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NEW IYENGAR-TYPE INEQUALITIES FOR FRACTIONAL INTEGRALS

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Abstract. In this paper, we obtain a new identity for fractional integrals which is a generalization of Alomari's result [10]. As an application of the identity, some Iyengar type inequalities for Riemann-Liouville fractional integral are established.

Keywords: Iyengar type inequality, s-convex function, quasi-convex function, Riemann-Liouville integral.

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1. Introduction

In 1938, Iyengar [1] proved the following theorem by using a geometric approach.

Theorem 1.1 Let $f: I \to \mathbb{R}$, where $I \subseteq R$ is an interval, be a mapping differentiable in the interior I^0 of I, and let $a, b \in I^0$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a,b]$, then the following inequality holds:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \leq \frac{M(b-a)}{4} - \frac{(f(b)-f(a))^2}{4M(b-a)}. \tag{1}$$

Recently, some generalizations and new results related to the inequality (1.1) are established, see ([2-7]).

We recall some basic definitions, which is well known in the literature. Let real function f be

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defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y) \tag{2}$$

holds for all $x,y \in I$, and $\lambda \in [0,1]$.

Convex functions play an important role in many branches of mathematics and other sciences, such as engineering, economics and optimization theory. Several extensions, generalizations and refinements have been presented by researchers. Now, we introduce the concept of the s-convex function and the quasi-convex function.

Definition 1.1([16]) A function $f:[0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\lambda x+(1-\lambda)y) \leq \lambda^s f(x)+(1-\lambda)^s f(y) \tag{3}$$

holds for all $x,y \in [0,\infty), \lambda \in [0,1]$ and for some fixed $s \in [0,1]$.

Obviously, if putting s=1 in Definition 1, then f is just the ordinary convex function on $[0,\infty)$.

Definition 1.2([21]) A function $f:[a,b] \to \mathbb{R}$ is said to be quasi-convex on [a,b] if

$$f(\lambda x+(1-\lambda)y) \leq \max(f(x),f(y)) \tag{4}$$

holds for all $x,y \in [a,b]$ and $\lambda \in [0,1]$.

Clearly, any convex function is a quasi-convex function. Furthermore, there exists quasi-convex functions which are not convex (see [20]).

In [18] and [5], the authors established the following inequalities of Iyengar type:

Theorem 1.2 Let $f:I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0, $a,b \in I^0$ with $a<b$. If $|f'|$ is convex on [a,b], then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_0^1 f(x)dx\right| \leq \frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{5}$$

Theorem 1.3 Let $f:I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^0, such that $f'' \in L_1[a,b]$, $a,b \in I^0$ with $a<b$. If $|f''|$ is quasi-convex on [a,b] for $q>1$, then the following inequality holds:

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_0^1 f(x)dx\right|\\ \le&\frac{(b-a)^2}{2}\left(\frac{q-1}{2q-p-1}\right)^{\frac{q-1}{q}}\left(\beta(p+1,q+1)\right)^{\frac{1}{q}}\left(\max\left(|f''(a)|^q,|f''(b)|^q\right)\right)^{\frac{1}{q}},\end{aligned}\tag{6}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\beta(,)$ is Euler Beta Function:

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, x,y>0.$$

The following lemmas play a crucial role in the proof of the above theorems.

Lemma 1.1 ([18]) Let $f: I^0\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^0, $a,b\in I^0$ with $a<b$. If $f'\in L_1[a,b]$, then the following equality holds:

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)\,dt.\tag{7}$$

Lemma 1.2([10]) Let $f: I^0\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I^0, $a,b\in I^0$ with $a<b$ and $f''\in L_1[a,b]$. Then the following equality holds:

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)\,dt.\tag{8}$$

In the sequel, we will give some necessary definitions of the fractional calculus which are used constantly in this paper. For more details, one can consult [20].

Definition 1.3 Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\geq 0$ are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt, x>a\tag{9}$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt, x<b\tag{10}$$

respectively. Where, $\Gamma(\alpha)$ is the Euler Gamma function and $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

Very recently, M.Z.Sarikaya et al [11] extend the identity (7) for fractional integrals. Then by making use of the established identity, they proved some new Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral. For other recent results concerning s-convex functions and quasi-convex functions , see [8-10,12-17,19].

Motivated by the results , in the present note, we extend the identity (8) for fractional integrals and then we develop some Iyengar-type inequalities for s-convex functions in the second sense and quasi-convex functions via Riemann-Liouville fractional integral.

2. Iyengar-type inequalities via fractional integrals

In order to prove our main results we need the following identity.

Lemma 2.1 Let $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^0, $a, b \in I^0$ with $a < b$ .If $f'' \in L_1[a,b]$, then for all $x \in [a,b]$ and $\alpha \geq 1$we have

$$\begin{aligned}
&\frac{f(a)+f(b)}{(b-a)^2}+\frac{\Gamma(\alpha+1)}{(b+a)^{\alpha+1}}\left(J_{a+}^{\alpha-1}f(b)+J_{b-}^{\alpha-1}f(a)\right)\\
&-\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a))\\
=&\int_0^1\left(u^\alpha+(1-u)^\alpha-u^{\alpha+1}-(1-u)^{\alpha+1}\right)f''(ua+(1-u)b)du.
\end{aligned}$$

Proof. From lemma 1.2, the case $\alpha = 1$ had been proved in [10]. Here we consider only the case $\alpha > 1$. By the integration by parts, we have

$$\begin{aligned}
I :=& \int_0^1\left(u^\alpha-u^{\alpha+1}+(1-u)^\alpha-(1-u)^{\alpha+1}\right)\cdot f''(ua+(1-u)b)\,du\\
=&\frac{1}{b-a}\left(u^{\alpha+1}+(1-u)^{\alpha+1}-u^\alpha-(1-u^\alpha)\right)\cdot f'(ua+(1-u)b)\Big|_0^1\\
+&\frac{1}{b-a}\int_0^1\left(\alpha u^{\alpha-1}-\alpha(1-u)^{\alpha-1}-(\alpha+1)u^\alpha+(\alpha+1)(1-u)^\alpha\right)\\
\cdot& f'(ua+(1-u)b)\,du\\
=&\frac{1}{(b-a)^2}\left((\alpha+1)u^\alpha+\alpha(1-u)^{\alpha-1}-\alpha u^{\alpha-1}-(\alpha+1)(1-u)^\alpha\right)\\
\cdot& f(ua+(1-u)b)\Big|_0^1+\frac{1}{(b-a)^2}\int_0^1\left(\alpha(\alpha-1)u^{\alpha-2}+\alpha(\alpha-1)(1-u)^{\alpha-2}\right.\\
-&\left.(\alpha+1)\alpha u^{\alpha-1}-(\alpha+1)\alpha(1-u)^{\alpha-1}\right)\cdot f(ua+(1-u)b)\,du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f(a)+f(b)}{(b-a)^2} + \frac{\alpha(\alpha-1)}{(b-a)^2}\int_0^1 \left(u^{\alpha-2}+(1-u)^{\alpha-2}\right) f\left(ua+(1-u)b\right) du \\
&- \frac{(\alpha+1)\alpha}{(b-a)^2}\int_0^1 (u^{\alpha-1}+(1-u)^{\alpha-1})\cdot f\left(ua+(1-u)b\right) du.
\end{aligned} \tag{11}
$$

Notice that

$$
\begin{aligned}
&\int_0^1 \left(u^{\alpha-2}+(1-u)^{\alpha-2}\right) f(ua+(1-u)b)du \\
&= \frac{1}{(b-a)^{\alpha-1}}\left(\int_a^b (b-t)^{\alpha-2} f(t)dt + \int_a^b (t-a)^{\alpha-2} f(t)dt\right)
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
&\int_0^1 \left(u^{\alpha-1}+(1-u)^{\alpha-1}\right) f(ua+(1-u)b)du \\
&= \frac{1}{(b-a)^{\alpha}}\left(\int_a^b (b-t)^{\alpha-1} f(t)dt + \int_a^b (t-a)^{\alpha-1} f(t)dt\right).
\end{aligned} \tag{13}
$$

Using Definition 1.3 and from (11), (12) and (13), we have

$$
\begin{aligned}
I = &\frac{f(a)+f(b)}{(b-a)^2} + \frac{\Gamma(\alpha+1)}{(b+a)^{\alpha+1}}\left(J_{a+}^{\alpha-1} f(b) + J_{b-}^{\alpha-1} f(a)\right) \\
&- \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}\left(J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right).
\end{aligned}
$$

This completes the proof.

By this lemma, we can obtain the following fractional integral inequalities.

Theorem 2.1 Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$, be a twice differentiable mapping on (a,b) with $a<b$ such that $f''\in L_1[a,b]$. If $|f''|^q$ is s-convex in the second sense on [a,b] for some fixed $s\in(0,1]$, $p,q>1$, then the following inequality for fractional integrals holds:

$$
\begin{aligned}
&\Big|\frac{f(a)+f(b)}{(b-a)^2} + \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left(J_{a+}^{\alpha-1} f(b) + J_{b-}^{\alpha-1} f(a)\right) \\
&- \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}\left(J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right)\Big| \\
\leq &2\left(\frac{q-1}{\alpha q-\alpha p+q-1}\right)^{\frac{q-1}{q}} \left(\beta(\alpha p+s+1,q+1)+\beta(\alpha p+1,s+q+1)\right)^{\frac{1}{q}} \\
&\cdot\left(\max\left(|f''(a)|^q,|f''(b)|^q\right)\right)^{\frac{1}{q}}.
\end{aligned} \tag{14}
$$

Where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha \geq 1$ and $\beta(,)$ is Euler Beta function.

Proof. From lemma 2.1 we get

$$
\begin{aligned}
&\Big|\frac{f(a)+f(b)}{(b-a)^2}+\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left(J_{a+}^{\alpha-1}f(b)+J_{b-}^{\alpha-1}f(a)\right) \\
&-\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}\left(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right)\Big| \\
\leq &\int_0^1 u^{\alpha}(1-u)|f''(ua+(1-u)b)|du+\int_0^1 u(1-u)^{\alpha} \\
&\cdot|f''(ua+(1-u)b)|du \\
= &I_1+I_2.
\end{aligned} \tag{15}
$$

By Hölder's inequality we have

$$
|I_1| \leq \left(\int_0^1 u^{\alpha(1-\frac{p}{q})p}du\right)^{\frac{1}{p}}\left(\int_0^1 u^{\alpha p}(1-u)^q|f''(ua+(1-u)b)|^q du\right)^{\frac{1}{q}}.
$$

Since $|f''|^q$ is s-convex in the second sense on [a,b], we get

$$
|f''(ua+(1-u)b)|^q \leq u^s|f''(a)|^q+(1-u)^s|f''(b)|^q.
$$

Therefore, we get

$$
\begin{aligned}
|I_1| \leq &\left(\frac{q-1}{\alpha q-\alpha p+q-1}\right)^{\frac{q-1}{q}}\left(\beta(\alpha p+s+1,q+1)+\beta(\alpha p+1,s+q+1)\right)^{\frac{1}{q}} \\
&\cdot\left(\max\left(|f''(a)|^q,|f''(b)|^q\right)\right)^{\frac{1}{q}}.
\end{aligned} \tag{16}
$$

Where we have used the fact that

$$
\int_0^1 u^{\alpha(1-\frac{p}{q})p}du=\frac{q-1}{\alpha q-\alpha p+q-1}
$$

and

$$
\int_0^1 u^{x-1}(1-u)^{y-1}du=\beta(x,y),x,y>0.
$$

Clearly, the above argument may be repeated in the term I_2.

Thus from (2.5) and (2.6), we conclude that

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{(b-a)^2}+\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left(J_{a+}^{\alpha-1}f(b)+J_{b-}^{\alpha-1}f(a)\right)\\
&-\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}\left(J_{a+}^{\alpha}f(a)+J_{b-}^{\alpha}f(a)\right)\Big|\\
\leq& 2\left(\frac{q-1}{\alpha q-\alpha p+q-1}\right)^{\frac{q-1}{q}}\left(\beta(\alpha p+s+1,q+1)+\beta(\alpha p+1,s+q+1)\right)^{\frac{1}{q}}\\
&\cdot\left(\max\left(|f''(a)|^q,|f''(b)|^q\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof.

Corollary 2.1 Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$,be a twice differentiable mapping on (a,b) with $a<b$ such that $f''\in L_1[a,b]$. If $|f''|^q$ is convex on [a,b], then the following inequality for fractional integrals holds:

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{(b-a)^2}+\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left(J_{a+}^{\alpha-1}f(b)+J_{b-}^{\alpha-1}f(a)\right)\\
&-\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}\left(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right)\Big|\\
\leq& 2\left(\frac{q-1}{\alpha q-\alpha p+q-1}\right)^{\frac{q-1}{q}}\left(\frac{\alpha pq\beta(\alpha p,q)}{(\alpha p+q)(\alpha p+q+1)}\right)^{\frac{1}{q}}\\
&\cdot\left(\max\left(|f''(a)|^q,|f''(b)|^q\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

Where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\geq 1$ and $\beta(,)$ is Euler Beta function.

Proof. Setting s=1 in (2.4),and using the fact that $\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, we get the required.

Theorem 2.2 With the assumption of Theorem 1.3, we have

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{(b-a)^2}+\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left(J_{a+}^{\alpha-1}f(b)+J_{b-}^{\alpha-1}f(a)\right)\\
&-\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}\left(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right)\Big|\\
\leq& 2\left(\frac{q-1}{\alpha q-\alpha p+q-1}\right)^{\frac{q-1}{q}}\left(\beta(\alpha p+1,q+1)\right)^{\frac{1}{q}}\\
&\cdot\left(\max\left(|f''(a)|^q,|f''(b)|^q\right)\right)^{\frac{1}{q}}.
\end{aligned}\tag{17}$$

Where $\alpha \geq 1$ and $\beta(,)$is Euler Beta function.

Proof. By similar way in proof of Theorem 2.1, and noting that $|f''|^q$ is quasi-convex, we get

$$\begin{aligned}
& \Big|\frac{f(a)+f(b)}{(b-a)^2}+\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left(J_{a+}^{\alpha-1}f(b)+J_{b-}^{\alpha-1}f(a)\right) \\
& -\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}\left(J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right)\Big| \\
\leq\ & \int_0^1 u^{\alpha}(1-u)|f''(ua+(1-u)b)|du+\int_0^1 u(1-u)^{\alpha} \\
& \cdot|f''(ua+(1-u)b)|du \\
\leq\ & 2\left(\int_0^1 u^{\alpha(1-\frac{p}{q})p}du\right)\left(\int_0^1 u^{\alpha p}(1-u)^q|f''(ua+(1-u)b)|^q du\right)^{\frac{1}{q}} \\
\leq\ & 2\left(\frac{q-1}{\alpha q-\alpha p+q-1}\right)^{\frac{q-1}{q}}\left(\max\left(|f''(a)|^q,|f''(b)|^q\right)\right)^{\frac{1}{q}}\left(\int_0^1 u^{\alpha p}(1-u)^q du\right)^{\frac{1}{q}} \\
=\ & 2\left(\frac{q-1}{\alpha q-\alpha p+q-1}\right)^{\frac{q-1}{q}}\left(\max\left(|f''(a)|^q,|f''(b)|^q\right)\right)^{\frac{1}{q}}\left(\beta(\alpha p+1,q+1)\right)^{\frac{1}{q}}.
\end{aligned}$$

Where we have used the fact that

$$\int_0^1 u^{\alpha(1-\frac{p}{q})p}du=\frac{q-1}{\alpha q-\alpha p+q-1}$$

and

$$\int_0^1 u^{\alpha p}(1-u)^q du=\beta(\alpha p+1,q+1).$$

This completes the proof.

Remark 2.1 In Theorem 2.2, if we choose $\alpha=1$, then (2.7) reduces inequality (1.6) of Theorem 1.3.

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