\phi(u(t)) \leq c+\sum_{s=0}^{t-1} a(s)\psi(u(s))\log\chi(u(s)),\ t \in N_0, \tag{1}$$

then

$$u(t) \leq \phi^{-1}\left[c^{\prod_{s=0}^{t-1}[1+a(s)]}\right],\ t \in N_0. \tag{2}$$

Proof. Define a function $z(t)$ by

$$z(t) = c + \sum_{s=0}^{t-1} a(s)\psi(u(s))\log(\chi(u(s))). \tag{3}$$

Then $z(0) = c$ and inequality (1) can be restated as

$$\phi(u(t)) \leq z(t). \tag{4}$$

Using the fact that $\psi(u), \chi(u) \leq \phi(u)$, we get

$$\Delta z(t) = a(t)\psi(u(t))\log(\chi(u(t))) \leq a(t)\phi(u(t))\log(\phi(u(t))) \leq a(t)z(t)\log z(t). \tag{5}$$

A suitable application of Theorem ([3], p.11) to (5) yields

$$z(t) \leq c\prod_{s=0}^{t-1}[1 + a(s)\log z(s)] \leq c\exp\left(\sum_{s=0}^{t-1} a(s)\log z(s)\right). \tag{6}$$

From (6), we observe that

$$\log z(t) \leq \log c + \sum_{s=0}^{t-1} a(s)\log z(s). \tag{7}$$

A suitable application of Theorem (1.3) to the inequality (7) implies the estimate

$$\log z(t) \leq \left(\prod_{s=0}^{t-1}[1 + a(s)]\right)\log c = \log c^{\prod_{s=0}^{t-1}[1+a(s)]}. \tag{8}$$

From (8), we have

$$z(t) \leq c^{\prod_{s=0}^{t-1}[1+a(s)]}. \tag{9}$$

Using (9) in (4), we get

$$u(t) \leq \phi^{-1}\left[c^{\prod_{s=0}^{t-1}[1+a(s)]}\right], \tag{10}$$

which is the desired inequality in (2). This proves our theorem. □

Corollary 2.2. Let $u(t) \geq 1$ be a function defined for $t \in N_0$ and $c \geq 1$ be a real constant. Let $a(t)$ be a nonnegative function defined for $t \in N_0$. If

$$u^p(t) \leq c + \sum_{s=0}^{t-1} a(s)u(s)\log(u(s)),\ t \in N_0, \tag{11}$$

then

$$u(t) \le \left[c^{\prod_{s=0}^{t-1}[1+a(s)]} \right]^{\frac{1}{p}}, \ t \in N_0. \tag{12}$$

Proof. Define $\phi(u) = u^p, \ p \ge 1$ and $\psi(u) = u = \chi(u)$ on $\mathbb{R}_+$ in Theorem 2.1. The rest of the proof can be easily completed by following the proof of the Theorem 2.1. □

Corollary 2.3. Let $u(t) \ge 1$ be a function defined for $t \in N_0$ and $c \ge 1$ be a real constant. Let $a(t)$ be a nonnegative function defined for $t \in N_0$. If

$$u^p(t) \le c + \sum_{s=0}^{t-1} a(s) u^q(s) \log(u^r(s)), \ t \in N_0, \tag{13}$$

where $p \ge q \ge 0, \ p \ge r \ge 0, \ p \ne 0$, then

$$u(t) \le \left[c^{\prod_{s=0}^{t-1}[1+a(s)]} \right]^{\frac{1}{p}}, \ t \in N_0. \tag{14}$$

Proof. Define $\phi(u) = u^p, \ \psi(u) = u^q$ and $\chi(u) = u^r$ on $\mathbb{R}_+$ with $p \ge q \ge 0, \ p \ge r \ge 0, \ p \ne 0$ in Theorem 2.1. The rest of the proof can be easily completed by following the proof of the Theorem 2.1. □

Remark 2.1. We note that Theorem 1.1 proved by Pachaptte in [5] can be obtained as a special case of Theorem 2.1 by taking $\phi(u) = u = \psi(u) = \chi(u), \ u \in \mathbb{R}_+$.

Theorem 2.4. *Let $u(t), a(t), c, \phi, \psi, \chi$ be as in Theorem 2.1. If*

$$\phi(u(t)) \le c + \sum_{s=t+1}^{\infty} a(s) \psi(u(s)) \log \chi(u(s)), \ t \in N_0, \tag{15}$$

then

$$u(t) \le \phi^{-1} \left[c^{\prod_{s=t+1}^{\infty}[1+a(s)]} \right], \ t \in N_0. \tag{16}$$

Proof. Define a function $v(t)$ by the right side of (15), then

$$v(t) - v(t+1) = a(t+1) \psi(u(t+1)) \log \chi(u(t+1)). \tag{17}$$

Using the facts that $\psi(u), \chi(u) \le \phi(u)$ and $\phi(u(t+1)) \le v(t+1)$, we get

$$v(t) \le [1 + a(t+1) \log(v(t+1))] v(t+1). \tag{18}$$

By setting $t = s$ in (18) and then setting $s = t, t+1, \ldots, m-1$ successively for some arbitrary $m \in N_0$, we obtain

$$v(t) \leq v(m) \prod_{s=t+1}^{m} [1 + a(s) \log v(s)]. \tag{19}$$

Noting that $\lim_{m\to\infty} v(m) = c$ and by letting $m \to \infty$ in (19), we get

$$v(t) \leq c \prod_{s=t+1}^{\infty} [1 + a(s) \log v(s)] \leq c \exp\left(\sum_{s=t+1}^{\infty} a(s) \log v(s)\right) \tag{20}$$

From (20), we observe that

$$\log v(t) \leq \log c + \sum_{s=t+1}^{\infty} a(s) \log v(s). \tag{21}$$

Define a function $z(t)$ by the right side of (21). Then $z(0) = \log c,\ \log v(t) \leq z(t),\ t \in N_0$ and

$$z(t) - z(t+1) = a(t+1) \log v(t+1) \leq a(t+1) z(t+1)$$

i.e.

$$z(t) \leq [1 + a(t+1)] z(t+1)$$

which by following the above arguments yields

$$z(t) \leq \left(\prod_{s=t+1}^{\infty} [1 + a(s)]\right) \log c = \log c^{\prod_{s=t+1}^{\infty} [1+a(s)]}. \tag{22}$$

Using (22) in (21), we have

$$\log v(t) \leq \log c^{\prod_{s=t+1}^{\infty} [1+a(s)]}. \tag{23}$$

The inequality (23) implies the estimate

$$v(t) \leq c^{\prod_{s=t+1}^{\infty} [1+a(s)]}. \tag{24}$$

Using (24) in $\phi(u(t)) \leq v(t)$ we get the required inequality in (16). □

Remark 2.2. We note that Theorem ([3], p.244) proved by Pachaptte in [6] can be obtained as a special case of Theorem 2.4 if we take $\phi(u) = u = \psi(u) = \chi(u),\ u \in \mathbb{R}_+$.

Theorem 2.5. *Let $u(t), a(t), c, \phi, \psi, \chi$ be as in Theorem 2.1 and $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing. If*

$$\phi(u(t)) \leq c + \sum_{s=0}^{t-1} a(s)\psi(u(s))g(\log \chi(u(s))),\ t \in N_0, \tag{25}$$

then

$$u(t) \leq \phi^{-1}\left(\exp\left[G^{-1}\left[G(\log c) + \sum_{s=0}^{t-1} a(s)\right]\right]\right) \tag{26}$$

where

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)},\ r > 0 \tag{27}$$

$r_0 > 0$ is arbitrary, G^{-1} is the inverse of G, and $t_1 \in N_0$ is chosen so that

$$G(\log c) + \sum_{s=0}^{t-1} a(s) \in Dom(G^{-1})$$

for all $t \in N_0$ such that $0 \leq t \leq t_1$.

Proof. Defining a function $v(t)$ by the right side of (25) and following the same arguments as in the proof of Theorem (2.1) upto the inequality (7) we get

$$\log v(t) \leq \log c + \sum_{s=0}^{t-1} a(s)g(\log v(s)). \tag{28}$$

A suitable application of Theorem (1.2) to the inequality (28) yields

$$\log v(t) \leq G^{-1}\left[G(\log c) + \sum_{s=0}^{t-1} a(s)\right]. \tag{29}$$

From (29), we observe that

$$v(t) \leq \exp\left[G^{-1}\left[G(\log c) + \sum_{s=0}^{t-1} a(s)\right]\right]. \tag{30}$$

Now using (30) in $\phi(u(t)) \leq v(t)$ we get the desired inequality in (26) for $0 \leq t \leq t_1$. □

Theorem 2.6. *Let $a(t),\ b(t)$ be nonnegative functions defined for $t \in N_0$ and $u(t), c, \phi, \psi, \chi$ be as in Theorem 2.1. If*

$$\phi(u(t)) \leq c + \sum_{s=0}^{t-1} a(s)\psi(u(s))\left(\log \chi(u(s)) + \sum_{\sigma=0}^{s-1} b(\sigma)\log \chi(u(\sigma))\right),\ t \in N_0, \tag{31}$$

then

$$u(t) \leq \phi^{-1}\left(c^{1+\sum_{s=0}^{t-1} a(s) \prod_{\sigma=0}^{s-1}[1+a(\sigma)+b(\sigma)]}\right), \text{ for } t \in N_0. \tag{32}$$

Proof. By seeting $v(t)$ equal to the right side of (31) and following the same arguments as in the proof of Theorem (2.1) upto the inequality (7) we obtain

$$\log v(t) \leq \log c + \sum_{s=0}^{t-1} a(s)\left[\log v(s) + \sum_{\sigma=0}^{s-1} b(\sigma)\log v(\sigma)\right]. \tag{33}$$

Now by applying Theorem ([3], p.26) to the inequality (33) and following the last arguments as in the proof of Theorem 2.1, we obtain the required inequality in (32). □

Theorem 2.7. *Let $u(t), a(t), c, g(u), \phi, \psi, \chi$ be as in Theorem 2.5. If*

$$\phi(u(t)) \leq c + \sum_{s=0}^{t-1} a(s)\psi(u(s))\left(\log\chi(u(s)) + \sum_{\sigma=0}^{s-1} a(\sigma)g(\log\chi(u(\sigma)))\right),\ t \in N_0,$$

then for $0 \leq t \leq t_1,\ t, t_1 \in N_0$,

$$u(t) \leq \phi^{-1}\left[c\exp\left(\sum_{s=0}^{t-1} a(s)\Omega^{-1}\left[\Omega(\log c) + \sum_{\sigma=0}^{s-1} a(\sigma)\right]\right)\right],$$

where Ω^{-1} is the inverse of Ω defined as,

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{s+g(s)},\ r > 0,\ r_0 > 0 \text{ is arbitrary},$$

and $t_1 \in N_0$ be chosen so that

$$\Omega(\log c) + \sum_{s=0}^{t-1} a(s) \in Dom(\Omega^{-1}), \text{ for } 0 \leq t \leq t_1,\ t, t_1 \in N_0.$$

Proof. The proof of this Theorem follows by the same arguments as in the proof of Theorem 2.5 and making use of the Theorem ([3], p.134). □

Theorem 2.8. *Let $u(t), a(t), b(t), c, \phi, \psi, \chi$ be as in Theorem 2.6. If*

$$\phi(u(t)) \leq c + \sum_{s=0}^{t-1} a(s)\psi(u(s))\left(\sum_{\sigma=0}^{s-1} b(\sigma)\log\chi(u(\sigma))\right), \text{ for } t \in N_0, \tag{34}$$

then

$$u(t) \leq \phi^{-1}\left[c^{\prod_{s=0}^{t-1}\left[1+a(s)\left(\sum_{\sigma=0}^{s-1} b(\sigma)\right)\right]}\right], \text{ for } t \in N_0. \tag{35}$$

Proof. Define a function $z(t)$ by the right side of (34). Then $z(0) = c,\ \phi(u(t)) \le z(t)$ and

$$\Delta z(t) = a(t)\psi(u(t))\left(\sum_{\sigma=0}^{t-1} b(\sigma)\log\chi(u(\sigma))\right). \tag{36}$$

Using the fact that $\psi(u), \chi(u) \le \phi(u)$ we get

$$\Delta z(t) \le a(t)z(t)\left(\sum_{\sigma=0}^{t-1} b(\sigma)\log z(\sigma)\right). \tag{37}$$

A suitable application of Theorem ([3], p.11) implies the estimate

$$z(t) \le c\prod_{s=0}^{t-1}\left[1 + a(s)\left(\sum_{\sigma=0}^{s-1} b(\sigma)\log z(\sigma)\right)\right] \le c\exp\left(\sum_{s=0}^{t-1} a(s)\left(\sum_{\sigma=0}^{s-1} b(\sigma)\log z(\sigma)\right)\right). \tag{38}$$

From (38), we observe that

$$\log z(t) \le \log c + \sum_{s=0}^{t-1} a(s)\left(\sum_{\sigma=0}^{s-1} b(\sigma)\log z(\sigma)\right). \tag{39}$$

Define a function $v(t)$ by the right side of (39).Then $v(0) = \log c,\ \log z(t) \le v(t)$ and $v(t)$ is nondecreasing in $t \in N_0$. Now

$$\Delta v(t) = a(t)\left(\sum_{\sigma=0}^{t-1} b(\sigma)\log z(\sigma)\right) \le a(t)\left(\sum_{\sigma=0}^{t-1} b(\sigma)v(\sigma)\right) \le a(t)\left(\sum_{\sigma=0}^{t-1} b(\sigma)\right)v(t). \tag{40}$$

On application of Theorem ([3], p.11) to the inequality (40) yields

$$\begin{aligned} v(t) &\le (\log c)\prod_{s=0}^{t-1}\left[1 + a(s)\left(\sum_{\sigma=0}^{s-1} b(\sigma)\right)\right] \\ &= \log c^{\prod_{s=0}^{t-1}\left[1+a(s)\left(\sum_{\sigma=0}^{s-1} b(\sigma)\right)\right]}. \end{aligned} \tag{41}$$

From (41), we observe that

$$z(t) \le c^{\prod_{s=0}^{t-1}\left[1+a(s)\left(\sum_{\sigma=0}^{s-1} b(\sigma)\right)\right]}. \tag{42}$$

Using (42) in $\phi(u(t)) \le z(t)$ we get the required inequality in (35). □

Theorem 2.9. *Let $u(t), a(t), b(t), c, \phi, \psi, \chi$ be as in Theorem 2.6 and $g(u)$ be as in Theorem 2.5. If*

$$\phi(u(t)) \le c + \sum_{s=0}^{t-1} a(s)\psi(u(s))\left(\sum_{\sigma=0}^{s-1} b(\sigma)g(\log\chi(u(\sigma)))\right), \textit{ for } t \in N_0, \tag{43}$$

then

$$u(t) \le \phi^{-1}\left[exp\left(G^{-1}\left[G(\log c)+\sum_{s=0}^{t-1}a(s)\left(\sum_{\sigma=0}^{s-1}b(\sigma)\right)\right]\right)\right], \textit{ for } t \in N_0. \tag{44}$$

where G, G^{-1} *are defined as in Theorem 2.5 and* $t_1 \in N_0$ *be chosen so that*

$$G(\log c)+\sum_{s=0}^{t-1}a(s)\left(\sum_{\sigma=0}^{s-1}b(\sigma)\right) \in Dom(G^{-1}) \tag{45}$$

for all $t \in N_0$ *such that* $0 \le t \le t_1$.

Proof. Define a function $z(t)$ by the right side of (43). Then $z(0) = c,\ \phi(u(t)) \le z(t)$ and

$$\Delta z(t) = a(t)\psi(u(t))\left(\sum_{\sigma=0}^{t-1}b(\sigma)g(\log\chi(u(\sigma)))\right). \tag{46}$$

Using the fact that $\psi(u), \chi(u) \le \phi(u)$ we get

$$\Delta z(t) \le a(t)z(t)\left(\sum_{\sigma=0}^{t-1}b(\sigma)g(\log z(\sigma))\right). \tag{47}$$

A suitable application of Theorem ([3], p.11) implies the estimate

$$z(t) \le c\prod_{s=0}^{t-1}\left[1+a(s)\left(\sum_{\sigma=0}^{s-1}b(\sigma)g(\log z(\sigma))\right)\right] \le ce^{\left(\sum_{s=0}^{t-1}a(s)\left(\sum_{\sigma=0}^{s-1}b(\sigma)g(\log z(\sigma))\right)\right)}. \tag{48}$$

From (48), we observe that

$$\log z(t) \le \log c+\sum_{s=0}^{t-1}a(s)\left(\sum_{\sigma=0}^{s-1}b(\sigma)g(\log z(\sigma))\right). \tag{49}$$

Define a function $v(t)$ by the right side of (49). Then $v(0) = \log c,\ \log z(t) \le v(t)$ and $v(t)$ is nondecreasing in $t \in N_0$. Now

$$\begin{aligned}
\Delta v(t) &= a(t)\left[\sum_{\sigma=0}^{t-1}b(\sigma)g(\log z(\sigma))\right] \\
&\le a(t)\left[\sum_{\sigma=0}^{t-1}b(\sigma)g(v(\sigma))\right] \\
&\le a(t)\left[\sum_{\sigma=0}^{t-1}b(\sigma)\right]g(v(t)).
\end{aligned} \tag{50}$$

From (45) and (50), we have

$$G(v(t+1)) - G(v(t)) = \int_{v(t)}^{v(t+1)} \frac{ds}{g(s)} \le \frac{\Delta v(t))}{g(v(t))} \le a(t) \left(\sum_{\sigma=0}^{t-1} b(\sigma) \right). \tag{51}$$

Now by setting $t = s$ in (51) and summing over s from 0 to $t-1$ we obtain the estimate

$$G(v(t)) \le G(\log c) + \sum_{s=0}^{t-1} a(s) \left(\sum_{\sigma=0}^{s-1} b(\sigma) \right). \tag{52}$$

Using (52),(49) and $\phi(u(t)) \le z(t)$ we get the required inequality in (44). □

Theorem 2.10. *Let $u(t), a(t), b(t), c, \phi, \psi, \chi$ be as in Theorem 2.6. If*

$$\phi(u(t)) \le c + \sum_{s=0}^{t-1} a(s)\psi(u(s)) \log\chi(u(s)) \left(\log\chi(u(s)) + \sum_{\sigma=0}^{s-1} b(\sigma) \log\chi(u(\sigma)) \right), \tag{53}$$

and $1 - (\log c) \sum_{s=0}^{t-1} a(s) \prod_{\tau=0}^{s} [1 + b(\tau)] > 0,$ *for $t \in N_0$ then*

$$u(t) \le \phi^{-1} \left(c^{\prod_{s=0}^{t-1} [1+a(s)Q(s)]} \right), \text{ for } t \in N_0. \tag{54}$$

where

$$Q(t) = \frac{(\log c) \prod_{s=0}^{t-1} [1 + b(s)]}{1 - (\log c) \sum_{s=0}^{t-1} a(s) \prod_{\tau=0}^{s} [1 + b(\tau)]}. \tag{55}$$

Proof. Define a function $z(t)$ by the right side of (53). Then $z(0) = c,\ \phi(u(t)) \le z(t)$ and

$$\Delta z(t) = a(t)\psi(u(t)) \log\chi(u(t)) \left(\log(\chi(u(t)) + \sum_{\sigma=0}^{t-1} b(\sigma) \log\chi(u(\sigma)) \right). \tag{56}$$

Using the fact that $\psi(u), \chi(u) \le \phi(u)$, we get

$$\Delta z(t) \le a(t)z(t) \log z(t) \left(\log z(t) + \sum_{\sigma=0}^{t-1} b(\sigma) \log z(\sigma) \right). \tag{57}$$

A suitable application of Theorem ([3], p.11) implies the estimate

$$\begin{aligned} z(t) &\le c \prod_{s=0}^{t-1} \left[1 + a(s) \log z(s) \left(\log z(s) + \sum_{\sigma=0}^{s-1} b(\sigma) \log z(\sigma) \right) \right] \\ &\le c \exp\left(\sum_{s=0}^{t-1} a(s) \log z(s) \left(\log z(s) + \sum_{\sigma=0}^{s-1} b(\sigma) \log z(\sigma) \right) \right). \end{aligned} \tag{58}$$

From (58), we observe that

$$\log z(t) \leq \log c + \sum_{s=0}^{t-1} a(s) \log z(s) \left(\log z(s) + \sum_{\sigma=0}^{s-1} b(\sigma) \log z(\sigma) \right) \tag{59}$$

Define a function $v(t)$ by the right side of (59). Then $v(0) = \log c,\ \log z(t) \leq v(t)$ and $v(t)$ is nondecreasing in $t \in N_0$. Now

$$\Delta v(t) = a(t) \log z(t) \left(\log z(t) + \sum_{\sigma=0}^{t-1} b(\sigma) \log z(\sigma) \right) \leq a(t) v(t) \left(v(t) + \sum_{\sigma=0}^{t-1} b(\sigma) v(\sigma) \right). \tag{60}$$

Put

$$r(t) = v(t) + \sum_{\sigma=0}^{t-1} b(\sigma) v(\sigma). \tag{61}$$

Then $r(0) = v(0) = \log c,\ v(t) \leq r(t)$ and

$$\Delta r(t) = \Delta v(t) + b(t) v(t) \leq a(t) v(t) r(t) + b(t) r(t) \leq a(t) r^2(t) + b(t) r(t). \tag{62}$$

A suitable application of Theorem ([3], p.108) implies the estimate

$$r(t) \leq \frac{(\log c) \prod_{s=0}^{t-1} [1 + b(s)]}{1 - (\log c) \sum_{s=0}^{t-1} a(s) \prod_{\tau=0}^{s} [1 + b(\tau)]} = Q(t), \tag{63}$$

Provided $1 - (\log c) \sum_{s=0}^{t-1} a(s) \prod_{\tau=0}^{s} [1 + b(\tau)] > 0,\ for\ t \in N_0$. Thus from (64) we have

$$\Delta v(t) \leq a(t) Q(t) v(t). \tag{64}$$

The inequality (64) implies the estimate

$$v(t) \leq (\log c) \prod_{s=0}^{t-1} [1 + a(s) Q(s)]. \tag{65}$$

Using this in $\log z(t) \leq v(t)$, we obtain

$$z(t) \leq c^{\prod_{s=0}^{t-1} [1 + a(s) Q(s)]}. \tag{66}$$

From (66) and $\phi(u(t)) \leq z(t)$ we get the desired inequality in (54). □

3. Applications

Example 3.1. Consider the nonlinear difference equation

$$x^2(t) = x_0 + \sum_{s=0}^{t-1} a(s)x(s)\log|x(s)|, \tag{67}$$

where $x(t), p(t)$ are real valued functions defined on N_0 and x_0 is a constant. Let $x(t),\ t \in N_0$ be a solution of (67). From (67), we observe that

$$\left|x^2(t)\right| \le |x_0| + \sum_{s=0}^{t-1} |a(s)|\,|x(s)|\,|\log|x(s)|| \le |x_0| + \sum_{s=0}^{t-1} |a(s)|\,(1+|x(s)|)\,|\log(1+|x(s)|)|\,. \tag{68}$$

Now using Lemma 1.4, we get

$$\begin{aligned}
(1+|x(t)|)^2 &= 1 + 2\,|x(t)| + \left|x^2(t)\right| \\
&\le 1 + 2\left(|x_0| + \sum_{s=0}^{t-1} |a(s)|\,(1+|x(s)|)\,|\log(1+|x(s)|)|\right)^{\frac{1}{2}} \\
&\quad + \left(|x_0| + \sum_{s=0}^{t-1} |a(s)|\,(1+|x(s)|)\,|\log(1+|x(s)|)|\right) \\
&\le 1 + 2\left[\frac{1}{2\sqrt{k}}\left(|x_0| + \sum_{s=0}^{t-1} |a(s)|\,(1+|x(s)|)\,|\log(1+|x(s)|)|\right) + \frac{\sqrt{k}}{2}\right] \\
&\quad + \left(|x_0| + \sum_{s=0}^{t-1} |a(s)|\,(1+|x(s)|)\,|\log(1+|x(s)|)|\right) \\
&= \left(1 + |x_0| + \frac{|x_0|}{\sqrt{k}} + \sqrt{k}\right) + \sum_{s=0}^{t-1}\left(\frac{1+\sqrt{k}}{\sqrt{k}}\,|a(s)|\right)(1+|x(s)|)\,|\log(1+|x(s)|)|\,,
\end{aligned} \tag{69}$$

for any $k > 0$.

Let $u(t) = 1 + |x(t)|$, then $u_0 = 1 + |x_0|$ and (69) takes the form

$$u^2(t) \le \left(u_0 + \frac{u_0 - 1}{\sqrt{k}} + \sqrt{k}\right) + \sum_{s=0}^{t-1}\left(\frac{1+\sqrt{k}}{\sqrt{k}}\,|a(s)|\right)u(s)\,|\log u(s)|\,,\ \text{for any } k > 0. \tag{70}$$

Application of Theorem 2.1 to (70) yields

$$u(t) \le \left[\left(u_0 + \frac{u_0 - 1}{\sqrt{k}} + \sqrt{k}\right)^{\prod_{s=0}^{t-1}\left[1 + \frac{1+\sqrt{k}}{\sqrt{k}}|a(s)|\right]}\right]^{\frac{1}{2}} \tag{71}$$

i.e.

$$|x(t)| \leq \left[\left(u_0 + \frac{u_0 - 1}{\sqrt{k}} + \sqrt{k}\right)^{\prod_{s=0}^{t-1}\left[1+\frac{1+\sqrt{k}}{\sqrt{k}}|a(s)|\right]}\right]^{\frac{1}{2}} - 1,\ k > 0. \tag{72}$$

The above inequality obtains bound on the solution of Equation (67) in terms of the known functions.

Example 3.2. Consider the nonlinear difference equation

$$x^3(t) = x_0 + \sum_{s=0}^{t-1} a(s)x^2(s)\left(\sum_{\sigma=0}^{s-1} b(\sigma)\log|x(\sigma)|\right), \tag{73}$$

where $x(t), a(t), b(t)$ are real valued functions defined on N_0 and x_0 is a constant. Let $x(t),\ t \in N_0$ be a solution of (73).

From (73), we observe that

$$\begin{aligned} \left|x^3(t)\right| &\leq |x_0| + \sum_{s=0}^{t-1} |a(s)|\left|x^2(s)\right|\left(\sum_{\sigma=0}^{s-1} |b(\sigma)|\left|\log|x(\sigma)|\right|\right) \\ &\leq |x_0| + \sum_{s=0}^{t-1} |a(s)|(1+|x(s)|)^2\left(\sum_{\sigma=0}^{s-1} |b(\sigma)|\left|\log(1+|x(\sigma)|)\right|\right). \end{aligned} \tag{74}$$

Now using Lemma 1.4, Lemma 1.5, we get

$$\begin{aligned} (1+|x(t)|)^3 &= 1 + 3|x(t)| + 3|x(t)|^2 + |x(t)|^3 \\ &\leq 1 + 3\left(|x_0| + \sum_{s=0}^{t-1} |a(s)|(1+|x(s)|)^2\left(\sum_{\sigma=0}^{s-1} |b(\sigma)|\left|\log(1+|x(\sigma)|)\right|\right)\right)^{\frac{1}{3}} \\ &\quad + \left(|x_0| + \sum_{s=0}^{t-1} |a(s)|(1+|x(s)|)^2\left(\sum_{\sigma=0}^{s-1} |b(\sigma)|\left|\log(1+|x(\sigma)|)\right|\right)\right)^{\frac{2}{3}} \\ &\quad + \left(|x_0| + \sum_{s=0}^{t-1} |a(s)|(1+|x(s)|)^2\left(\sum_{\sigma=0}^{s-1} |b(\sigma)|\left|\log(1+|x(\sigma)|)\right|\right)\right) \\ &\leq 1 + 3\left[\frac{1}{3(\sqrt[3]{k})^2}\left(|x_0| + \sum_{s=0}^{t-1} |a(s)|(1+|x(s)|)^2\left(\sum_{\sigma=0}^{s-1} |b(\sigma)|\left|\log(1+|x(\sigma)|)\right|\right)\right) + \frac{2\sqrt[3]{k}}{3}\right] \\ &\quad + 3\left[\frac{2}{3\sqrt[3]{k}}\left(|x_0| + \sum_{s=0}^{t-1} |a(s)|(1+|x(s)|)^2\left(\sum_{\sigma=0}^{s-1} |b(\sigma)|\left|\log(1+|x(\sigma)|)\right|\right)\right) + \frac{(\sqrt[3]{k})^2}{3}\right] \\ &\quad + \left[|x_0| + \sum_{s=0}^{t-1} |a(s)|(1+|x(s)|)^2\left(\sum_{\sigma=0}^{s-1} |b(\sigma)|\left|\log(1+|x(\sigma)|)\right|\right)\right] \end{aligned}$$

$$= \left[1+|x_0|+\frac{|x_0|}{(\sqrt[3]{k})^2}+2\sqrt[3]{k}+\frac{2|x_0|}{\sqrt[3]{k}}+(\sqrt[3]{k})^2\right]$$
$$+\sum_{s=0}^{t-1}\left[\left(1+\frac{1}{(\sqrt[3]{k})^2}+\frac{2}{\sqrt[3]{k}}\right)|a(s)|\right]|(1+|x(s)|)^2\left(\sum_{\sigma=0}^{s-1}|b(\sigma)|\,|\log(1+|x(\sigma)|)|\right),$$

for any $k>0$.

Let $u(t)=1+|x(t)|$, then $u_0=1+|x_0|$ and above inequality takes the form

$$u^3(t)\le\left[1+|x_0|+\frac{|x_0|}{(\sqrt[3]{k})^2}+2\sqrt[3]{k}+\frac{2|x_0|}{\sqrt[3]{k}}+(\sqrt[3]{k})^2\right]$$
$$+\sum_{s=0}^{t-1}\left[\left(1+\frac{1}{(\sqrt[3]{k})^2}+\frac{2}{\sqrt[3]{k}}\right)|a(s)|\right]u^2(s)\left(\sum_{\sigma=0}^{s-1}|b(\sigma)|\,|\log u(\sigma)|\right),\text{ for any } k>0. \tag{75}$$

Application of Theorem 2.8 to the inequality (75) yields

$$|x(t)|\le\left[\left(1+|x_0|+\frac{|x_0|}{(\sqrt[3]{k})^2}+2\sqrt[3]{k}+\frac{2|x_0|}{\sqrt[3]{k}}+(\sqrt[3]{k})^2\right)^{\prod_{s=0}^{t-1}\left[1+\left[\left(1+\frac{1}{(\sqrt[3]{k})^2}+\frac{2}{\sqrt[3]{k}}\right)|a(s)|\right]\left(\sum_{\sigma=0}^{s-1}b(\sigma)\right)\right]}\right]^{\frac{1}{3}}-1,\ k>0.$$

The above inequality obtains bound on the solution of equation (73) in terms of the known functions.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Hull, T.E. and Luxemburg, W.A.J., Numerical methods and existence theorems for ordinary differential equations, Numer. Math., 2 (1960), 30-41.

[2] Sugiyama, S., On the stability of difference equations, Bull. Sci. Engr. Research Lab., Waseda Univ. , 45 (1969), 140-144.

[3] B. G. Pachpatte, Inequalities for Finite Difference Equations, Marcel Dekker, Inc., New York, (2002).

[4] B. G. Pachpatte, On some discrete inequalities of the Bellman-Bihari type, Indian J. Pure Appl. Math., 6 (1975), 1479-1487.

[5] B. G. Pachpatte, Inequalities related to certain integral inequality arising in the theory of differential equations, Studia. Univ. Babes-Bolyai, Math., 39 (1994), 33-50.

[6] B. G. Pachpatte, On certain new finite difference inequalities, Indian J. Pure Appl. Math., 24 (1993), 373-384.

[7] B. G. Pachpatte, On Bihari like integral and discrete inequalities, Soochow J. Math., 17 (1991), 213-232.

[8] B. G. Pachpatte, Finite difference inequalities and their applications, Proc. Nat. Acad. Sci. India, 43(A) (1973), 348-356.

[9] Yang Enhao, On Some Nonlinear Integral and Discrete Inequalities Related to Ou-Iang's Inequality, Acta Math. Sinica, 14(3) (1998), 353-360.

[10] Zhao C. J., Some integral inequalities for differential equations, J. Binzhou Teachers College, 17(4) (2001), 41-46.