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# CAPUTO FRACTIONAL DERIVATIVE INEQUALITIES FOR REFINED MODIFIED $(h,m)-$CONVEX FUNCTIONS

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**Abstract.** In this paper, we introduce a new convex function called refined modified $(h,m)-$convex function and establish Hadamard-type inequalities for this new definition via Caputo $k-$fractional derivatives. Then by applying two integral identities including the nth order derivatives of given functions we prove estimation of Hadamard type inequalities for the Caputo $k-$fractional derivatives.

**Keywords:** Caputo $k-$fractional derivatives; refined modified $(h,m)-$convex function; Hadamard inequality.

**2010 AMS Subject Classification:** 26A51, 26A33, 26D15.

## 1. Introduction

Fractional order of differentiation and integration are bases of fractional calculus. The history of fractional calculus goes for back to the seventeenth century, when G. W. Leibniz and Marquis de l'Hospital started a discussion on semi-derivatives. Fractional derivatives and fractional integral operators are two main key features, which plays a very important role in the development of fractional calculus. There are a lot of contributions of Riemaan and Liouville in filed of fractional calculus, the first integral operator named as Riemaan-Liouville fractional integral operator. Further by the help of Riemaan-Liouville fractional integral operator, Riemaan-Liouville

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fractional derivative was obtained. Further Caputo improved the Riemaan-Liouville fractional derivative formula, which well known formula known as Caputo fractional derivatives. After the existence of these classical definitions, many authors gave other definitions of fractional derivatives and fractional integrals. In [8] Authors discussed about application of Caputo k-fractional derivatives. In [9] Authors proves some results related to Hadamard inequalities for k-fractional derivatives. In [13, 14, 15] Authors proves certain inequalities with different integral operators. And also see [19, 21] for further detail study. In order to solve the some difficult problems in Algebraic Geometry cause the invention of $k-$theory. Alexander Grothendieck is the founder of $k-$theory [3]. $k-$theory also rapidly used in other directions in mathematics, for example Number Theory [10], Topology [1, 2] and Functional Analysis [4].

The aim of this paper is to apply $k-$anlogue of definition of Caputo fractional derivative named Caputo $k-$fractional derivative [7] for establishing Hadamard type inequalities for modified $(h-m)-$Convex functions. Next we give the definition of Caputo fractional derivatives.

**Definition 1.** [6] *Let $f \in AC^n[a,b]$. The Caputo fractional derivatives of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ of $f$ are defined as follows:*

$$ {}^{C}D^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > a, \tag{1.1}$$

$$ {}^{C}D^{\alpha}_{b^-} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \quad x < b, \tag{1.2}$$

*where $n = [\alpha] + 1$. If $\alpha = n \in \{1,2,3,...\}$ and usual derivatives of order n exists, then Caputo fractional derivatives $({}^{C}D^{\alpha}_{a^+} f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^{C}D^{\alpha}_{b^-} f)(x)$ coincides with $f^{(n)}(x)$, with exactness to a constant multiplier $(-1)^n$.*
*In particular we have*

$$ ({}^{C}D^{0}_{a^+} f)(x) = ({}^{C}D^{0}_{b^-} f)(x) = f(x), \tag{1.3}$$

*where $n = 1$ and $\alpha = 0$.*

A $k-$fractional analogue of Caputo fractional derivatives is given is the upcoming definition.

**Definition 2.** [7] *Let $f \in AC^n[a,b]$. The Caputo $k-$fractional derivatives of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ of $f$ are defined as follows:*

$$ {}^{C}D^{\alpha,k}_{a^+} f(x) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, \quad x > a, \tag{1.4} $$

$$ {}^{C}D^{\alpha,k}_{b^-} f(x) = \frac{(-1)^n}{k\Gamma_k(n-\frac{\alpha}{k})} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, \quad x < b, \tag{1.5} $$

*where $k \geq 1$, $n = [\alpha] + 1$ and $\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{\frac{-t^k}{k}} dt$, $\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$.*

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivatives of order n exists, then Caputo $k-$fractional derivatives $({}^{C}D^{\alpha,1}_{a^+} f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^{C}D^{\alpha,1}_{b^-} f)(x)$ coincides with $f^{(n)}(x)$, with exactness to a constant multiplier $(-1)^n$. A very famous beta function also used frequently in this paper.

**Definition 3.** [5] *The beta function of two variable u and v is defined as follows:*

$$ \beta(u,v) = \int_0^1 \phi^{u-1}(1-\phi)^{v-1} d\phi, $$

*for $Re(u) > 0$, $Re(v) > 0$.*

The purpose of this paper is the study of Caputo fractional derivatives for a generalized convex function. Analytic representation of convex functions motivates to define new notions and concepts. Many of the inequalities are straightforward consequences of convex functions, the Hadamard inequality is one of them. We will study the Hadamard inequality for Caputo $k-$fractional derivatives of generalized convex functions called refined modified $(h,m)-$convex functions. The Hadamard inequality for convex functions is stated in undermentioned theorem:

**Theorem 1.** [18] *If $f : J \to \mathbb{R}$ is a convex function defined on the interval J, then for $a, b \in J$, $a < b$ we have*

$$ f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. $$

In the following we give definition of convex, $h-$convex, modified $h-$convex, modified $(h-m)$ convex, refined modified $(h-m)$ convex functions.

**Definition 4.** [17] *A function $f : J \to \mathbb{R}$, where $J$ is an interval in $\mathbb{R}$, is said to convex if for $x, y \in J$, $\alpha \in [0,1]$, we have*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

**Definition 5.** [20] *$h-$convex function is defined as a non-negative function $f : J \to R$ which satisfies $f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y)$, where $h$ is a non-negative function, $\alpha \in (0,1)$ and $x, y \in J$.*

In [19], Toader gave the concept of modified h-convex functions as follows.

**Definition 6.** [19] *Let $f, h : J \subseteq R \to R$ be a non negative function, then $f$ is called modified h-convex function, if $f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + (1-h(\alpha))f(y)$, $\alpha \in [0,1]$ and $x, y \in J$.*

**Definition 7.** *Let $f, h : J \subseteq R \to R$ be a non negative function, then $f$ is called modified $(h-m)$ convex function, if $f(\alpha x + m(1-\alpha)y) \leq h(\alpha)f(x) + m(1-h(\alpha))f(y)$, $m, \alpha \in [0,1]$ and $x, y \in J$.*

**Definition 8.** *Let $f, h : J \subseteq R \to R$ be a non negative function, then $f$ is called refined modified $(h-m)$ convex function, if*

$$f(\alpha x + m(1-\alpha)y) \leq h(\alpha)(1-h(\alpha))(f(x) + mf(y)),$$

*where $m, \alpha \in [0,1]$ and $x, y \in J$.*

In Section 2, we present two versions of Hadamard inequality for refined modified $(h,m)-$convex functions via the Caputo $k-$fractional derivatives. From these inequalities various special cases are analyzed in the form of corollaries.

In Section 3, we use two integral identities including arbitrary order derivative of function $f$, in establishing error estimates of Hadamard type inequalities.

## 2. Main Results

**Theorem 2.2.** *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $f \in AC^n[a,b], 0 \leq a < b$. Also let $f^{(n)}$ be a refined modified $(h,m)-$convex function on $[a, mb]$ with $m \in (0,1]$. Then the following inequality for Caputo k-fractional derivatives hold:*

$$f^{(n)}\left(\frac{bm+a}{2}\right)$$

$$
\begin{aligned}
&\leq \frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\\
&\quad\times\left[m^{n-\frac{\alpha}{k}+1}(-1)^n({}^CD^{\alpha,k}_{b^-}f(\frac{a}{m}))+({}^CD^{\alpha,k}_{a^+}f(mb)\right]\\
&\leq\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\quad\times\left[\left\{f^{(n)}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right\}\right.\\
&\quad\left.\times\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right].
\end{aligned}
\tag{2.1}
$$

*Proof.* Since $f^{(n)}$ is refined modified $(h,m)-$ convex function on $[a,mb]$, we have

$$
f^{(n)}\left(\frac{um+v}{2}\right)\leq h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(f^{(n)}(v)+mf^{(n)}(u)\right),\quad u,v\in[a,b].
\tag{2.2}
$$

By setting $u=(1-t)\frac{a}{m}+tb\leq b$ and $v=m(1-t)b+ta\geq a$ in the above inequality for $t\in[0,1]$, then by integrating over $[0,1]$ after multiplying with $t^{n-\frac{\alpha}{k}-1}$ we have

$$
\begin{aligned}
f^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt&\leq h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\times\left[\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(m(1-t)b+ta\right)dt+m\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left((1-t)\frac{a}{m}+tb\right)dt\right].
\end{aligned}
$$

Now, if we set $w=(1-t)\frac{a}{m}+tb$ and $z=m(1-t)b+ta$ in the right hand side of above inequality, we get

$$
\begin{aligned}
f^{(n)}\left(\frac{bm+a}{2}\right)\frac{1}{n-\frac{\alpha}{k}}&\leq h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\times\left[\int_a^{mb}\left(\frac{mb-z}{mb-a}\right)^{n-\frac{\alpha}{k}-1}\frac{f^{(n)}(z)dz}{(mb-a)}+m\int_{\frac{a}{m}}^{b}\left(\frac{w-\frac{a}{m}}{b-\frac{b}{m}}\right)^{n-\frac{\alpha}{k}-1}\frac{mf^{(n)}(w)dw}{(b-\frac{a}{m})}\right].
\end{aligned}
$$

From which we can write the following inequality:

$$
\begin{aligned}
f^{(n)}\left(\frac{bm+a}{2}\right)&\leq h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\\
&\times\left[{}^CD^{\alpha,k}_{a^+}f(mb)+m^{n-\frac{\alpha}{k}+1}(-1)^n({}^CD^{\alpha,k}_{b^-}f(\frac{a}{m}))\right].
\end{aligned}
\tag{2.3}
$$

On the other hand, by using refined modified $(h,m)-$ convexity of $f^{(n)}$, we have

$$
mf^{(n)}\left((1-t)\frac{a}{m}+tb\right)\leq h(t)(1-h(t))\left(m^2f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right).
$$

Multiplying both side by $(n-\frac{\alpha}{k})(h(\frac{1}{2}))(1-h(\frac{1}{2}))t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, after some calculation we get

$$\begin{aligned}
&\frac{m^{n-\frac{\alpha}{k}+1}\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)}{(mb-a)^{n-\frac{\alpha}{k}}}k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)(-1)^n\left({}^{C}D^{\alpha,k}_{b^-}f(\frac{a}{m})\right)\\
&\leq m\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\times\left[\left(mf^{(n)}\left(\frac{a}{m^2}\right)+f^{(n)}(b)\right)\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right]. \qquad (2.4)
\end{aligned}$$

By using refined modified $(h,m)-$ convexity of $f^{(n)}$, we have

$$f^{(n)}\left(m(1-t)b+ta\right)\leq h(t)(1-h(t))\left(mf^{(n)}(b)+f^{(n)}(a)\right).$$

Multiplying both side by $(n-\frac{\alpha}{k})\left(h\left(\frac{1}{2}\right)\right)(1-h(\frac{1}{2}))t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, after some calculation we get

$$\begin{aligned}
&\frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)}{(mb-a)^{n-\frac{\alpha}{k}}}k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)\left({}^{C}D^{\alpha,k}_{a^+}f(mb)\right)\\
&\leq\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\times\left[\left(mf^{(n)}(b)+f^{(n)}(a)\right)\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right]. \qquad (2.5)
\end{aligned}$$

Addition of (2.4) and (2.5) yields.

$$\begin{aligned}
&\frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\\
&\times\left[m^{n-\frac{\alpha}{k}+1}(-1)^n({}^{C}D^{\alpha,k}_{b^-}f(\frac{a}{m}))+({}^{C}D^{\alpha,k}_{a^+}f(mb)\right]\\
&\leq\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\times\left[\left\{f^{(n)}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right\}\right.\\
&\left.\times\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right]. \qquad (2.6)
\end{aligned}$$

By combining the inequalities (2.3) and (2.6), we get the required result.

$$f^{(n)}\left(\frac{bm+a}{2}\right)$$

$$
\begin{aligned}
&\leq \frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\\
&\times\left[m^{n-\frac{\alpha}{k}+1}(-1)^n({}^CD^{\alpha,k}_{b^-}f(\frac{a}{m}))+({}^CD^{\alpha,k}_{a^+}f(mb)\right]\\
&\leq\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\times\left[\left\{f^{(n)}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right\}\right.\\
&\left.\times\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right].
\end{aligned}
$$

□

**Corollary 2.1.** *By setting $k=1$ in the inequality* (2.1)*, the following Caputo fractional derivatives inequality holds:*

$$
\begin{aligned}
&f^{(n)}\left(\frac{bm+a}{2}\right)\\
&\leq\frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\\
&\times\left[m^{n-\alpha+1}(-1)^n\left({}^CD^{\alpha}_{b^-}f\left(\frac{a}{m}\right)\right)+\left({}^CD^{\alpha}_{a^+}f(mb)\right)\right]\\
&\leq(n-\alpha)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\times\left[\left\{f^{(n)}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right\}\right.\\
&\left.\times\int_0^1 h(t)(1-h(t))t^{n-\alpha-1}dt\right].
\end{aligned}
\tag{2.7}
$$

**Corollary 2.2.** *By setting $m=1$ in the inequality* (2.1)*, the following Caputo $k-$fractional derivatives inequality holds for refined modified $h-$convex functions:*

$$
\begin{aligned}
&f^{(n)}\left(\frac{b+a}{2}\right)\\
&\leq\frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}\\
&\times\left[(-1)^n({}^CD^{\alpha,k}_{b^-}f(a))+({}^CD^{\alpha,k}_{a^+}f(b)\right]\\
&\leq\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)
\end{aligned}
$$

$$\times 2\left[\left(f^{(n)}(a)+f^{(n)}(b)+\right)\int_0^1 h(t)(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right]. \tag{2.8}$$

**Corollary 2.3.** *By setting $k=1$ and $m=1$ in the inequality (2.1), the following Caputo fractional derivatives inequality holds:*

$$\begin{aligned}
&f^{(n)}\left(\frac{b+a}{2}\right)\\
&\leq \frac{\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\\
&\quad\times\left[(-1)^n({}^CD^{\alpha}_{b^-}f(a))+({}^CD^{\alpha}_{a^+}f(b)\right]\\
&\leq (n-\alpha)\left(h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\quad\times 2\left[\left(f^{(n)}(a)+f^{(n)}(b)\right)\int_0^1 h(t)(1-h(t))t^{n-\alpha-1}dt\right].
\end{aligned} \tag{2.9}$$

**Corollary 2.4.** *If we choose " h" is identity function in (2.1) and usind the definition of beta function, the following Caputo k−fractional derivatives inequality hold:*

$$\begin{aligned}
&f^{(n)}\left(\frac{bm+a}{2}\right)\\
&\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[m^{n-\frac{\alpha}{k}+1}(-1)^n({}^CD^{\alpha,k}_{b^-}f(\frac{a}{m}))+({}^CD^{\alpha,k}_{a^+}f(mb)\right]\\
&\leq\left(n-\frac{\alpha}{k}\right)\left[\left\{f^{(n)}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right\}\beta(2,n-\frac{\alpha}{k}+1)\right].
\end{aligned} \tag{2.10}$$

**Corollary 2.5.** *If " h" is identity function and set m=1 in (2.1) and using the definition of beta function the following Caputo k−fractional derivatives inequality hold for convex function:*

$$\begin{aligned}
&f^{(n)}\left(\frac{b+a}{2}\right)\\
&\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}\left[(-1)^n\left({}^CD^{\alpha,k}_{b^-}f(a)\right)+\left({}^CD^{\alpha,k}_{a^+}f(b)\right)\right]\\
&\leq 2\left(n-\frac{\alpha}{k}\right)\left[\left\{f^{(n)}(a)+f^{(n)}(b)\right\}\beta\left(2,n-\frac{\alpha}{k}+1\right)\right].
\end{aligned} \tag{2.11}$$

**Corollary 2.6.** *If " h" is identity function and set k=1 in (2.1) and using the definition of beta function the following Caputo fractional derivatives inequality hold for convex function:*

$$
\begin{aligned}
& f^{(n)}\left(\frac{bm+a}{2}\right) \\
& \leq \frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^{C}D^{\alpha}_{b^-}f\left(\frac{a}{m}\right)\right)+\left({}^{C}D^{\alpha}_{a^+}f(mb)\right)\right] \\
& \leq (n-\alpha)\left[\left\{f^{(n)}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right\}\beta\left(2,n-\alpha+1\right)\right]. \qquad (2.12)
\end{aligned}
$$

**Corollary 2.7.** *If " h" is identity function and set k=1, m=1 in* (2.1) *and using the definition of beta function the following Caputo fractional derivatives inequality hold for convex function:*

$$
\begin{aligned}
& f^{(n)}\left(\frac{b+a}{2}\right) \\
& \leq \frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[(-1)^n\left({}^{C}D^{\alpha}_{b^-}f(a)\right)+\left({}^{C}D^{\alpha,k}_{a^+}f(b)\right)\right] \\
& \leq 2\left(n-\alpha\right)\left[\left\{f^{(n)}(a)+f^{(n)}(b)\right\}\beta\left(2,n-\alpha+1\right)\right]. \qquad (2.13)
\end{aligned}
$$

**Theorem 2.3.** *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in AC^n[a,b], 0\leq a<b$. Also let $f^{(n)}$ be a refined modified $(h,m)-$convex function on $[a,mb]$ with $m\in(0,1]$. Then the following inequality for Caputo k-fractional derivatives hold:*

$$
\begin{aligned}
& f^{(n)}\left(\frac{bm+a}{2}\right) \\
& \leq 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right) \\
& \times\left[\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)+m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right] \\
& \leq h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right) \\
& \times\left[f^{n}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right]. \qquad (2.14)
\end{aligned}
$$

*Proof.* By putting $u=\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}$ and $v=\frac{t}{2}a+m\frac{(2-t)}{2}b$ in (2.2) where $t\in[0,1]$, and multiplying with $t^{n-\frac{\alpha}{k}-1}$, then integrating over $[0,1]$ one can have

$$
\begin{aligned}
f^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt \leq h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right) \\
\times\left[\left\{f^{(n)}\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)dt+mf^{(n)}\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)dt\right\}\int_0^1 t^{n-\frac{\alpha}{k}-1}\right].
\end{aligned}
$$

By change of variables, as we did to get (2.3), one can also get

$$f^{(n)}\left(\frac{bm+a}{2}\right) \leq 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right) \times\left[\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)+m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m^2}\right)\right)\right]. \tag{2.15}$$

On the other hand, by using refined modified $(h,m)-$ convexity of $f^{(n)}$, we have

$$f^{(n)}\left(\frac{t}{2}a+m\left(\frac{2-t}{2}b\right)\right) \leq h\left(\frac{t}{2}\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[f^{(n)}(a)+mf^{(n)}(b)\right].$$

Multiplying both side by $(n-\frac{\alpha}{k})(h(\frac{1}{2}))(1-h(\frac{1}{2}))t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, after some calculation we get

$$\frac{2^{(n-\frac{\alpha}{k})}}{(mb-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)k\Gamma_k(n-\frac{\alpha}{k}+k)({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)) \leq h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[f^{(n)}(a)+mf^{(n)}(b)\right]. \tag{2.16}$$

By using refined modified $(h,m)-$ convexity of $f^{(n)}$, we have

$$mf^{(n)}\left(\frac{t}{2}b+m\left(\frac{2-t}{2}\right)\frac{a}{m^2}\right) \leq h\left(\frac{t}{2}\right)\left(1-h\left(\frac{t}{2}\right)\right)\left[mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right].$$

Multiplying both side by $(n-\frac{\alpha}{k})(h(\frac{1}{2}))(1-h(\frac{1}{2}))t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, after some calculation we get

$$\begin{aligned}&\frac{2^{(n-\frac{\alpha}{k})}m^{n-\frac{\alpha}{k}+1}(h(\frac{1}{2}))\left(1-h\left(\frac{1}{2}\right)\right)}{(mb-a)^{n-\frac{\alpha}{k}}}\\ &\times k\Gamma_k(n-\frac{\alpha}{k}+k)(-1)^n\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m^2}\right)\right)\\ &\leq h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\\ &\times\left[mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right].\end{aligned} \tag{2.17}$$

Addition of (2.16) and (2.17) yields.

$$\begin{aligned}&2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\\ &\times\left[\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)+m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\leq h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\\
&\times\left[f^{n}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right]. \qquad (2.18)
\end{aligned}
$$

By combining the inequalities (2.15) and (2.18), we get the required result.

$$
\begin{aligned}
&f^{(n)}\left(\frac{bm+a}{2}\right)\\
&\leq 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\times\left[\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)+m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]\\
&\leq h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\\
&\times\left[f^{n}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right]. \qquad (2.19)
\end{aligned}
$$

□

**Corollary 2.8.** *By setting $k=1$ in inequality (2.19), the following inequality holds for caputo fractional derivatives:*

$$
\begin{aligned}
&f^{(n)}\left(\frac{bm+a}{2}\right)\\
&\leq 2^{(n-\alpha)}\frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\\
&\times\left[\left({}^{C}D^{\alpha}_{(\frac{a+bm}{2})^+}f(mb)\right)+m^{n-\alpha+1}(-1)^n\left({}^{C}D^{\alpha}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]\\
&\leq h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)\\
&\times\left[f^{n}(a)+2mf^{(n)}(b)+m^2f^{(n)}\left(\frac{a}{m^2}\right)\right]. \qquad (2.20)
\end{aligned}
$$

**Corollary 2.9.** *By setting $m=1$ in inequality (2.19), the following Caputo $k-$fractional derivatives holds:*

$$
\begin{aligned}
&f^{(n)}\left(\frac{b+a}{2}\right)\\
&\leq 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)
\end{aligned}
$$

$$\times\left[\left({}^{C}D^{\alpha,k}_{(\frac{a+b}{2})^{+}}f(b)\right)+m^{n-\frac{\alpha}{k}+1}(-1)^{n}\left({}^{C}D^{\alpha,k}_{(\frac{a+b}{2})^{-}}f(a)\right)\right]$$
$$\leq h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)$$
$$\times 2\left[f^{n}(a)+2f^{(n)}(b)\right]. \tag{2.21}$$

**Corollary 2.10.** *By setting $m=1$, $k=1$ in inequality* (2.19)*, the following inequality holds for convex function via Caputo fractional derivatives:*

$$f^{(n)}\left(\frac{b+a}{2}\right)$$
$$\leq 2^{(n-\alpha)}\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}h\left(\frac{1}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)$$
$$\times\left[\left({}^{C}D^{\alpha}_{(\frac{a+b}{2})^{+}}f(b)\right)+(-1)^{n}\left({}^{C}D^{\alpha}_{(\frac{a+b}{2})^{-}}f(a)\right)\right]$$
$$\leq h\left(\frac{1}{2}\right)h\left(\frac{t}{2}\right)\left(1-h\left(\frac{1}{2}\right)\right)\left(1-h\left(\frac{t}{2}\right)\right)$$
$$\times 2\left[f^{n}(a)+f^{(n)}(b)\right]. \tag{2.22}$$

**Corollary 2.11.** *If we choose " $h$" is identity function in* (2.19)*, the following Caputo $k-$fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{bm+a}{2}\right)$$
$$\leq 2^{(n-\frac{\alpha}{k}-2)}\frac{k\Gamma_{k}(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}$$
$$\times\left[\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2})^{+}}f(mb)\right)+m^{n-\frac{\alpha}{k}+1}(-1)^{n}\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2m})^{-}}f\left(\frac{a}{m}\right)\right)\right]$$
$$\leq 2^{-4}t(2-t)\left[f^{n}(a)+2mf^{(n)}(b)+m^{2}f^{(n)}\left(\frac{a}{m^{2}}\right)\right]. \tag{2.23}$$

**Corollary 2.12.** *If we choose " $h$" is identity function $k=1$ in* (2.19)*, the following Caputo fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{bm+a}{2}\right)$$
$$\leq 2^{(n-\alpha-2)}\frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}$$
$$\times\left[\left({}^{C}D^{\alpha}_{(\frac{a+bm}{2})^{+}}f(mb)\right)+m^{n-\alpha+1}(-1)^{n}\left({}^{C}D^{\alpha}_{(\frac{a+bm}{2m})^{-}}f\left(\frac{a}{m}\right)\right)\right]$$

$$\leq 2^{-4}t(2-t)\left[f^{n}(a)+2mf^{(n)}(b)+m^{2}f^{(n)}\left(\frac{a}{m^{2}}\right)\right]. \tag{2.24}$$

**Corollary 2.13.** *If we choose " h" is identity function and $m=1$ in (2.19), the following Caputo $k-$fractional derivatives inequality holds:*

$$\begin{aligned}
& f^{(n)}\left(\frac{b+a}{2}\right) \\
& \leq 2^{(n-\frac{\alpha}{k}-2)}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}} \\
& \times\left[\left({}^{C}D^{\alpha,k}_{(\frac{a+b}{2})^{+}}f(b)\right)+(-1)^{n}\left({}^{C}D^{\alpha,k}_{(\frac{a+b}{2})^{-}}f(a)\right)\right] \\
& \leq 2^{-3}t(2-t)\left[f^{n}(a)+f^{(n)}(b)\right].
\end{aligned} \tag{2.25}$$

**Corollary 2.14.** *If we choose " h" is identity function and $m=1$, $k=1$ in (2.19), the following Caputo fractional derivatives inequality holds:*

$$\begin{aligned}
& f^{(n)}\left(\frac{b+a}{2}\right) \\
& \leq 2^{(n-\alpha-2)}\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}} \\
& \times\left[\left({}^{C}D^{\alpha,k}_{(\frac{a+b}{2})^{+}}f(b)\right)+(-1)^{n}\left({}^{C}D^{\alpha,k}_{(\frac{a+b}{2})^{-}}f(a)\right)\right] \\
& \leq 2^{-3}t(2-t)\left[f^{n}(a)+f^{(n)}(b)\right].
\end{aligned} \tag{2.26}$$

**Theorem 2.4.** *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in AC^{n}[a,b], 0\leq a<b$. Also let $f^{(n)}$ be a refined modified $(h,m)-$convex function on $[a,mb]$ with $m\in(0,1]$. Then the following inequalities for Caputo k-fractional derivatives hold:*

$$\begin{aligned}
& \frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\left\{({}^{C}D^{\alpha,k}_{a^{+}}f(b))+(-1)^{n}({}^{C}D^{\alpha,k}_{b^{-}}f(a))\right\} \\
& \leq\left[f^{(n)}(a)+f^{(n)}(b)+m\left\{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right\}\right] \\
& \times\int_{0}^{1}t^{n-\frac{\alpha}{k}-1}h(t)(1-h(t))dt \\
& \leq\left[f^{(n)}(a)+f^{(n)}(b)+m\left\{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right\}\right]
\end{aligned}$$

$$\times \frac{\left(\int_0^1 \{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\left(np - \frac{\alpha p}{k} - p + 1\right)^{\frac{1}{p}}}, \tag{2.27}$$

*where* $p^{-1} + q^{-1} = 1$ *and* $p > 1$.

*Proof.* Since $f^{(n)}$ is refined modified $(h,m)-$convex function on $[a,mb]$ then for $m \in (0,1]$ and $t \in [0,1]$, we have

$$f^{(n)}(ta + (1-t)b) + f^{(n)}((1-t)a + tb)$$
$$\leq h(t)(1-h(t))\left[f^{(n)}(a) + f^{(n)}(b) + m\left\{f^{(n)}\left(\frac{a}{m}\right) + f^{(n)}\left(\frac{b}{m}\right)\right\}\right].$$

By multiplying both side of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating the above inequality with respect to t on $[0,1]$, we have

$$\int_0^1 t^{n-\frac{\alpha}{k}-1}\left\{f^{(n)}(ta + (1-t)b) + f^{(n)}((1-t)a + tb)\right\}.$$
$$\leq \left[f^{(n)}(a) + f^{(n)}(b) + m\left\{f^{(n)}\left(\frac{a}{m}\right) + f^{(n)}\left(\frac{b}{m}\right)\right\}\right]$$
$$\times \int_0^1 t^{n-\frac{\alpha}{k}-1} h(t)(1-h(t))dt.$$

If we set $x = ta + (1-t)b$ in the left side of above inequality, we get the following inequality

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\left\{({}^C D_{a^+}^{\alpha,k} f(b)) + (-1)^n({}^C D_{b^-}^{\alpha,k} f(a))\right\}$$
$$\leq \left[f^{(n)}(a) + f^{(n)}(b) + m\left\{f^{(n)}\left(\frac{a}{m}\right) + f^{(n)}\left(\frac{b}{m}\right)\right\}\right]$$
$$\times \int_0^1 t^{n-\frac{\alpha}{k}-1} h(t)(1-h(t))dt. \tag{2.28}$$

Thus, we get the first inequality of (2.27). The second inequality of (2.27) follows from the fact by using the H$\ddot{o}$lder inequality

$$\int_0^1 t^{n-\frac{\alpha}{k}-1} h(t)(1-h(t))dt \leq \frac{\left(\int_0^1 \{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\left(np - \frac{\alpha p}{k} - p + 1\right)^{\frac{1}{p}}}, \tag{2.29}$$

and from (2.28) and (2.29) we get the required result.

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\left\{({}^C D_{a^+}^{\alpha,k} f(b)) + (-1)^n({}^C D_{b^-}^{\alpha,k} f(a))\right\}$$

$$\leq \left[f^{(n)}(a)+f^{(n)}(b)+m\left\{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right\}\right]$$
$$\times\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)(1-h(t))dt.$$
$$\leq \left[f^{(n)}(a)+f^{(n)}(b)+m\left\{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right\}\right]$$
$$\times\frac{\left(\int_0^1\{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\left(np-\frac{\alpha p}{k}-p+1\right)^{\frac{1}{p}}}.$$

□

**Corollary 2.15.** *By setting* $k=1$ *in inequality* (2.27), *the following inequalities holds via Caputo fractional derivatives:*

$$\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left\{({}^CD^{\alpha}_{a^+}f(b))+(-1)^n({}^CD^{\alpha}_{b^-}f(a))\right\}$$
$$\leq \left[f^{(n)}(a)+f^{(n)}(b)+m\left\{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right\}\right]$$
$$\times\int_0^1 t^{n-\alpha-1}h(t)(1-h(t))dt.$$
$$\leq \left[f^{(n)}(a)+f^{(n)}(b)+m\left\{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right\}\right]$$
$$\times\frac{\left(\int_0^1\{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\{p(n-\alpha-1)+1\}^{\frac{1}{p}}}. \tag{2.30}$$

**Corollary 2.16.** *By setting* $m=1$ *in inequality* (2.27), *the following inequality holds via Caputo k-fractional derivatives:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\{({}^CD^{\alpha,k}_{a^+}f(b))+(-1)^n({}^CD^{\alpha,k}_{b^-}f(a))\}$$
$$\leq 2\left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)(1-h(t))dt$$
$$\leq 2\left[f^{(n)}(a)+f^{(n)}(b)\right]\frac{\left(\int_0^1\{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\left(np-\frac{\alpha p}{k}-p+1\right)^{\frac{1}{p}}}.$$

**Corollary 2.17.** *By setting $m=1$, $k=1$ in inequality (2.27), the following inequality holds via Caputo fractional derivatives:*

$$\begin{aligned}
&\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\{({}^{C}D^{\alpha}_{a^+}f(b))+(-1)^n({}^{C}D^{\alpha}_{b^-}f(a))\}\\
&\leq 2\left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^1 t^{n-\alpha-1}h(t)(1-h(t))dt\\
&\leq 2\left[f^{(n)}(a)+f^{(n)}(b)\right]\frac{\left(\int_0^1\{h(t)(1-h(t)\}^q dt\right)^{\frac{1}{q}}}{\{p(n-\alpha-1)+1\}^{\frac{1}{p}}}.
\end{aligned}$$

## 3. Caputo k-Fractional Derivatives for Functions Whose Nth Derivative in Absolute Value Are Modified $(h,m)$-Convex

The following lemma is helpful to prove the next result.

**Lemma 3.1.** [12] *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable mapping on interval $(a,mb)$ with $a\leq mb$.If $f\in C^{n+1}[a.mb]$, then the following equality for Caputo k-fractional integrals holds*

$$\begin{aligned}
&\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha,k}_{mb^-}f(a))\right)\\
&=\frac{mb-a}{2}\int_0^1((1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}})f^{(n+1)}(m(1-t)b+ta)dt.
\end{aligned}$$

**Theorem 3.1.** *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,b]$. If $|f^{(n+1)}|$ is modified (h,m)-convex function on [a,mb] with $m\in(0,1]$. Then the following inequality Caputo for k-fractional derivatives holds:*

$$\begin{aligned}
&\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha,k}_{mb^-}f(a))\right)\right|\\
&\leq\frac{mb-a}{2}\left\{\frac{\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left((1-2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}-(2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}\right)\right.\\
&\left.\times\left[\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}\tag{3.1}$$

*where $p^{-1}+q^{-1}=1$.*

*Proof.* From lemma 3.1 and by using the property of modulus, we get

$$\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)$$
$$\leq\frac{mb-a}{2}\int_0^1|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}|\left|f^{(n+1)}(m(1-t)b+ta)\right|dt.$$

By refined modified (h,m)-convexity of $|f^{(n+1)}|$, we have

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{mb-a}{2}\left\{\int_0^{\frac{1}{2}}((1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}})h(t)(1-h(t))\left(|f^{(n+1)}(a)|+m|f^{(n+1)}(b)|\right)dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1((1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}})h(t)(1-h(t))\left(|f^{(n+1)}(a)|+m|f^{(n+1)}(b)|\right)dt\right\}$$
$$=\frac{mb-a}{2}\left\{\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\int_0^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}}h(t)(1-h(t))dt\right)\right.$$
$$-\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\int_0^{\frac{1}{2}}t^{n-\frac{\alpha}{k}}h(t)(1-h(t))dt\right)$$
$$+\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\int_{\frac{1}{2}}^1(1-t)^{n-\frac{\alpha}{k}}h(t)(1-h(t))dt\right)$$
$$\left.-\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\int_{\frac{1}{2}}^1t^{n-\frac{\alpha}{k}}h(t)(1-h(t))dt\right)\right\}.$$

Now, by using the Hölder's inequality in the right hand side of above inequality, we get

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{mb-a}{2}\left\{\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\frac{1-2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right.$$
$$-\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\frac{2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}$$
$$+\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\frac{1-2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}$$
$$\left.-\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)\left(\frac{2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right\}.$$

After some calculation we get the desired result.

$$\begin{aligned}
&\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha,k}_{mb^-}f(a))\right)\right|\\
&\leq \frac{mb-a}{2}\left\{\frac{\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left((1-2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}-(2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}\right)\right.\\
&\left.\times\left[\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

□

**Corollary 3.1.** *By setting $k=1$ in inequality (3.1), the following Caputo fractional derivative inequality holds:*

$$\begin{aligned}
&\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left(({}^{C}D^{\alpha}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha}_{mb^-}f(a))\right)\right|\\
&\leq \frac{mb-a}{2}\left\{\frac{\left(\left|f^{(n+1)}(a)+mf^{(n+1)}(b)\right|\right)}{(np-\alpha p+1)^{\frac{1}{p}}}\left((1-2^{\alpha p-np-1})^{\frac{1}{p}}-(2^{\alpha p-np-1})^{\frac{1}{p}}\right)\right.\\
&\left.\times\left[\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

**Corollary 3.2.** *By setting $m=1$ in inequality (3.1), the following Caputo $k-$fractional derivative inequality holds:*

$$\begin{aligned}
&\left|\frac{f^{(n)}(b)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(b))+(-1)^n({}^{C}D^{\alpha,k}_{b^-}f(a))\right)\right|\\
&\leq \frac{b-a}{2}\left\{\frac{\left(\left|f^{(n+1)}(a)+f^{(n+1)}(b)\right|\right)}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left((1-2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}-(2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}\right)\right.\\
&\left.\times\left[\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

**Corollary 3.3.** *By setting $k=1$ and $m=1$ in inequality* (3.1), *the following holds for convex function via Caputo fractional derivative:*

$$\left|\frac{f^{(n)}(b)+f^{(n)}(a)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left(({}^{C}D^{\alpha}_{a^+}f(b))+(-1)^n({}^{C}D^{\alpha}_{b^-}f(a))\right)\right|$$
$$\leq \frac{b-a}{2}\left\{\frac{\left(\left|f^{(n+1)}(a)+f^{(n+1)}(b)\right|\right)}{(np-\alpha p+1)^{\frac{1}{p}}}\left((1-2^{\alpha p-np-1})^{\frac{1}{p}}-(2^{\alpha p-np-1})^{\frac{1}{p}}\right)\right.$$
$$\left.\times\left[\left(\int_0^{\frac{1}{2}}\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\{h(t)(1-h(t))\}^q\right)^{\frac{1}{q}}\right]\right\}.$$

**Lemma 3.2.** [12] *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable mapping on interval $(a,mb)$ with $a\leq mb$. If $f\in C^{n+2}[a.mb]$, then the following equality for Caputo k-fractional integrals holds:*

$$\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha,k}_{mb^-}f(a))\right)$$
$$=\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}f^{(n+2)}(ta+m(1-t)b)dt.$$

**Theorem 3.2.** *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in C^{n+2}[a,b]$. If $|f^{(n+2)}|$ is modified (h,m)-convex function on [a,mb] with $m\in(0,1]$. Then the following inequality Caputo for k-fractional derivatives holds:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{(mb-a)^2}{2(n-\frac{\alpha}{k}+1)}\left(\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right)$$
$$\times\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1\{h(t)(1-h(t))\}^q\,dt\right)^{\frac{1}{q}}. \tag{3.2}$$

*Proof.* Using lemma 3.2 and refined modified (h,m)-convexity of $|f^{(n+2)}|$, we find □

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\left|f^{(n+2)}(ta+m(1-t)b)\right|dt$$

$$\leq \frac{(mb-a)^2}{2}\int_0^1 \frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}$$
$$\times\left\{h(t)(1-h(t))\left(\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right)\right\}dt$$
$$=\frac{(mb-a)^2}{2(n-\frac{\alpha}{k}+1)}\left\{\left(\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right)\right.$$
$$\left.\times\int_0^1(1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1})h(t)(1-h(t))dt\right\}.$$

Now, by using H$\ddot{o}$lder inequality, we have

$$\int_0^1(1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1})h(t)dt$$
$$\leq\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1\{h(t)(1-h(t))\}^q dt\right)^{\frac{1}{q}}.$$

This implies

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{(mb-a)^2}{2(n-\frac{\alpha}{k}+1)}\left(\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right)$$
$$\times\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1\{h(t)(1-h(t))\}^q dt\right)^{\frac{1}{q}}.$$

**Corollary 3.4.** *By setting $k=1$ in inequality* (3.2)*, the following Caputo fractional derivatives inequality holds:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left(({}^CD^{\alpha}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{(mb-a)^2}{2(n-\alpha+1)}\left(\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right)$$
$$\times\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1\{h(t)(1-h(t))\}^q dt\right)^{\frac{1}{q}}.$$

**Corollary 3.5.** *By setting $m=1$ in inequality* (3.2)*, the following Caputo $k-$fractional derivatives inequality holds:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(b))+(-1)^n({}^CD^{\alpha,k}_{b^-}f(a))\right)\right|$$

$$\leq \frac{(b-a)^2}{2(n-\frac{\alpha}{k}+1)}\left(\left|f^{(n+2)}(a)\right|+\left|f^{(n+2)}(b)\right|\right)$$
$$\times\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1\{h(t)(1-h(t))\}^q\,dt\right)^{\frac{1}{q}}.$$

**Corollary 3.6.** *By setting $k=1$ and $m=1$ in inequality (3.2), the following inequality holds for convex function via Caputo fractional derivatives:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left(({}^{C}D^{\alpha}_{a^+}f(b))+(-1)^n({}^{C}D^{\alpha}_{b^-}f(a))\right)\right|$$
$$\leq \frac{(b-a)^2}{2(n-\alpha+1)}\left(\left|f^{(n+2)}(a)\right|+\left|f^{(n+2)}(b)\right|\right)$$
$$\times\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1\{h(t)(1-h(t))\}^q\,dt\right)^{\frac{1}{q}}.$$

## Conflict of Interests

The author declares that there is no conflict of interests.

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