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THE PERSISTENCE AND BIFURCATION CONDITIONS OF AN ECO-EPIDEMIOLOGICAL MODEL INVOLVING TOXIN AND FEAR WITH TWO TYPES OF DISEASE

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Abstract: In this paper, we investigate the conditions in which local bifurcations might occur in an eco-epidemiological model that considers the combined effects of human behavior and fear. The model takes into account the SI (Susceptible–Infected) infection structure in the predator population and the SIS (Susceptible–Infected–Susceptible) disease dynamics in the prey population. The effect of fear on prey behavior and human intervention methods, including harvesting, treatment, and environmental management, is given special attention. The existence and stability of equilibrium points as well as the conditions around local bifurcations are the main topics of the qualitative study. It appears that the system can display Hopf bifurcation around the positive equilibrium, which results in the formation of periodic oscillations, as well as transcritical and saddle-node bifurcations close to certain equilibrium points. The dynamics and stability of the system are significantly shaped by the interplay between the degree of fear and the parameters of human involvement. Additionally, the presence of local bifurcations is convincingly verified using Sotomayor's theorem. Transitions between disease persistence, extinction, and coexistence stages may be brought about by changes in important factors, according to the acquired analytical results. In order to verify the theoretical results, numerical simulations are carried out for certain parameter settings and initial conditions. The results demonstrate a high degree of agreement with the system's expected dynamical behavior.

Keywords: prey–predator; eco-epidemiological model; local bifurcation; Hopf bifurcation; fear effect; human intervention; Sotomayor's theorem.

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1. INTRODUCTION

Mathematical modeling of ecological systems has long been a central topic in applied mathematics, particularly for understanding predator–prey interactions, food-chain structures, and food-web dynamics. Classical predator–prey models provide a fundamental framework for analyzing interactions among biological populations and have subsequently been extended to incorporate more realistic ecological mechanisms, including nonlinear functional responses, predator interference, and multi-trophic interactions. Such extensions are important because the inclusion of additional ecological mechanisms can significantly alter equilibrium structure, stability, persistence, and other complex dynamical properties of the system. Recent studies have demonstrated that incorporating mixed functional and numerical responses, predator interference, and food-chain structures can substantially enrich ecological dynamics and improve the mathematical description of interacting populations [1]–[4].

Among the behavioral mechanisms incorporated into contemporary ecological models, the fear effect has received considerable attention. Fear represents an indirect effect of predators on prey, whereby the presence or perceived risk of predation may alter prey behavior and consequently reduce feeding activity, reproduction, movement, or other demographic processes. Several studies have demonstrated that incorporating fear into predator–prey systems can substantially modify equilibrium stability and the long-term dynamical behavior of interacting populations [5], [6], [9], [10]. Moreover, combining fear with other behavioral mechanisms, including prey refuge and anti-predator behavior, provides a more realistic representation of predator–prey interactions and may generate richer dynamical behavior [8], [12], [32]. The interaction of fear with toxicity and other nonlinear dynamical effects further demonstrates the important role of behavioral responses in determining ecological stability and long-term population dynamics [7], [11].

In addition to fear, several demographic and behavioral mechanisms can substantially modify the dynamics of ecological systems. The Allee effect, which describes reduced population performance at low population densities, is particularly important because it may influence population persistence, equilibrium structure, and bifurcation behavior [14]. Predator cooperation represents another important ecological mechanism, since cooperative hunting can modify predation pressure and consequently affect the stability and persistence of interacting populations. Such effects have been investigated in food-web eco-epidemiological models incorporating additional food for predators [15]. Eco-epidemiological models further extend classical ecological systems by incorporating disease transmission into predator–prey or food-web interactions,

thereby coupling epidemiological processes with population dynamics. Previous investigations have shown that disease can significantly modify ecological stability and coexistence [16] and may even generate counterintuitive outcomes, such as a reduction in the healthy predator population despite an increase in prey abundance [17]. Consequently, incorporating behavioral, demographic, and epidemiological mechanisms provides a more comprehensive mathematical framework for describing complex ecological interactions [8], [14]–[17].

Human activities and environmental disturbances are also major factors influencing ecological dynamics. Harvesting is particularly important because different harvesting strategies can alter equilibrium stability and produce significant qualitative changes in predator–prey dynamics [18]–[20]. Environmental pollution represents another major ecological stressor. Toxic substances can influence survival, reproduction, and trophic interactions and may consequently produce substantial changes in the qualitative behavior of ecological systems. Several studies have incorporated toxicants into ecological and food-chain models and demonstrated their effects on equilibrium structure, stability, persistence, and bifurcation behavior [13], [21]–[25]. Furthermore, interactions between toxic effects and behavioral responses, particularly fear and anti-predator behavior, may produce additional dynamical complexity and provide a more comprehensive representation of populations simultaneously exposed to environmental and behavioral stressors [7], [25], [26].

From the viewpoint of dynamical systems theory, stability and bifurcation analyses are fundamental mathematical tools for identifying qualitative changes in ecological dynamics. In particular, Hopf bifurcation characterizes the transition from a stable equilibrium to periodic oscillations when a critical parameter crosses a threshold value, while transcritical and saddle-node bifurcations describe other important changes in equilibrium structure and stability. Classical dynamical systems theory provides rigorous mathematical criteria for identifying and classifying such bifurcations [27]. More recent ecological studies have extended these approaches to systems incorporating behavioral mechanisms, nonlinear interactions, discrete dynamics, and time delays. In particular, the interaction between fear and delay can modify stability thresholds and the conditions for the occurrence of Hopf bifurcation [28]. Stability, local bifurcation, and nonlinear dynamical analyses have also been employed to investigate qualitative transitions and complex dynamical behavior in various predator–prey and food-web systems [11], [13], [25], [26], [29], [31].

Despite these substantial developments, most existing studies have investigated fear, toxicants,

harvesting, disease transmission, Allee effects, predator cooperation, and other ecological mechanisms either separately or through particular combinations. Although these studies have provided important insights into the stability and long-term behavior of ecological and eco-epidemiological systems, a detailed characterization of the critical parameter thresholds at which qualitative changes in system dynamics occur remains an important issue. In particular, bifurcation analysis is essential for identifying transitions between different dynamical regimes and for determining how variations in biologically relevant parameters may lead to changes in equilibrium stability, the emergence of periodic solutions, or the exchange and creation of equilibrium states. More recently, Lafta and Majeed [30] investigated the influence of human intervention and fear on the dynamics of an eco-epidemiological model involving SIS disease in prey and SI disease in predators. The present study is a continuation of the mathematical investigation of the model introduced in [30], with particular emphasis on its bifurcation structure. Specifically, the conditions for the occurrence of Hopf, transcritical, and saddle-node bifurcations are derived and analyzed. The present bifurcation study therefore complements the dynamical analysis reported in [30] by identifying critical parameter values at which qualitative changes in the behavior of the system occur, thereby providing a deeper understanding of the mechanisms governing transitions among different dynamical states.

2. MODEL FORMULATION

A system (1) which is given, [30] creates a nonlinear predator-prey model with susceptible and infected classes, including disease transmission, fear effects, cooperative hunting, Allee effects, environmental toxicity, harvesting, and natural death. The model depicts the ecological and epidemiological interactions that influence population persistence and long-term dynamics. Population densities at time (t) are represented by μ_{1s} , μ_{1I} , μ_{2s} , and μ_{2I} .

$$\begin{aligned}
\frac{d\mu_{1s}}{dt} &= m_1\mu_{1s} \left(1 - \frac{\mu_{1s} + \mu_{1I}}{L}\right) \left(\frac{1}{1+f\mu_{2s}}\right) - \alpha_1\mu_{1s}\mu_{1I} - \rho_1\mu_{2s}^2\mu_{1s} - \delta_1\mu_{1s}^2 - \frac{Eq\mu_{1s}}{a_1E + a_2\mu_{1s}} \\
&\quad + \beta_1\mu_{1I} - d_1\mu_{1s} - \frac{\alpha_3\mu_{1s}\mu_{2s}}{b_1 + \mu_{1s}}, \\
\frac{d\mu_{1I}}{dt} &= \alpha_1\mu_{1s}\mu_{1I} - \beta_1\mu_{1I} - \delta_2\mu_{1I}^2 - d_2\mu_{1I}, \\
\frac{d\mu_{2s}}{dt} &= m_2\mu_{2s} \left(1 - \frac{\alpha_4\mu_{2s}}{b_2 + \mu_{1s}}\right) \left(\frac{\mu_{2s}}{\mu_{2s} + A}\right) + \rho_2\mu_{2s}^2\mu_{1s} - \delta_3\mu_{2s} - \alpha_2\mu_{2s}\mu_{2I} - d_3\mu_{2s}, \\
\frac{d\mu_{2I}}{dt} &= \alpha_2\mu_{2s}\mu_{2I} - \delta_4\mu_{2I} - d_4\mu_{2I}.
\end{aligned} \tag{1}$$

Table 1 illustrates the biological description of the parameters in system (1).

Parameters	Biological description	Unit
$m_1 > 0,$	Intrinsic growth rate of susceptible prey.	time^{-1}
$m_2 > 0,$	Intrinsic growth rate of susceptible predators.	time^{-1}
$\alpha_1 > 0,$	Transmission rate of disease in prey populations (SIS-type).	$\text{population}^{-1} \cdot \text{time}^{-1}$
$\alpha_2 > 0,$	Transmission rate of disease in predator populations (SI-type).	$\text{population}^{-1} \cdot \text{time}^{-1}$
$\alpha_3 > 0,$	Predation rate of susceptible predators on susceptible prey.	time^{-1}
$\alpha_4 > 0,$	Predator growth parameter in modified Leslie–Gower	dimensionless
$\delta_1 > 0,$	Rates of toxicity affecting susceptible prey.	$\text{population}^{-1} \cdot \text{time}^{-1}$
$\delta_2 > 0,$	Rates of toxicity affecting infected prey	$\text{population}^{-1} \cdot \text{time}^{-1}$
$\delta_3 > 0,$	Rates of toxicity affecting susceptible predators.	time^{-1}
$\delta_4 > 0,$	Rates of toxicity affecting infected predator.	time^{-1}
$d_i > 0, i = 1, 2, 3, 4.$	Natural death rates for the prey and predator populations, respectively.	time^{-1}
$\rho_1 > 0,$	Cooperative hunting coefficients affecting prey.	$\text{population}^{-2} \cdot \text{time}^{-1}$
$\rho_2 > 0,$	Cooperative hunting coefficients affecting predators.	$\text{population}^{-2} \cdot \text{time}^{-1}$
$b_1 > 0,$	Half-saturation constant for predation on susceptible prey.	population
$b_2 > 0,$	Half-saturation constant for predator interference.	population
$a_1 > 0,$	Maximum harvesting rate constant.	population
$a_2 > 0,$	Harvesting the half-saturation constant coefficient.	dimensionless
$q > 0,$	Harvesting effort.	$\text{population} \cdot \text{time}^{-1}$
$E > 0,$	Harvesting effort intensity.	dimensionless
$f > 0,$	Fear effect coefficient.	population^{-1}
$L > 0,$	Environment carrying capacity for prey.	population
$A > 0,$	Environmental constant in predator functional response.	population
$\beta_1 > 0.$	Recovery rate of infected prey populations.	time^{-1}

Introducing suitable dimensionless variables and parameters, the system (1) can be rewritten in the following dimensionless form:

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 \left[(1 - x_1 - x_2) \left(\frac{1}{1+f^*x_3} \right) - \alpha_1^*x_2 - \rho_1^*x_3^2 - \delta_1^*x_1 - \frac{S}{1+kx_1} - d_1^* - \frac{\alpha_3^*x_3}{b_1^*+x_1} \right] + \beta_1^*x_2, \\ \frac{dx_2}{dt} &= x_2 [\alpha_1^*x_1 - \delta_2^*x_2 - (\beta_1^* + d_2^*)], \\ \frac{dx_3}{dt} &= x_3 \left[r \left(1 - \frac{\alpha_4^*x_3}{b_2^*+x_1} \right) \left(\frac{x_3}{x_3+1} \right) + \rho_2^*x_1x_3 - \alpha_2^*x_4 - (\delta_3^* + d_3^*) \right], \\ \frac{dx_4}{dt} &= x_4 [\alpha_2^*x_3 - (\delta_4^* + d_4^*)],\end{aligned}\tag{2}$$

where:

$$\begin{aligned}T &= m_1 t, \quad x_1 = \frac{\mu_1 S}{L}, \quad x_2 = \frac{\mu_1 I}{L}, \quad x_3 = \frac{\mu_2 S}{L}, \quad x_4 = \frac{\mu_2 I}{L}, \quad f^* = fL, \quad \alpha_i^* = \frac{\alpha_i L}{m_1}, \quad k = \frac{a_2 L}{a_1 E}, \\ S &= \frac{q}{m_1 a_1}, \quad r = \frac{m_2}{m_1}, \quad \delta_i^* = \frac{\delta_i L}{m_1}, \quad \rho_i^* = \frac{\rho_i L^2}{m_1}, \quad d_j^* = \frac{d_j}{m_1}, \quad b_i^* = \frac{b_i}{L}, \quad \beta_1^* = \frac{\beta_1}{m_1}, \quad \alpha_3^* = \frac{\alpha_3}{m_1}, \\ \alpha_4^* &= \alpha_4, \quad \delta_k^* = \frac{\delta_k}{m_1}, \quad A^* = \frac{A}{L}.\end{aligned}$$

where $i = 1, 2, j = 1, 2, 3, 4$, and $k = 3, 4$.

For simplicity, we assume, $A = L$, which results in the dimensionless term $\left(\frac{x_3}{x_3+1} \right)$, with the initial condition, $x_1(0) \geq 0, x_2(0) \geq 0, x_3(0) \geq 0$, and $x_4(0) \geq 0$.

3. LOCAL BIFURCATION ANALYSIS

In this section, we investigate the local bifurcation (LB) behavior of model (2) with particular emphasis on variations near equilibrium points resulting from changes in system parameters. Using Sotomayor's theorem, we establish higher-order conditions ensuring the emergence of the most common types of local bifurcations.

The Jacobian matrix of the system (2) which is given in [30], is

$$J = [\psi_{ij}]_{4 \times 4},\tag{3}$$

where:

$$\begin{aligned}\psi_{11} &= (1 - 2x_1 - x_2) \left(\frac{1}{1+f^*x_3} \right) - \alpha_1^*x_2 - \rho_1^*x_3^2 - 2\delta_1^*x_1 - \frac{S}{(1+kx_1)^2} - d_1^* - \frac{\alpha_3^*b_1^*x_3}{(b_1^*+x_1)^2}, \\ \psi_{12} &= \left(\frac{-x_1}{1+f^*x_3} \right) - \alpha_1^*x_1 + \beta_1^*, \quad \psi_{13} = - \left(\frac{f^*x_1(1-x_1-x_2)}{(1+f^*x_3)^2} \right) - 2\rho_1^*x_1x_3 - \frac{\alpha_3^*x_1}{b_1^*+x_1}, \quad \psi_{14} = 0, \\ \psi_{21} &= \alpha_1^*x_2, \quad \psi_{22} = \alpha_1^*x_1 - 2\delta_2^*x_2 - (\beta_1^* + d_2^*), \quad \psi_{23} = \psi_{24} = 0, \quad \psi_{31} = \\ &\frac{r\alpha_4^*x_3^3}{(b_2^*+x_1)^2(x_3+1)} + \rho_2^*x_3^2,\end{aligned}$$

$$\begin{aligned}\psi_{32} &= 0, \psi_{33} = r \left(\left(1 - \frac{2\alpha_4^* x_3}{b_2^* + x_1} \right) \left(\frac{x_3}{x_3 + 1} \right) + x_3 \left(1 - \frac{\alpha_4^* x_3}{b_2^* + x_1} \right) \frac{1}{(x_3 + 1)^2} \right) + 2\rho_2^* x_1 x_3 - \\ &(\delta_3^* + \alpha_2^* x_4 + d_3^*), \\ \psi_{34} &= -\alpha_2^* x_3, \psi_{41} = \psi_{42} = 0, \psi_{43} = \alpha_2^* x_4, \psi_{44} = \alpha_2^* x_3 - (\delta_4^* + d_4^*).\end{aligned}$$

It can be a verity for any nonzero vector that $P = (p_1, p_2, p_3, p_4)^T$ we have

$$D^2 f(X, \mu)(P, P) = [M_{i1}]_{4 \times 1}, \quad (4)$$

where:

$$\begin{aligned}M_{11} &= A(x) p_1^2 + B(x) p_1 p_2 + C(x) p_1 p_3, \\ M_{21} &= 2\alpha_1^* p_1 p_2 - 2\delta_2^* p_2^2, \\ M_{31} &= D(x) p_1^2 + E(x) p_1 p_3 + F(x) p_3^2 - \alpha_2^* p_3 p_4, \\ M_{41} &= 2\alpha_2^* p_3 p_4.\end{aligned}$$

where:

$$\begin{aligned}A(x) &= \frac{-2}{(1+f^* x_3)} - 2\delta_1^* + \frac{2Sk}{(1+kx_1)^3} + \frac{2\alpha_3^* b_1^* x_3}{(b_1^* + x_1)^3} - \alpha_1^* - \frac{1}{(1+f^* x_3)} + \frac{-f^*(1-2x_1-x_2)}{(1+f^* x_3)^2} - \\ &2\rho_1^* x_3 - \frac{\alpha_3^* b_1^*}{(b_1^* + x_1)^2}, \\ B(x) &= -\alpha_1^* - \frac{1}{(1+f^* x_3)} + \frac{f^* x_1}{(1+f^* x_3)^2}, \\ C(x) &= \frac{-f^*(1-3x_1-x_2)}{(1+f^* x_3)^2} - \frac{\alpha_3^* b_1^*}{(b_1^* + x_1)^2} - 2\rho_1^* x_1 - 2\rho_1^* x_3 + \frac{2f^{*2} x_1 (1-x_1-x_2)}{(1+f^* x_3)^3}, \\ D(x) &= \frac{-2r\alpha_4^* x_3^3}{(b_2^* + x_1)^3 (x_3 + 1)}, \\ E(x) &= 4\rho_2^* x_3 - \frac{2r\alpha_4^* x_3^3}{(b_2^* + x_1)^3 (x_3 + 1)} - \frac{2r\alpha_4^* x_3^3}{(b_2^* + x_1)^2 (x_3 + 1)^3} - \frac{r\alpha_4^* x_3^3}{(b_2^* + x_1)^2 (x_3 + 1)^2} + \frac{3r\alpha_4^* x_3^2}{(b_2^* + x_1)^2 (x_3 + 1)}, \\ F(x) &= 2\rho_2^* x_1 - \frac{2r}{(x_3 + 1)^3} - \frac{6r\alpha_4^* x_3}{(b_2^* + x_1)(x_3 + 1)^2},\end{aligned}$$

$$\text{and, } D^3 f_\mu(X, \mu)(P, P, P) = [N_{i1}]_{4 \times 1}, \quad (5)$$

where:

$$\begin{aligned}N_{11} &= A_{x_1} p_1^3 + (A_{x_2} + B_{x_1}) p_1^2 p_2 + (A_{x_3} + C_{x_1}) p_1^2 p_3 + (B_{x_3} + C_{x_2}) p_1 p_2 p_3 + C_{x_3} p_1 p_3^2, \\ A_{x_1} &= \frac{-6Sk^2}{(1+kx_1)^4} - \frac{6\alpha_3^* b_1^* x_3}{(b_1^* + x_1)^4} + \frac{2\alpha_3^* b_1^*}{(b_1^* + x_1)^3} + \frac{2f^*}{(1+f^* x_3)^2}, \quad A_{x_2} = \frac{f^*}{(1+f^* x_3)^2}, \quad B_{x_1} = \frac{f^*}{(1+f^* x_3)^2}, \\ A_{x_3} &= \frac{3f^*}{(1+f^* x_3)^2} + \frac{2f^{*2}(1-2x_1-x_2)}{(1+f^* x_3)^3} - 2\rho_1^* + \frac{2\alpha_3^* b_1^*}{(b_1^* + x_1)^3},\end{aligned}$$

$$C_{x_1} = \frac{3f^*}{(1+f^*x_3)^2} + \frac{2f^{*2}(1-2x_1-x_2)}{(1+f^*x_3)^3} - 2\rho_1^* + \frac{2\alpha_3^*b_1^*}{(b_1^*+x_1)^3},$$

$$B_{x_3} = \frac{f^*}{(1+f^*x_3)^2} - \frac{2f^{*2}x_1}{(1+f^*x_3)^3}, \quad C_{x_2} = \frac{f^*}{(1+f^*x_3)^2} - \frac{2f^{*2}x_1}{(1+f^*x_3)^3},$$

$$C_{x_3} = \frac{2f^{*2}(1-3x_1-x_2)}{(1+f^*x_3)^3} - \frac{6f^{*3}(1-x_1-x_2)}{(1+f^*x_3)^4} - 2\rho_1^*,$$

$$N_{21} = 0,$$

$$N_{31} = D_{x_1}p_1^3 + (D_{x_3} + E_{x_1})p_1^2p_3 + (E_{x_3} + F_{x_1})p_1p_3^2 + F_{x_3}p_3^3,$$

$$D_{x_1} = \frac{6r\alpha_4^*x_3^3}{(b_2^*+x_1)^4(x_3+1)},$$

$$D_{x_3} = -\frac{2r\alpha_4^*x_3^2(2x_3+3)}{(b_2^*+x_1)^3(x_3+1)^2},$$

$$E_{x_1} = \frac{6r\alpha_4^*x_3^3}{(b_2^*+x_1)^4(x_3+1)} + \frac{4r\alpha_4^*x_3^3}{(b_2^*+x_1)^3(x_3+1)^3} + \frac{2r\alpha_4^*x_3^3}{(b_2^*+x_1)^3(x_3+1)^2} - \frac{6r\alpha_4^*x_3^2}{(b_2^*+x_1)^3(x_3+1)},$$

$$E_{x_3} = 4\rho_2^* - \frac{2r\alpha_4^*x_3^2(2x_3+3)}{(b_2^*+x_1)^3(x_3+1)^2} - \frac{6r\alpha_4^*x_3^2}{(b_2^*+x_1)^2(x_3+1)^4} - \frac{r\alpha_4^*(x_3+3)}{(b_2^*+x_1)^2(x_3+1)^3} + \frac{3r\alpha_4^*(x_3+2)}{(b_2^*+x_1)^2(x_3+1)^2},$$

$$F_{x_1} = 2\rho_2^* + \frac{6r\alpha_4^*x_3}{(b_2^*+x_1)^2(x_3+1)^2},$$

$$F_{x_3} = \frac{6r}{(x_3+1)^4} + \frac{6r\alpha_4^*(x_3-1)}{(b_2^*+x_1)(x_3+1)^3},$$

$$N_{41} = 0.$$

Theorem 1: If a parameter S goes through the bifurcation value $S^\circ = 1 - d_1^*$, with $d_1^* < 1$ then system (2) at $E_0 = (0, 0, 0, 0)$ satisfies the following:

- No saddle-node bifurcation occurs.
- A transcritical bifurcation occurs if:

$$2Sk \neq 2(1 + \delta_1^*) + \alpha_1^* + 1 + f^* + \frac{\alpha_3^*}{b_1^*}, \quad (6)$$

- Pitchfork bifurcation if the condition (6) fails and

$$3Sk^2 \neq f^* + \frac{\alpha_3^*}{b_1^{*2}}, \quad (7)$$

passes through S° .

Proof: By substituting $E_0 = (0, 0, 0, 0)$ into the Jacobian matrix (JM) given in equation (11) of [30], with the parameter value $((S^\circ = S = 1 - d_1^*))$, we obtain the Jacobian matrix (J_0°) .

The corresponding characteristic equation of (J_0°) admits a zero eigenvalue, denoted by $(\lambda_{0x_1} = 0)$ at $S^\circ = S$.

Let $P^{[0]} = (p_1^{[0]}, p_2^{[0]}, p_3^{[0]}, p_4^{[0]})^T$ be the eigenvector relating with the eigenvalue $(\lambda_{0x_1} = 0)$.

Thus, $(J_0^\circ - \lambda_{0x_1} I) P^{[0]} = 0$, where $J_0^\circ = J(E_0, S^\circ)$.

So, $p_2^{[0]} = p_3^{[0]} = p_4^{[0]} = 0$, and $p_1^{[0]} \neq 0$ real number.

Let $Q^{[0]} = (q_1^{[0]}, q_2^{[0]}, q_3^{[0]}, q_4^{[0]})^T$ be the eigenvector correlated with the eigenvalue $\lambda_{0x_1} = 0$, of the matrix $J_0^{\circ T}$. So, by $(J_0^{\circ T} - \lambda_{0x_1} I) Q^{[0]} = 0$. That gives, $q_2^{[0]} = A^\circ q_1^{[0]}, q_3^{[0]} = q_4^{[0]} = 0$, where $A^\circ = \frac{\beta_1^*}{\beta_1^* + d_2^*}$, and $q_1^{[0]} \neq 0$ real number.

Since, $\frac{\partial f}{\partial S} = f_S(X, S) = \left(\frac{\partial f_1}{\partial S}, \frac{\partial f_2}{\partial S}, \frac{\partial f_3}{\partial S}, \frac{\partial f_4}{\partial S} \right)^T = \left(-\frac{x_1}{(1+kx_1)}, 0, 0, 0 \right)^T$.

Then, $f_S(E_0, S^\circ) = (0, 0, 0, 0)^T$ and so $(Q^{[0]})^T f_S(E_0, S^\circ) = 0$.

According to Sotomayor's theorem, a (SNB) cannot occur at (E_0) . This implies that the first condition required for the existence of a (TB) is fulfilled. Moreover,

$$D f_S(X, S) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Here, $D f_S(X, S)$ denotes the derivative of $f_S(X, S)$ with $X = (x_1, x_2, x_3, x_4)$, Furthermore, it is noted that:

$$D f_S(E_0, S^\circ) P^{[0]} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1^{[0]} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -p_1^{[0]} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

So, we obtain that, $(Q^{[0]})^T [D f_S(E_0, S^\circ) P^{[0]}] = -q_1^{[0]} p_1^{[0]} \neq 0$,

Moreover, by substituting E_0, S° and $P^{[0]}$ in eq.(4) get:

$$D^2 f(E_0, S^\circ)(P^{[0]}, P^{[0]}) = \begin{bmatrix} K_1^\circ (p_1^{[0]})^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{where:}$$

$$K_1^\circ = 2Sk - 2(1 + \delta_1^*) - \alpha_1^* - 1 - f^* - \frac{\alpha_3^*}{b_1^{*2}},$$

Hence, it is obtained that

$$\text{So, } (Q^{[0]})^T D^2 f(E_0, S^\circ)(P^{[0]}, P^{[0]}) = K_1^\circ q_1^{[0]} (p_1^{[0]})^2 \neq 0,$$

Hence, in accordance with condition (6) of the Sotomayor theorem, system (2) exhibits a (TB) at the equilibrium point (E_0) for $(S^\circ = S)$. Otherwise, when condition (6) fails to hold, substituting $(P^{[0]})$ into equation (5) yields:

$$D^3 f(E_0, S^\circ)(P^{[0]}, P^{[0]}, P^{[0]}) = \begin{bmatrix} \Gamma_1^\circ (p_1^{[0]})^3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{where}$$

$$\Gamma_1^\circ = 2(-3Sk^2 + f^* + \frac{\alpha_3^*}{b_1^{*2}}),$$

Hence, it is obtained that by condition (7)

$$(Q^{[0]})^T D^3 f(E_0, S^\circ)(P^{[0]}, P^{[0]}, P^{[0]}) = \Gamma_1^\circ q_1^{[0]} (p_1^{[0]})^3 \neq 0,$$

Therefore, by Sotomayor's theorem for the local bifurcation model (2) has a (PB) bifurcation at E_0 and the parameter $S^\circ = S$. Hence, the proof is complete.

Theorem 2: For the parameter $\hat{\alpha}_1^* = \alpha_1^* = \frac{\beta_1^* + d_2^*}{\hat{x}_1}$, system (2) exhibits a (TB) at the (EP) $E_1 = (\hat{x}_1, 0, 0, 0)$. Nonetheless, neither a (SNB) nor a (PB) can transpire at E_1 .

Proof: By substituting $E_1 = (\hat{x}_1, 0, 0, 0)$ into the (JM) in equation (12) of reference [30] with the parameter value $\hat{\alpha}_1^* = \alpha_1^*$.

Then, the characteristic equation of $\hat{J}_1$ where $\hat{J}_1 = J_1(E_1, \hat{\alpha}_1^*)$, possesses an eigenvalue zero, eigenvalue $(\lambda_{1x_2} = 0)$ at $\hat{\alpha}_1^* = \alpha_1^*$.

Now, let $\hat{P}^{[1]} = (\hat{p}_1^{[1]}, \hat{p}_2^{[1]}, \hat{p}_3^{[1]}, \hat{p}_4^{[1]})^T$ be the eigenvector equivalent to an eigenvalue $\lambda_{1x_2} = 0$.

Thus $(\hat{J}_1 - \lambda_{1x_2} I) \hat{P}^{[1]} = 0$, this gives: $\hat{p}_1^{[1]} = \hat{p}_3^{[1]} = \hat{p}_4^{[1]} = 0$, and $\hat{p}_2^{[1]} \neq 0$ real number.

Therefore, $\hat{Q}^{[1]} = (\hat{q}_1^{[1]}, \hat{q}_2^{[1]}, \hat{q}_3^{[1]}, \hat{q}_4^{[1]})^T$ be the eigenvector related with the eigenvalue $\lambda_{1x_2} = 0$ of the matrix $\hat{J}_1^T$. Then we have:

$(\hat{J}_1^T - \lambda_{1x_2} I) \hat{Q}^{[1]} = 0$. Then, $\hat{q}_1^{[1]} = \hat{q}_3^{[1]} = \hat{q}_4^{[1]} = 0$, and $\hat{q}_2^{[1]} \neq 0$, a real number.

Now, consider:

$$\frac{\partial f}{\partial \alpha_1^*} = f_{\alpha_1^*}(X, \alpha_1^*) = \left(\frac{\partial f_1}{\partial \alpha_1^*}, \frac{\partial f_2}{\partial \alpha_1^*}, \frac{\partial f_3}{\partial \alpha_1^*}, \frac{\partial f_4}{\partial \alpha_1^*} \right)^T = (-x_1 x_2, x_1 x_2, 0, 0)^T.$$

So, $f_{\alpha_1^*}(E_1, \hat{\alpha}_1^*) = (0, 0, 0, 0)^T$, and so $(\hat{Q}^{[1]})^T f_{\alpha_1^*}(E_1, \hat{\alpha}_1^*) = 0$,

This confirms that no Saddle-Node bifurcation occurs at $\hat{\alpha}_1^* = \alpha_1^*$. Additionally, it is getting that

Now, since
$$Df_{\alpha_1^*}(X, \hat{\alpha}_1^*) = \begin{bmatrix} -x_2 & -x_1 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Here, $Df_{\alpha_1^*}(X, \hat{\alpha}_1^*)$ denotes the derivative of $f_{\alpha_1^*}(X, \alpha_1^*)$ with $X = (x_1, x_2, x_3, x_4)$, Furthermore, it is noted that:

$$Df_{\alpha_1^*}(E_1, \hat{\alpha}_1^*) P^{[1]} = \begin{bmatrix} 0 & -\hat{x}_1 & 0 & 0 \\ 0 & \hat{x}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \hat{p}_2^{[1]} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\hat{x}_1 \hat{p}_2^{[1]} \\ \hat{x}_1 \hat{p}_2^{[1]} \\ 0 \\ 0 \end{bmatrix},$$

So, we obtain that, $(\hat{Q}^{[1]})^T [Df_{\alpha_1^*}(E_1, \hat{\alpha}_1^*) P^{[1]}] = \hat{x}_1 \hat{p}_2^{[1]} \hat{q}_2^{[1]} \neq 0$,

Moreover, by replacing $\hat{P}^{[1]}$ in eq. (4), we get:

$$D^2 f(E_1, \hat{\alpha}_1^*)(\hat{P}^{[1]}, \hat{P}^{[1]}) = \begin{bmatrix} 0 \\ -2\delta_2^* (\hat{p}_2^{[1]})^2 \\ 0 \\ 0 \end{bmatrix},$$

Hence, it is obtained that: $(\hat{Q}^{[1]})^T D^2 f(E_1, \hat{\alpha}_1^*)(\hat{P}^{[1]}, \hat{P}^{[1]}) = -2\delta_2^* (\hat{p}_2^{[1]})^2 \hat{q}_2^{[1]} \neq 0$.

According to the Sotomayor theorem, system (2) indicates a (TB) at E_1 with $\hat{\alpha}_1^* = \alpha_1^*$, and (PB) cannot transpire.

Theorem (3): For the parameter $\bar{r} = r = \frac{\delta_3^* + d_3^*}{H(\bar{x}_3)}$, $H(\bar{x}_3) \neq 0$ where

$$H(\bar{x}_3) = \frac{\bar{x}_3}{\bar{x}_3 + 1} \left(1 + \frac{1}{\bar{x}_3 + 1} \right) - \frac{\alpha_4^* \bar{x}_3^2}{b_2^* (\bar{x}_3 + 1)} \left(2 + \bar{x}_3 + \frac{1}{\bar{x}_3 + 1} \right), \text{ system (2) exhibits a(SNB) and (TB) at}$$

the (EP) $E_2 = (0, 0, \bar{x}_3, 0)$ satisfies the following:

$$\frac{\bar{x}_3}{\bar{x}_3 + 1} \left(1 + \frac{1}{\bar{x}_3 + 1} \right) > \frac{\alpha_4^* \bar{x}_3^2}{b_2^* (\bar{x}_3 + 1)} \left(2 + \bar{x}_3 + \frac{1}{\bar{x}_3 + 1} \right), \quad (8)$$

$$\frac{\alpha_4^* \bar{x}_3}{b_2^*} \neq 1, \quad (9)$$

while (PB) not present near E_2 at $\bar{r}$.

Proof: by putting $E_2 = (0, 0, \bar{x}_3, 0)$ in the (JM) in equation (13) in [30],

$$\bar{J}_2 = J_2(E_2, \bar{r}) = [\bar{b}_{ij}]_{4 \times 4}, \text{ where } \bar{b}_{ij} = b_{ij} \text{ except } \bar{b}_{33} = 0.$$

Then, the characteristic equation of $\bar{J}_2$ where $\bar{J}_2 = J_2(E_2, \bar{r})$, possesses an eigenvalue zero, eigenvalue $(\lambda_{2x_3} = 0)$ at $\bar{r} = \frac{\delta_3^* + d_3^*}{H(\bar{x}_3)}$, $H(\bar{x}_3) > 0$.

Now, let $\bar{P}^{[2]} = (\bar{p}_1^{[2]}, \bar{p}_2^{[2]}, \bar{p}_3^{[2]}, \bar{p}_4^{[2]})^T$ be the eigenvector equivalent to an eigenvalue $\lambda_{2x_3} = 0$

Thus $(\bar{J}_2 - \lambda_{2x_3} I) \bar{P}^{[2]} = 0$, then $\bar{p}_1^{[2]} = \bar{p}_2^{[2]} = \bar{p}_4^{[2]} = 0$, and $\bar{p}_3^{[2]} \neq 0$ real number.

Let $\bar{Q}^{[2]} = (\bar{q}_1^{[2]}, \bar{q}_2^{[2]}, \bar{q}_3^{[2]}, \bar{q}_4^{[2]})^T$ be the eigenvector related with the eigenvalue $\lambda_{2x_3} = 0$ of the

matrix $\bar{J}_2^T$. Then we have: $(\bar{J}_2^T - \lambda_{2x_3} I) \bar{Q}^{[2]} = 0$. which gives,

$$\bar{q}_1^{[2]} = \bar{A}_1 \bar{q}_3^{[2]}, \bar{q}_2^{[2]} = \bar{A}_2 \bar{q}_3^{[2]}, \bar{q}_4^{[2]} = \bar{A}_4 \bar{q}_3^{[2]}, \quad \text{where} \quad \bar{A}_1 = -\frac{\bar{b}_{31}}{\bar{b}_{11}}, \bar{A}_2 = \frac{\bar{b}_{12} \bar{b}_{31}}{\bar{b}_{11} \bar{b}_{22}}, \bar{A}_4 = -\frac{\bar{b}_{34}}{\bar{b}_{44}}$$

and $\bar{q}_3^{[2]} \neq 0$ real number.

Now, consider:

$$\frac{\partial f}{\partial r} = f_r(X, \bar{r}) = \left(\frac{\partial f_1}{\partial r}, \frac{\partial f_2}{\partial r}, \frac{\partial f_3}{\partial r}, \frac{\partial f_4}{\partial r} \right)^T = \left(0, 0, \left(1 - \frac{\alpha_4^* x_3}{b_2^* + x_1} \frac{x_3^2}{x_3 + 1} \right), 0 \right)^T.$$

$$\text{So, } f_r(E_2, \bar{r}) = \left(0, 0, \left(1 - \frac{\alpha_4^* x_3}{b_2^*} \frac{x_3^2}{x_3 + 1} \right), 0 \right)^T,$$

$$\text{Therefore } (\bar{Q}^{[2]})^T f_r(E_2, \bar{r}) = \left(1 - \frac{\alpha_4^* x_3}{b_2^*} \right) \frac{x_3^2}{x_3 + 1} \bar{q}_3^{[2]} \neq 0.$$

Hence, in view of condition (9) of the Sotomayor theorem, system (2) exhibits a saddle-node bifurcation (SNB) at the equilibrium point (E_2) for $(\bar{r} = r)$. However, when condition (9) fails to hold, the application of $(\bar{P}^{[2]})$ in equation (4) yields

$$D^2 f(E_2, \bar{r})(\bar{P}^{[2]}, \bar{P}^{[2]}) = \begin{bmatrix} 0 \\ 0 \\ \bar{Q}_2 (\bar{p}_3^{[2]})^2 \\ 0 \end{bmatrix},$$

$$\text{where } \bar{Q}_2 = \frac{-2\bar{r}}{(\bar{x}_3 + 1)^2} \left(\frac{1}{(x_3 + 1)} + \frac{3\alpha_4^* \bar{x}_3}{b_2^*} \right),$$

$$\text{Hence, it is obtained that: } (Q^{[2]})^T D^2 f(E_2, \bar{r})(\bar{P}^{[2]}, \bar{P}^{[2]}) = -2 \bar{Q}_2 \bar{q}_3^{[2]} (\bar{p}_3^{[2]})^2 \neq 0,$$

Therefore, by Sotomayor's theorem for the local bifurcation model, (2) has a (TB) bifurcation at E_2 and the parameter $\bar{r} = r$. Hence, the proof is complete. Similarly, for $E_3 = (0, 0, \bar{x}_3^\circ, 0)$.

Theorem 4: Assume that condition (14.3) and (14.4) which are given in [30] and the next conditions hold.

$$2\ddot{x}_1 + \ddot{x}_2 < 1, \tag{10}$$

$$(1 - 2\ddot{x}_1 - \ddot{x}_2) + \frac{\alpha_1^* \ddot{x}_2 ((1 + \alpha_1^*) \ddot{x}_1 - \beta_1^*)}{\alpha_1^* \ddot{x}_1 - 2\delta_2^* \ddot{x}_2 - (\beta_1^* + d_2^*)} > \alpha_1^* \ddot{x}_2 + 2\delta_1^* \ddot{x}_1 \frac{s}{(1 + k\ddot{x}_1)^2}, \tag{11}$$

$$\frac{Sk}{(1 + kx_1)^3} \neq 1 + \delta_1^* + \alpha_1^* + 1 + f^*(1 - 2x_1 - x_2) + \frac{\alpha_3^* b_1^*}{(b_1^* + x_1)^2}, \tag{12}$$

$$\frac{3Sk^2}{(1 + kx_1)^4} \neq \frac{\alpha_3^* b_1^*}{(b_1^* + x_1)^3} + f^*, \tag{13}$$

are hold. then system (2) near E_4 exhibits a(SNB) and (TB) at the parameter value

$$(\ddot{d}_1^* = d_1^* = (1 - 2\ddot{x}_1 - \ddot{x}_2) - \alpha_1^* \ddot{x}_2 - 2\delta_1^* \ddot{x}_1 - \frac{s}{(1 + k\ddot{x}_1)^2} + \frac{\alpha_1^* \ddot{x}_2 ((1 + \alpha_1^*) \ddot{x}_1 - \beta_1^*)}{\alpha_1^* \ddot{x}_1 - 2\delta_2^* \ddot{x}_2 - (\beta_1^* + d_2^*)}), \text{ while (PB)}$$

not present near E_4 at $\ddot{d}_1^*$.

Proof: by putting $E_4 = (\ddot{x}_1, \ddot{x}_2, 0, 0)$ in the (JM) in eq. (14) given in [30],

$$\ddot{J}_4 = J_4(E_4, \ddot{d}_1^*) = [\ddot{c}_{ij}]_{4 \times 4}, \text{ where } c_{ij} = \ddot{c}_{ij}, \text{ except } c_{11} = 0,$$

Now, let $\ddot{P}^{[4]} = (\ddot{p}_1^{[4]}, \ddot{p}_2^{[4]}, \ddot{p}_3^{[4]}, \ddot{p}_4^{[4]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{4x_1} = 0$,

Thus $(\ddot{J}_4 - \lambda_{4x_1} I) \ddot{P}^{[4]} = 0$, then, $\ddot{p}_2^{[4]} = \ddot{p}_3^{[4]} = \ddot{p}_4^{[4]} = 0, \ddot{p}_1^{[4]}$ any real number.

Let $\ddot{Q}^{[4]} = (\ddot{q}_1^{[4]}, \ddot{q}_2^{[4]}, \ddot{q}_3^{[4]}, \ddot{q}_4^{[4]})^T$ be the eigenvector related with the eigenvalue $\lambda_{4x_1} = 0$,

$(\ddot{J}_4^T - \lambda_{4x_1} I) \ddot{Q}^{[4]} = 0$, of the matrix $\ddot{J}_4^T$. Then we have:

$$\ddot{q}_3^{[4]} = \frac{\ddot{c}_{13}}{\ddot{c}_{33}}, \ddot{q}_2^{[4]} = \ddot{q}_4^{[4]} = 0, \ddot{q}_1^{[4]} \neq 0 \text{ any real number.}$$

Now, consider:

$$\frac{\partial f}{\partial d_1^*} = f_{d_1^*}(X, \ddot{d}_1^*) = \left(\frac{\partial f_1}{\partial d_1^*}, \frac{\partial f_2}{\partial d_1^*}, \frac{\partial f_3}{\partial d_1^*}, \frac{\partial f_4}{\partial d_1^*} \right)^T = (-x_1, 0, 0, 0)^T.$$

$$\text{So, } f_{d_1^*}(E_4, \ddot{d}_1^*) = (-\ddot{x}_1, 0, 0, 0)^T,$$

$$\text{Therefore } (\ddot{Q}^{[4]})^T f_{d_1^*}(E_4, \ddot{d}_1^*) = -\ddot{x}_1 \ddot{q}_1^{[4]} \neq 0,$$

Now, by replacing $\ddot{P}^{[4]}$ in eq. (4), we get:

$$D^2 f(E_4, \ddot{d}_1^*)(\ddot{P}^{[4]}, \ddot{P}^{[4]}) = \begin{bmatrix} \ddot{r}_1 (\ddot{p}_1^{[4]})^2 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\ddot{r}_1 = -2 \left(1 + \delta_1^* - \frac{Sk}{(1+kx_1)^3} \right) - \alpha_1^* - 1 - f^*(1 - 2x_1 - x_2) - \frac{\alpha_3^* b_1^*}{(b_1^* + x_1)^2},$$

Hence, it is obtained that

$$(\ddot{Q}^{[4]})^T D^2 f(E_4, \ddot{d}_1^*)(\ddot{P}^{[4]}, \ddot{P}^{[4]}) = \ddot{r}_1 (\ddot{p}_1^{[4]})^2 \ddot{q}_1^{[4]} \neq 0,$$

Hence, in accordance with condition (12) of the Sotomayor theorem, system (2) exhibits a saddle-node bifurcation (SNB) at the equilibrium point (E_4) for $(\ddot{d}_1^* = d_1^*)$. Otherwise, when condition (12) fails to hold, the substitution of $(\ddot{P}^{[4]})$ into equation (5) yields:

$$D^3 f(E_4, \check{d}_1^*)(\check{p}^{[4]}, \check{p}^{[4]}, \check{p}^{[4]}) = \begin{bmatrix} \check{K}_1 (\check{p}_1^{[4]})^3 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\check{K}_1 = 2 \left(\frac{-3Sk^2}{(1+kx_1)^4} + \frac{\alpha_3^* b_1^*}{(b_1^* + x_1)^3} + f^* \right),$$

Hence, it is obtained that by condition (13)

$$(Q^{[4]})^T D^3 f(E_4, \check{d}_1^*)(\check{p}^{[4]}, \check{p}^{[4]}, \check{p}^{[4]}) = \check{K}_1 (\check{p}_1^{[4]})^3 \check{q}_1^{[4]} \neq 0.$$

Therefore, by Sotomayor's theorem for the local bifurcation model, (2) has a (PB) bifurcation at E_4 and the parameter $\check{d}_1^* = d_1^*$. Hence, the proof is complete. Similarly, for $E_5 = (\check{x}_1^\circ, \check{x}_2^\circ, 0, 0)$.

Theorem 5: Assume that condition (15.4) – (15.7) which are given in [30] hold. Then system (2) at the equilibrium point $E_6 = (\check{x}_1, 0, \check{x}_3, 0)$ undergoes only a (TB) at the parameter value $(\check{\alpha}_2^* = \alpha_2^* = \frac{\delta_4^* + d_4^*}{\check{x}_3})$. In contrast, neither a (SNB) nor a (PB) can occur at this parameter value.

Proof: By putting $E_6 = (\check{x}_1, 0, \check{x}_3, 0)$ in the (JM) in eq. (15) given in [30],

$$\check{J}_6 = J_6(E_6, \check{\alpha}_2^*) = [\check{d}_{ij}]_{4 \times 4}, \text{ where } \check{d}_{ij} = d_{ij}, \text{ except } \check{d}_{44} = 0,$$

Now, let $\check{P}^{[6]} = (\check{p}_1^{[6]}, \check{p}_2^{[6]}, \check{p}_3^{[6]}, \check{p}_4^{[6]})^T$ be the eigenvector equivalent to an eigenvalue $\lambda_{6x_4} = 0$.

Thus $(\check{J}_6 - \lambda_{6x_4} I) \check{P}^{[6]} = 0$, Then, $\check{p}_1^{[6]} = \check{A}_1 \check{p}_4^{[6]}, \check{p}_2^{[6]} = 0, \check{p}_3^{[6]} = \check{A}_3 \check{p}_4^{[6]}$, where:

$$\check{A}_1 = \frac{\check{d}_{13} \check{d}_{34}}{\check{d}_{11}(\check{d}_{11} \check{d}_{33} - \check{d}_{13} \check{d}_{31})}, \check{A}_3 = \frac{-\check{d}_{34}}{(\check{d}_{11} \check{d}_{33} - \check{d}_{13} \check{d}_{31})}, \text{ and } \check{p}_4^{[6]} \neq 0, \text{ real number.}$$

Let $\check{Q}^{[6]} = (\check{q}_1^{[6]}, \check{q}_2^{[6]}, \check{q}_3^{[6]}, \check{q}_4^{[6]})^T$ be the eigenvector related with the eigenvalue $\lambda_{6x_4} = 0$, of the matrix $\check{J}_6^T$. Then we have:

$$(\check{J}_6^T - \lambda_{6x_4} I) \check{Q}^{[6]} = 0. \text{ Then, } \check{q}_1^{[6]} = \check{q}_2^{[6]} = \check{q}_3^{[6]} = 0, \text{ and } \check{q}_4^{[6]} \neq 0, \text{ real number.}$$

Now, consider:

$$\frac{\partial f}{\partial \alpha_2^*} = f_{\alpha_2^*}(X, \check{\alpha}_2^*) = \left(\frac{\partial f_1}{\partial \alpha_2^*}, \frac{\partial f_2}{\partial \alpha_2^*}, \frac{\partial f_3}{\partial \alpha_2^*}, \frac{\partial f_4}{\partial \alpha_2^*} \right)^T = (0, 0, -x_3 x_4, x_3 x_4)^T.$$

So, $f_{\alpha_2^*}(E_6, \check{\alpha}_2^*) = (0, 0, 0, 0)^T$. Therefore $(\check{Q}^{[6]})^T f_{\alpha_2^*}(E_6, \check{\alpha}_2^*) = 0$,

Therefore, by using Sotomayor's theorem the (SNB) condition can not satisfy at E_6 . But the first condition of (TB) is verified, as below,

$$Df_{\alpha_2^*}(X, \check{\alpha}_2^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x_4 & -x_3 \\ 0 & 0 & x_4 & x_3 \end{bmatrix},$$

Here, $Df_{\alpha_2^*}(X, \check{\alpha}_2^*)$ denotes the derivative of $f_{\alpha_2^*}(X, \check{\alpha}_2^*)$ with $X = (x_1, x_2, x_3, x_4)$

furthermore, it is noted that:

$$Df_{\alpha_2^*}(E_6, \check{\alpha}_2^*)\check{P}^{[6]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\check{x}_3 \\ 0 & 0 & 0 & \check{x}_3 \end{bmatrix} \begin{bmatrix} \check{A}_1\check{p}_4^{[6]} \\ 0 \\ \check{A}_3\check{p}_4^{[6]} \\ \check{p}_4^{[6]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\check{x}_3\check{p}_4^{[6]} \\ \check{x}_3\check{p}_4^{[6]} \end{bmatrix},$$

So, we obtain that, $(\check{Q}^{[6]})^T [Df_{\alpha_2^*}(E_6, \check{\alpha}_2^*)\check{P}^{[6]}] = \check{x}_3\check{p}_4^{[6]}\check{q}_4^{[6]} \neq 0$.

Moreover, by replacing $\check{P}^{[6]}$ in (4), we get:

$$D^2f(E_6, \check{\alpha}_2^*)(\check{P}^{[6]}, \check{P}^{[6]}) = \begin{bmatrix} \check{K}_1 (\check{p}_4^{[6]})^2 \\ 0 \\ \check{K}_3 (\check{p}_4^{[6]})^2 \\ \alpha_2^* \check{A}_3 (\check{p}_4^{[6]})^2 \end{bmatrix}, \text{ where}$$

$\check{K}_1 = (A(\check{x}_1, \check{x}_3)(\check{A}_1)^2 - C(\check{x}_1, \check{x}_3)\check{A}_1\check{A}_3)$, where

$$\begin{aligned} A(\check{x}_1, \check{x}_3) &= \frac{-2}{(1+f^*\check{x}_3)} - 2\delta_1^* + \frac{2Sk}{(1+k\check{x}_1)^3} + \frac{2\alpha_3^*b_1^*\check{x}_3}{(b_1^*+\check{x}_1)^3} - \alpha_1^* - \frac{1}{(1+f^*\check{x}_3)} + \frac{-f^*(1-2\check{x}_1)}{(1+f^*\check{x}_3)^2} - \\ &2\rho_1^*\check{x}_3 - \frac{\alpha_3^*b_1^*}{(b_1^*+\check{x}_1)^2}, \\ C(\check{x}_1, \check{x}_3) &= \frac{-f^*(1-3\check{x}_1)}{(1+f^*\check{x}_3)^2} - \frac{\alpha_3^*b_1^*}{(b_1^*+\check{x}_1)^2} - 2\rho_1^*\check{x}_1 - 2\rho_1^*\check{x}_3 + \frac{2f^{*2}\check{x}_1(1-\check{x}_1)}{(1+f^*\check{x}_3)^3}, \end{aligned}$$

$$\tilde{K}_3 = \left(D(\tilde{x}_1, \tilde{x}_3)(\tilde{A}_1)^2 - E(\tilde{x}_1, \tilde{x}_3)\tilde{A}_1\tilde{A}_3 + F(\tilde{x}_1, \tilde{x}_3)(\tilde{A}_3)^2 \right) - \alpha_2^* \tilde{A}_3,$$

where

$$D(\tilde{x}_1, \tilde{x}_3) = \frac{-2r\alpha_4^* \tilde{x}_3^3}{(b_2^* + \tilde{x}_1)^3(\tilde{x}_3 + 1)},$$

$$E(\tilde{x}_1, \tilde{x}_3) = 4\rho_2^* \tilde{x}_3 - \frac{2r\alpha_4^* \tilde{x}_3^3}{(b_2^* + \tilde{x}_1)^3(\tilde{x}_3 + 1)} - \frac{2r\alpha_4^* \tilde{x}_3^3}{(b_2^* + \tilde{x}_1)^2(\tilde{x}_3 + 1)^3} - \frac{r\alpha_4^* \tilde{x}_3^3}{(b_2^* + \tilde{x}_1)^2(\tilde{x}_3 + 1)^2} + \frac{3r\alpha_4^* \tilde{x}_3^2}{(b_2^* + \tilde{x}_1)^2(\tilde{x}_3 + 1)},$$

$$F(\tilde{x}_1, \tilde{x}_3) = 2\rho_2^* \tilde{x}_1 - \frac{2r}{(\tilde{x}_3 + 1)^3} - \frac{6r\alpha_4^* \tilde{x}_3}{(b_2^* + \tilde{x}_1)(\tilde{x}_3 + 1)^2},$$

Hence, it is obtained that:

$$(\tilde{Q}^{[6]})^T D^2 f(E_6, \tilde{\alpha}_2^*)(\tilde{P}^{[6]}, \tilde{P}^{[6]}) = \alpha_2^* \tilde{A}_3 (\tilde{p}_4^{[6]})^2 \tilde{q}_4^{[6]} \neq 0.$$

Hence, by the Sotomayor theorem, system (2) exhibits a (TB) at the equilibrium point (E_6) for $(\tilde{\alpha}_2^* = \alpha_2^*)$ under conditions (15.4) – (15.7). Similarly, for $E_7 = (\tilde{x}_1, 0, \tilde{x}_3^{(2)}, 0)$.

Theorem 6: Assume that conditions (16.4) and (16.5) given in [30] hold, while condition (16.6) is reversed, together with the following additional conditions:

$$\frac{\alpha_4^* \tilde{x}_3}{b_2^*} \neq 1, \quad (14)$$

$$2\tilde{\tau}_3 \neq \alpha_2^* \tilde{\eta}_3, \quad (15)$$

then system (2) near $E_8 = (0, 0, \tilde{x}_3, \tilde{x}_4)$ exhibits a (SNB) and (TB) at the parameter value

$$\tilde{r} = r = \frac{\delta_3^* + d_3^* \alpha_2^* x_4}{\left(\left(1 - \frac{2\alpha_4^* x_3}{b_2^*} \right) \left(\frac{x_3}{x_3 + 1} \right) + x_3 \left(1 - \frac{\alpha_4^* x_3}{b_2^*} \right) \frac{1}{(x_3 + 1)^2} \right)},$$

Proof: by putting $E_8 = (0, 0, \tilde{x}_3, \tilde{x}_4)$, in the (JM) in equation (16) given in [30],

$$\tilde{J}_8 = J_8(E_8, \tilde{r}) = [\tilde{e}_{ij}]_{4 \times 4}, \text{ where } e_{ij} = \tilde{e}_{ij}, \text{ except } e_{33} = 0,$$

Now, let $\tilde{P}^{[8]} = (\tilde{p}_1^{[8]}, \tilde{p}_2^{[8]}, \tilde{p}_3^{[8]}, \tilde{p}_4^{[8]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{8x_3} = 0$,

Thus $(\tilde{J}_8 - \lambda_{8x_3} I) \tilde{P}^{[8]} = 0$, Then, $\tilde{p}_1^{[8]} = \tilde{p}_2^{[8]} = 0$, $\tilde{p}_4^{[8]} = \frac{-\tilde{e}_{33}}{\tilde{e}_{34}} \tilde{p}_3^{[8]}$, $\tilde{p}_3^{[8]} \neq 0$, real number.

Let $\tilde{Q}^{[8]} = (\tilde{q}_1^{[8]}, \tilde{q}_2^{[8]}, \tilde{q}_3^{[8]}, \tilde{q}_4^{[8]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{8x_3} = 0$,

$(\tilde{J}_8^T - \lambda_{8x_3} I) \tilde{Q}^{[8]} = 0$, of the matrix $\tilde{J}_8^T$. Then $\tilde{q}_1^{[8]} = \tilde{\eta}_1 \tilde{q}_3^{[8]}$, $\tilde{q}_2^{[8]} = \tilde{\eta}_2 \tilde{q}_3^{[8]}$, $\tilde{q}_4^{[8]} = \tilde{\eta}_3 \tilde{q}_3^{[8]}$,

where: $\tilde{\eta}_1 = -\frac{\tilde{e}_{31}}{\tilde{e}_{11}}$, $\tilde{\eta}_2 = \frac{\tilde{e}_{12}\tilde{e}_{31}}{\tilde{e}_{11}\tilde{e}_{22}}$, $\tilde{\eta}_3 = -\frac{\tilde{e}_{34}}{\tilde{e}_{44}}$, and $\tilde{q}_3^{[8]} \neq 0$, real number.

Now, consider:

$$\frac{\partial f}{\partial r} = f_r(X, \tilde{r}) = \left(\frac{\partial f_1}{\partial r}, \frac{\partial f_2}{\partial r}, \frac{\partial f_3}{\partial r}, \frac{\partial f_4}{\partial r} \right)^T = \left(0, 0, \left(1 - \frac{\alpha_4^* \tilde{x}_3}{b_2^* + x_1} \right) \left(\frac{\tilde{x}_3}{\tilde{x}_3 + 1} \right), 0 \right)^T.$$

$$\text{So, } f_r(E_8, \tilde{r}) = \left(0, 0, \left(1 - \frac{\alpha_4^* \tilde{x}_3}{b_2^*} \right) \frac{\tilde{x}_3}{\tilde{x}_3 + 1}, 0 \right)^T,$$

Therefore

$$(\tilde{Q}^{[8]})^T f_r(E_8, \tilde{r}) = \left(1 - \frac{\alpha_4^* \tilde{x}_3}{b_2^*} \right) \left(\frac{\tilde{x}_3}{\tilde{x}_3 + 1} \right) \tilde{q}_3^{[8]} \neq 0,$$

Hence, in accordance with condition (14) of the Sotomayor theorem, system (2) exhibits a (SNB) at the equilibrium point (E_8) for ($\tilde{r} = r$). Otherwise, when condition (14) fails to hold, the application of ($\tilde{P}^{[8]}$) to equation (4) yields:

$$D^2 f(E_8, \tilde{r})(\tilde{P}^{[8]}, \tilde{P}^{[8]}) = \begin{bmatrix} 0 \\ 0 \\ -(2\tilde{\tau}_3 + \alpha_2^* \tilde{\eta}_3) (\tilde{p}_3^{[8]})^2 \\ 2\alpha_2^* \tilde{\eta}_3 (\tilde{p}_3^{[8]})^2 \end{bmatrix}, \text{ where}$$

$$\tilde{\tau}_3 = \left(\frac{r}{(\tilde{x}_3 + 1)^3} + \frac{3r\alpha_4^* \tilde{x}_3}{b_2^* (\tilde{x}_3 + 1)^2} \right),$$

Hence, it is obtained that:

$$(\tilde{Q}^{[8]})^T D^2 f(E_8, \tilde{r})(\tilde{P}^{[8]}, \tilde{P}^{[8]}) = (-2\tilde{\tau}_3 + \alpha_2^* \tilde{\eta}_3) (\tilde{p}_3^{[8]})^2 \tilde{q}_3^{[8]} \neq 0,$$

On the other hand, under the reversed form of condition (15) in the Sotomayor theorem, system (2) exhibits a (TB) at the equilibrium point (E_8) for ($\tilde{r} = r$). Furthermore, when condition (15) holds, substituting ($\tilde{P}^{[8]}$) into equation (5) yields:

$$D^3 f(E_8, \tilde{r})(\tilde{P}^{[8]}, \tilde{P}^{[8]}, \tilde{P}^{[8]}) = \begin{bmatrix} 0 \\ 0 \\ \tilde{K}_3 (\tilde{p}_3^{[8]})^3 \\ 0 \end{bmatrix}, \text{ where}$$

$$\tilde{K}_3 = 6 \left(\frac{\tilde{r}}{(\tilde{x}_3 + 1)^4} + \frac{\tilde{r}\alpha_4^*}{b_2^* (\tilde{x}_3 + 1)^2} \right),$$

Hence, it is obtained that

$$(Q^{[8]})^T D^3 f(E_8, \tilde{r})(\tilde{P}^{[8]}, \tilde{P}^{[8]}, \tilde{P}^{[8]}) = \tilde{K}_3 \tilde{q}_3^{[8]} (\tilde{p}_3^{[8]})^3 \neq 0.$$

Therefore, by Sotomayor's theorem for the local bifurcation model, (2) has a (PB) bifurcation at

E_8 and the parameter $r = \tilde{r}$. Hence, the proof is complete.

Theorem 7: Assume that condition (17.4), (17.5) and (17.8) which are given in [30] hold. Therefore, system (2) exhibits exclusively a (TB) at the equilibrium point $E_9 = (\bar{x}_1, \bar{x}_2, \bar{x}_3, 0)$ for the critical parameter value $(\bar{\alpha}_2^* = \alpha_2^* = \frac{\delta_4^* + d_4^*}{\bar{x}_3})$. In contrast, neither a (SNB) nor a (PB) is possible at this parameter value.

Proof: By putting $E_9 = (\bar{x}_1, \bar{x}_2, \bar{x}_3, 0)$ in the (JM) in equation (17) given in [30],

$$\bar{J}_9 = J_9(E_9, \bar{\alpha}_2^*) = [\bar{f}_{ij}]_{4 \times 4}, \text{ where } \bar{f}_{ij} = f_{ij}, \text{ except } \bar{f}_{44} = 0,$$

Now, let $\bar{P}^{[9]} = (\bar{p}_1^{[9]}, \bar{p}_2^{[9]}, \bar{p}_3^{[9]}, \bar{p}_4^{[9]})^T$ be the eigenvector equivalent to an eigenvalue $\lambda_{9x_4} = 0$.

Thus $(\bar{J}_9 - \lambda_{9x_4} I) \bar{P}^{[9]} = 0$, Then, $\bar{p}_1^{[9]} = \bar{A}_1$, $\bar{p}_2^{[9]} = \bar{A}_2$, $\bar{p}_3^{[9]} = \bar{A}_3$, , and $\bar{p}_4^{[9]} \neq 0$ real number. where:

$$\bar{A}_1 = -\bar{f}_{13}\bar{f}_{22}, \bar{A}_2 = \bar{f}_{13}\bar{f}_{21}, \bar{A}_3 = \bar{f}_{11}\bar{f}_{22} - \bar{f}_{12}\bar{f}_{21},$$

Let $\bar{Q}^{[9]} = (\bar{q}_1^{[9]}, \bar{q}_2^{[9]}, \bar{q}_3^{[9]}, \bar{q}_4^{[9]})^T$ be the eigenvector related with the eigenvalue $\lambda_{9x_4} = 0$ of the matrix J_9^T . Then we have:

$$(\bar{J}_9^T - \lambda_{9x_4} I) \bar{Q}^{[9]} = 0 \text{ Then, } \bar{q}_1^{[9]} = \bar{q}_2^{[9]} = \bar{q}_3^{[9]} = 0, \text{ and } \bar{q}_4^{[9]} \neq 0 \text{ real number.}$$

Now, consider:

$$\frac{\partial f}{\partial \alpha_2^*} = f_{\alpha_2^*}(X, \bar{\alpha}_2^*) = \left(\frac{\partial f_1}{\partial \alpha_2^*}, \frac{\partial f_2}{\partial \alpha_2^*}, \frac{\partial f_3}{\partial \alpha_2^*}, \frac{\partial f_4}{\partial \alpha_2^*} \right)^T = (0, 0, -x_3 x_4, x_3 x_4)^T.$$

$$\text{So, } f_{\alpha_2^*}(E_9, \bar{\alpha}_2^*) = (0, 0, 0, 0)^T,$$

$$\text{Therefore } (\bar{Q}^{[9]})^T f_{\alpha_2^*}(E_9, \bar{\alpha}_2^*) = 0,$$

Therefore, by using Sotomayor's theorem the (SNB) condition can not satisfy at E_9 . But the first condition of (TB) is verified, as below

$$Df_{\alpha_2^*}(X, \bar{\alpha}_2^*) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x_4 & -x_3 \\ 0 & 0 & x_4 & x_3 \end{bmatrix},$$

Here, $Df_{\alpha_2^*}(X, \bar{\alpha}_2^*)$ denotes the derivative of $f_{\alpha_2^*}(X, \bar{\alpha}_2^*)$ with $X = (x_1, x_2, x_3, x_4)$. Furthermore, it is noted that:

$$Df_{\alpha_2^*}(E_9, \bar{\alpha}_2^*)\bar{P}^{[9]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{x}_3 \\ 0 & 0 & 0 & \bar{x}_3 \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{p}_4^{[9]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\bar{x}_3\bar{p}_4^{[9]} \\ \bar{x}_3\bar{p}_4^{[9]} \end{bmatrix},$$

So, we obtain that, $(\bar{Q}^{[9]})^T [Df_{\alpha_2^*}(E_9, \bar{\alpha}_2^*)\bar{P}^{[9]}] = \bar{x}_3\bar{p}_4^{[9]}\bar{q}_4^{[9]} \neq 0$.

Now, by replacing $\bar{P}^{[9]}$ in (4), we get:

$$D^2f(E_9, \bar{\alpha}_2^*)(\bar{P}^{[9]}, \bar{P}^{[9]}) = \begin{bmatrix} \bar{K}_1 \\ \bar{K}_2 \\ \bar{K}_3 \\ 2\alpha_2^*\bar{A}_3\bar{p}_4^{[9]} \end{bmatrix}. \text{ where:}$$

$$\bar{K}_1 = (A(\bar{x}_1, \bar{x}_3)(\bar{A}_1)^2 + B(\bar{x}_1, \bar{x}_3)\bar{A}_1\bar{A}_2 - C(\bar{x}_1, \bar{x}_3)\bar{A}_1\bar{A}_3),$$

$$\bar{K}_2 = 2\alpha_1^*\bar{A}_1\bar{A}_2 - 2\delta_2^*(\bar{A}_2)^2,$$

$$\bar{K}_3 = D(\bar{x}_1, \bar{x}_3)(\bar{A}_1)^2 - E(\bar{x}_1, \bar{x}_3)\bar{A}_1\bar{A}_3 + F(\bar{x}_1, \bar{x}_3)(\bar{A}_3)^2 - \alpha_2^*\bar{A}_3\bar{p}_4^{[9]},$$

where

$$A(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{-2}{(1+f^*\bar{x}_3)} - 2\delta_1^* + \frac{2Sk}{(1+k\bar{x}_1)^3} + \frac{2\alpha_3^*b_1^*\bar{x}_3}{(b_1^*+\bar{x}_1)^3} - \alpha_1^* - \frac{1}{(1+f^*\bar{x}_3)} + \frac{-f^*(1-\bar{x}_1-\bar{x}_2)}{(1+f^*\bar{x}_3)^2} -$$

$$2\rho_1^*\bar{x}_3 - \frac{\alpha_3^*b_1^*}{(b_1^*+\bar{x}_1)^2},$$

$$B(\bar{x}_1, \bar{x}_3) = -\alpha_1^* - \frac{1}{(1+f^*\bar{x}_3)} + \frac{f^*\bar{x}_1}{(1+f^*\bar{x}_3)^2},$$

$$C(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{-f^*(1-3\bar{x}_1-\bar{x}_2)}{(1+f^*\bar{x}_3)^2} - \frac{\alpha_3^*b_1^*}{(b_1^*+\bar{x}_1)^2} - 2\rho_1^*\bar{x}_1 - 2\rho_1^*\bar{x}_3 + \frac{2f^{*2}\bar{x}_1(1-\bar{x}_1-\bar{x}_2)}{(1+f^*\bar{x}_3)^3},$$

$$D(\bar{x}_1, \bar{x}_3) = \frac{-2r\alpha_4^*\bar{x}_3^3}{(b_2^*+\bar{x}_1)^3(\bar{x}_3+1)},$$

$$E(\bar{x}_1, \bar{x}_3) = 4\rho_2^* \bar{x}_3 - \frac{2r\alpha_4^* \bar{x}_3^3}{(b_2^* + \bar{x}_1)^3(\bar{x}_3 + 1)} - \frac{2r\alpha_4^* \bar{x}_3^3}{(b_2^* + \bar{x}_1)^2(\bar{x}_3 + 1)^3} - \frac{r\alpha_4^* \bar{x}_3^3}{(b_2^* + \bar{x}_1)^2(\bar{x}_3 + 1)^2} +$$

$$\frac{3r\alpha_4^* \bar{x}_3^2}{(b_2^* + \bar{x}_1)^2(\bar{x}_3 + 1)},$$

$$F(\bar{x}_1, \bar{x}_3) = 2\rho_2^* \bar{x}_1 - \frac{2r}{(\bar{x}_3 + 1)^3} - \frac{6r\alpha_4^* x_3}{(b_2^* + \bar{x}_1)(\bar{x}_3 + 1)^2}.$$

Hence, it is obtained that:

$$(\bar{Q}^{[9]})^T D^2 f(E_9, \bar{\alpha}_2^*) (\bar{P}^{[9]}, \bar{P}^{[9]}) = 2\alpha_2^* \bar{A}_3 \bar{p}_4^{[9]} \bar{q}_4^{[9]} \neq 0,$$

Hence, by the Sotomayor theorem, system (2) exhibits a (TB) at the equilibrium point (E_9) for $(\bar{\alpha}_2^* = \alpha_2^*)$, under conditions (17.4), (17.5), and (17.8). Moreover, the system does not undergo a (PB) at this critical parameter value.

Theorem 8: For the parameter $\ddot{\alpha}_1^* = \frac{\beta_1^* + d_2^*}{\ddot{x}_1}$, system (2) exhibits a (TB) at the (EP) $E_{10} = (\ddot{x}_1, 0, \ddot{x}_3, \ddot{x}_4)$. In contrast, at the equilibrium point $E_{10} = (\ddot{x}_1, 0, \ddot{x}_3, \ddot{x}_4)$, neither a (SNB) nor a (PB) is possible.

Proof: By putting $E_{10} = (\ddot{x}_1, 0, \ddot{x}_3, \ddot{x}_4)$ in the (JM) in equation (18) in [30],

$$\ddot{J}_{10} = J_{10}(E_{10}, \ddot{\alpha}_1^*) = [\ddot{j}_{ij}]_{4 \times 4}, \text{ where } \ddot{j}_{ij} = j_{ij}, \text{ except } \ddot{j}_{22} = 0,$$

Now, let $\ddot{P}^{[10]} = (\ddot{p}_1^{[10]}, \ddot{p}_2^{[10]}, \ddot{p}_3^{[10]}, \ddot{p}_4^{[10]})^T$ be the eigenvector equivalent to an eigenvalue $\lambda_{10x_2} = 0$.

Thus $(\ddot{J}_{10} - \lambda_{10x_2} I) \ddot{P}^{[10]} = 0$, Then, $\ddot{p}_1^{[10]} = \ddot{p}_3^{[10]} = \ddot{p}_4^{[10]} = 0$, and $\ddot{p}_2^{[10]} \neq 0$, real number.

Let $\ddot{Q}^{[10]} = (\ddot{q}_1^{[10]}, \ddot{q}_2^{[10]}, \ddot{q}_3^{[10]}, \ddot{q}_4^{[10]})^T$ be the eigenvector related with the eigenvalue $\lambda_{10x_2} =$

0 of the matrix $\ddot{J}_{10}^T$. Thus: $(\ddot{J}_{10}^T - \lambda_{10x_2} I) \ddot{Q}^{[10]} = 0$, Then, $\ddot{q}_1^{[10]} = \ddot{q}_3^{[10]} = \ddot{q}_4^{[10]} = 0$,

and $\ddot{q}_2^{[10]} \neq 0$, real number.

Now, consider:

$$\frac{\partial f}{\partial \alpha_1^*} = f_{\alpha_1^*}(X, \ddot{\alpha}_1^*) = \left(\frac{\partial f_1}{\partial \alpha_1^*}, \frac{\partial f_2}{\partial \alpha_1^*}, \frac{\partial f_3}{\partial \alpha_1^*}, \frac{\partial f_4}{\partial \alpha_1^*} \right)^T = (-x_1 x_2, x_1 x_2, 0, 0)^T.$$

So, $f_{\alpha_1^*}(E_{10}, \ddot{\alpha}_1^*) = (0, 0, 0, 0)^T$,

$$\text{Therefore } (\ddot{Q}^{[10]})^T f_{\alpha_1^*}(E_{10}, \ddot{\alpha}_1^*) = 0,$$

Therefore, by using Sotomayor's theorem the (SNB) condition can not satisfy at E_{10} . But the

first condition of (TB) is verified, as below:

$$\text{Now, since } Df_{\alpha_1^*}(X, \ddot{\alpha}_1^*) = \begin{bmatrix} -x_2 & -x_1 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Here, $Df_{\alpha_1^*}(X, \ddot{\alpha}_1^*)$ denotes the derivative of $f_{\alpha_1^*}(X, \ddot{\alpha}_1^*)$ with $X = (x_1, x_2, x_3, x_4)$,

Furthermore, it is noted that:

$$Df_{\alpha_1^*}(E_{10}, \ddot{\alpha}_1^*)\ddot{p}^{[10]} = \begin{bmatrix} 0 & -\ddot{x}_1 & 0 & 0 \\ 0 & \ddot{x}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \ddot{p}_2^{[10]} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\ddot{x}_1 \ddot{p}_2^{[10]} \\ \ddot{x}_1 \ddot{p}_2^{[10]} \\ 0 \\ 0 \end{bmatrix},$$

So, we obtain that, $(\ddot{Q}^{[10]})^T [Df_{\alpha_1^*}(E_{10}, \ddot{\alpha}_1^*)\ddot{p}^{[10]}] = \ddot{x}_1 \ddot{p}_2^{[10]} \ddot{q}_2^{[10]} \neq 0$,

Now, by replacing $\ddot{p}^{[10]}$ in (4), we get:

$$D^2 f(E_{10}, \ddot{\alpha}_1^*)(\ddot{p}^{[10]}, \ddot{p}^{[10]}) = \begin{bmatrix} 0 \\ -2\delta_2^* (\ddot{p}_2^{[10]})^2 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, it is obtained that:

$$(\ddot{Q}^{[10]})^T D^2 f(E_{10}, \ddot{\alpha}_1^*)(\ddot{p}^{[10]}, \ddot{p}^{[10]}) = -2\delta_2^* (\ddot{p}_2^{[10]})^2 \ddot{q}_2^{[10]} \neq 0,$$

Hence, by the Sotomayor theorem, system (2) exhibits a (TB) at the equilibrium point (E_{10}) for $(\ddot{\alpha}_1^* = \alpha_1^*)$ while no (PB) occurs.

Theorem 9: Assume that condition (19.3) – (19.6) which are given in [30] hold, while condition (19.2) is reversed, together with the following additional conditions:

$$1 - 2x_1^* - x_2^* > (1 + f^* x_3^*)(\alpha_1^* x_2^* + \rho_1^* x_3^{*2} + 2\delta_1^* x_1^* + S + d_1^* + \alpha_3^* x_3^*) \quad (16)$$

$$\frac{\alpha_4^* x_3^*}{b_2^* + x_1^*} \neq 1, \quad (17)$$

Then system (2) at the equilibrium point $E_{11} = (x_1^*, x_2^*, x_3^*, x_4^*)$, exhibits a (SNB) and (TB) with the parameter value $r^* = r = \frac{p_{12}p_{34}}{k_{13}p_{24}\Phi}$, with $\Phi = \frac{\alpha_4^* x_3^{*3}}{(b_2^* + x_1^*)(x_3^* + 1)} + \rho_2^* x_3^{*2}$,

Proof: From the characteristic equation $\lambda^4 + \hat{E}_1 \lambda^3 + \hat{E}_2 \lambda^2 + \hat{E}_3 \lambda + \hat{E}_4 = 0$, associated with the Jacobian matrix J_{11} , (see Eq. (19) in [30]), it follows that the system (2) at the (EP) E_{11} possesses a zero eigenvalue, denoted by $(\lambda_{11x_3} = 0)$. Then E_{11} will be non-hyperbolic equilibrium point if and only if $\hat{E}_4 = 0$. Then J_{11} with $r^* = r$ becomes $J_{11}^* = J_{11}(E_{11}, r^*) = [k_{ij}^*]_{4 \times 4}$, where $k_{ij}^* = k_{ij}$; $i, j = 1, 2, 3$ which are given in equation (19) in [30] except

$$k_{ij}^* = \frac{k_{12}^* k_{21}^* + (k_{11}^* k_{22}^* - k_{12}^* k_{21}^*)}{k_{22}^*}.$$

Now, let $P^{*[11]} = (p_1^{*[11]}, p_2^{*[11]}, p_3^{*[11]}, p_4^{*[11]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{11x_3} = 0$.

Thus $J_{11}^* P^{*[11]} = 0$, Then, $p_2^{*[11]} = A_1^* p_1^{*[11]}$, $p_3^{*[11]} = A_2^* p_1^{*[11]}$, $p_4^{*[11]} = A_3^* p_1^{*[11]}$, $p_1^{*[11]} \neq 0$, where

$$A_1^* = -\frac{k_{21}^*}{k_{22}^*}, \quad A_2^* = \frac{k_{31}^* k_{44}^*}{k_{34}^* k_{43}^*}, \quad A_3^* = -\frac{k_{31}^*}{k_{34}^*},$$

Let $Q^{*[11]} = (q_1^{*[11]}, q_2^{*[11]}, q_3^{*[11]}, q_4^{*[11]})^T$ be the eigenvector related with the eigenvalue.

$(J_{11}^*)^T Q^{*[11]} = 0$, of the matrix J_{11}^{*T} . Then $q_1^{*[11]} = \eta_1^* q_3^{*[11]}$, $q_2^{*[11]} = \eta_2^* q_3^{*[11]}$, $q_4^{*[11]} = \eta_3^* q_3^{*[11]}$,

$q_3^{*[11]} \neq 0$, where

$$\eta_1^* = \frac{k_{34}^* k_{43}^*}{k_{31}^* k_{44}^*}, \quad \eta_2^* = \frac{k_{12}^* k_{34}^* k_{43}^*}{k_{13}^* k_{22}^* k_{44}^*}, \quad \eta_3^* = -\frac{k_{43}^*}{k_{44}^*},$$

Now, consider:

$$\frac{\partial f}{\partial r} = f_r(X, r^*) = \left(\frac{\partial f_1}{\partial r}, \frac{\partial f_2}{\partial r}, \frac{\partial f_3}{\partial r}, \frac{\partial f_4}{\partial r} \right)^T = \left(0, 0, \left(1 - \frac{\alpha_4^* x_3^*}{b_2^* + x_1^*} \right) \left(\frac{x_3^*}{x_3^* + 1} \right), 0 \right)^T.$$

$$\text{So, } f_r(E_{11}, r^*) = \left(0, 0, \left(1 - \frac{\alpha_4^* x_3^*}{b_2^* + x_1^*} \right) \left(\frac{x_3^*}{x_3^* + 1} \right), 0 \right)^T,$$

$$\text{Therefore } (Q^{*[11]})^T f_r(E_{11}, r^*) = \left(1 - \frac{\alpha_4^* x_3^*}{b_2^* + x_1^*} \right) \left(\frac{x_3^*}{x_3^* + 1} \right) q_3^{*[11]} \neq 0,$$

Hence, in accordance with condition (17) of the Sotomayor theorem, system (2) exhibits a (SNB) at the equilibrium point (E_{11}) for $(r^* = r)$. Otherwise, when condition (17) fails to hold, substituting $P^{*[11]}$ into equation (4) yields:

$$D^2 f(E_{11}, r^*) (P^{*[11]}, P^{*[11]}) = \begin{bmatrix} M_{11} (p_1^{*[11]})^2 \\ M_{21} (p_1^{*[11]})^2 \\ M_{31} (p_1^{*[11]})^2 \\ M_{41} (p_1^{*[11]})^2 \end{bmatrix},$$

where:

$$M_{11} = A(x^*) + B(x^*)A_1^* + C(x^*)A_2^*,$$

$$M_{21} = 2\alpha_1^* A_1^* - 2\delta_2^* A_1^{*2},$$

$$M_{31} = D(x^*) + E(x^*)A_2^* + F(x^*)A_2^{*2} - \alpha_2^* A_2^* A_3^*,$$

$$M_{41} = 2\alpha_2^* A_2^* A_3^*.$$

where:

$$A(x^*) = \frac{-2}{(1+f^*x_3^*)} - 2\delta_1^* + \frac{2Sk}{(1+kx_1^*)^3} + \frac{2\alpha_3^* b_1^* x_3^*}{(b_1^* + x_1^*)^3} - \alpha_1^* - \frac{1}{(1+f^*x_3^*)} + \frac{-f^*(1-2x_1^*-x_2^*)}{(1+f^*x_3^*)^2} -$$

$$2\rho_1^* x_3^* - \frac{\alpha_3^* b_1^*}{(b_1^* + x_1^*)^2},$$

$$B(x^*) = -\alpha_1^* - \frac{1}{(1+f^*x_3^*)} + \frac{f^*x_1^*}{(1+f^*x_3^*)^2},$$

$$C(x^*) = \frac{-f^*(1-3x_1^*-x_2^*)}{(1+f^*x_3^*)^2} - \frac{\alpha_3^* b_1^*}{(b_1^* + x_1^*)^2} - 2\rho_1^* x_1^* - 2\rho_1^* x_3^* + \frac{2f^{*2}x_1^*(1-x_1^*-x_2^*)}{(1+f^*x_3^*)^3},$$

$$D(x^*) = \frac{-2r\alpha_4^* x_3^{*3}}{(b_2^* + x_1^*)^3 (x_3^* + 1)},$$

$$E(x^*) = 4\rho_2^* x_3 - \frac{2r\alpha_4^* x_3^{*3}}{(b_2^* + x_1^*)^3 (x_3^* + 1)} - \frac{2r\alpha_4^* x_3^{*3}}{(b_2^* + x_1^*)^2 (x_3^* + 1)^3} - \frac{r\alpha_4^* x_3^{*3}}{(b_2^* + x_1^*)^2 (x_3^* + 1)^2} + \frac{3r\alpha_4^* x_3^{*2}}{(b_2^* + x_1^*)^2 (x_3^* + 1)},$$

$$F(x^*) = 2\rho_2^* x_1^* - \frac{2r}{(x_3^* + 1)^3} - \frac{6r\alpha_4^* x_3^*}{(b_2^* + x_1^*) (x_3^* + 1)^2}.$$

Hence, it is obtained that:

$$(Q^{*[11]})^T D^2 f(E_{11}, r^*) (P^{*[11]}, P^{*[11]}) = (M_{11}\eta_1^* + M_{21}\eta_2^* + M_{31} + M_{41}\eta_3^*) q_3^{*[11]} (p_1^{*[11]})^2 \neq 0$$

Hence, by the Sotomayor theorem, system (2) exhibits a (TB) at the equilibrium point E_{11} for $(r^* = r)$, under conditions (19.3) – (19.6), together with the reversed form of condition (19.2). then by $P^{*[11]}$ in eq. (5), we get:

$$D^3 f(E_{11}, r^*)(P^{*[11]}, P^{*[11]}, P^{*[11]}) = \begin{bmatrix} N_{11} (p_1^{*[11]})^3 \\ 0 \\ N_{31} (p_1^{*[11]})^3 \\ 0 \end{bmatrix}, \text{ where}$$

$$N_{11} = A_{x_1} + (A_{x_2} + B_{x_1})A_1^* + (A_{x_3} + C_{x_1})A_2^* + (B_{x_3} + C_{x_2})A_1^*A_2^* + C_{x_3}A_2^{*2}, \text{ where}$$

$$A_{x_1} = \frac{-6Sk^2}{(1+kx_1^*)^4} - \frac{6\alpha_3^*b_1^*x_3^*}{(b_1^*+x_1^*)^4} + \frac{2\alpha_3^*b_1^*}{(b_1^*+x_1^*)^3} + \frac{2f^*}{(1+f^*x_3^*)^2}, \quad A_{x_2} = \frac{f^*}{(1+f^*x_3^*)^2}, \quad B_{x_1} = \frac{f^*}{(1+f^*x_3^*)^2},$$

$$A_{x_3} = \frac{3f^*}{(1+f^*x_3^*)^2} + \frac{2f^{*2}(1-2x_1^*-x_2^*)}{(1+f^*x_3^*)^3} - 2\rho_1^* + \frac{2\alpha_3^*b_1^*}{(b_1^*+x_1^*)^3},$$

$$C_{x_1} = \frac{3f^*}{(1+f^*x_3^*)^2} + \frac{2f^{*2}(1-2x_1^*-x_2^*)}{(1+f^*x_3^*)^3} - 2\rho_1^* + \frac{2\alpha_3^*b_1^*}{(b_1^*+x_1^*)^3},$$

$$B_{x_3} = \frac{f^*}{(1+f^*x_3^*)^2} - \frac{2f^{*2}x_1^*}{(1+f^*x_3^*)^3}, \quad C_{x_2} = \frac{f^*}{(1+f^*x_3^*)^2} - \frac{2f^{*2}x_1^*}{(1+f^*x_3^*)^3},$$

$$C_{x_3} = \frac{2f^{*2}(1-3x_1^*-x_2^*)}{(1+f^*x_3^*)^3} - 2\rho_1^* - \frac{6f^{*3}(1-x_1^*-x_2^*)}{(1+f^*x_3^*)^4},$$

$$N_{31} = D_{x_1} + (D_{x_3} + E_{x_1})A_2^* + (E_{x_3} + F_{x_1})A_2^{*2} + F_{x_3}A_2^{*3},$$

$$D_{x_1} = \frac{6r\alpha_4^*x_3^{*3}}{(b_2^*+x_1^*)^4(x_3^*+1)},$$

$$D_{x_3} = -\frac{2r\alpha_4^*x_3^{*2}(2x_3^*+3)}{(b_2^*+x_1^*)^3(x_3^*+1)^2},$$

$$E_{x_1} = \frac{6r\alpha_4^*x_3^{*3}}{(b_2^*+x_1^*)^4(x_3^*+1)} + \frac{4r\alpha_4^*x_3^{*3}}{(b_2^*+x_1^*)^3(x_3^*+1)^3} + \frac{2r\alpha_4^*x_3^{*3}}{(b_2^*+x_1^*)^3(x_3^*+1)^2} - \frac{6r\alpha_4^*x_3^{*2}}{(b_2^*+x_1^*)^3(x_3^*+1)},$$

$$E_{x_3} = 4\rho_2^* - \frac{2r\alpha_4^*x_3^{*2}(2x_3^*+3)}{(b_2^*+x_1^*)^3(x_3^*+1)^2} - \frac{6r\alpha_4^*x_3^{*2}}{(b_2^*+x_1^*)^2(x_3^*+1)^4} - \frac{r\alpha_4^*(x_3^*+3)}{(b_2^*+x_1^*)^2(x_3^*+1)^3} + \frac{3r\alpha_4^*(x_3^*+2)}{(b_2^*+x_1^*)^2(x_3^*+1)^2},$$

$$F_{x_1} = 2\rho_2^* + \frac{6r\alpha_4^*x_3^*}{(b_2^*+x_1^*)^2(x_3^*+1)^2},$$

$$F_{x_3} = \frac{6r}{(x_3^*+1)^4} + \frac{6r\alpha_4^*(x_3^*-1)}{(b_2^*+x_1^*)(x_3^*+1)^3},$$

Hence, it is obtained that

$$(Q^{*[11]})^T D^3 f(E_{11}, r^*)(P^{*[11]}, P^{*[11]}, P^{*[11]}) = (N_{11}\eta_1^* + N_{31})(p_1^{*[11]})^3 q_3^{*[11]} \neq 0,$$

Hence, by the Sotomayor theorem, system (2) exhibits a (PB) at the equilibrium point E_{11} for $r^* = r$, under conditions (19.3) – (19.6). Hence, the proof is complete. Similarly, for $E_{12} =$

$$(x_1^{\circ}, x_2^{\circ}, x_3^{\circ}, x_4^{\circ}).$$

4. HOPF BIFURCATION ANALYSIS

In this section, the possibility of a Hopf bifurcation of near to Positive equilibrium of the system (2) is explored as seen in the following theorem.

Theorem 10: assume that the local condition (19.2) – (19.6) in [30] with the following condition is satisfied:

$$(k_{13})^2 k_{31}^{(r)} k_{31}^{(0)} k_{44} + k_{13} k_{31}^{(r)} [s_{12} k_{31}^{(0)} + s_{24} k_{33}^{(r)}] > k_{33}^{(r)} k_{31}^{(r)} [s_{12} k_{44} + s_{12} s_{34}^{(0)} + p_{12}], \quad (18)$$

$$\hat{E}_3 > \max \left\{ \frac{\hat{E}_1 \hat{E}_4}{\hat{E}_2}, \frac{4\hat{E}_1 \hat{E}_2 - \hat{E}_1^3}{4} \right\}, \quad (19)$$

Then at parameter value $r = r^*$, the system (2) has a (HB) at E_{11} .

Proof: The characteristic equation of system (2) evaluated at the equilibrium point E_{11} , which is presented in Eq. (19.1) of [30], is given by:

$$[\lambda^4 + \hat{E}_1 \lambda^3 + \hat{E}_2 \lambda^2 + \hat{E}_3 \lambda + \hat{E}_4] = 0, \quad (20)$$

where:

$$\hat{E}_1 = -(s_{12} + s_{34}), \quad \hat{E}_2 = p_{12} + p_{34} + s_{12} s_{34} - k_{13} k_{31},$$

$$\hat{E}_3 = -(s_{12} p_{34} + s_{34} p_{12} - k_{13} k_{31} s_{24}), \quad \hat{E}_4 = p_{12} p_{34} - k_{13} k_{31} p_{24},$$

with:

$$s_{12} = k_{11} + k_{22}, p_{12} = k_{11} k_{22} - k_{12} k_{21}, s_{34} = k_{33} + k_{44}, p_{34} = k_{33} k_{44} - k_{34} k_{43},$$

$$s_{24} = k_{22} + k_{44}, p_{24} = k_{22} k_{44}.$$

The parameter (r^*) plays a crucial role in establishing the necessary and sufficient conditions for the emergence of a Hopf bifurcation (HB) at the positive equilibrium point (PEP), which satisfies the following conditions: $\hat{E}_i(r^*) > 0$; $i = 1, 3, 4$, $\Delta_1(r^*) > 0$,

$$\hat{E}_1^3(r^*) - 4 \Delta_1(r^*) > 0, \text{ and } \Delta_2(r^*) = 0, \text{ where } \Delta_1, \Delta_2 \text{ are Hurwitz determinants.}$$

Provided conditions of locally (19.2) – (19.6) given in [30], $\hat{E}_i(r^*) > 0$; $i = 1, 3, 4$, $\Delta_1(r^*) > 0$.

while, $\hat{E}_1^3(r^*) - 4 \Delta_1(r^*) > 0$, provided that the condition (21) holds.

$$\text{It is examined that } \Delta_2 = 0 \text{ gets that } \hat{E}_3 (\hat{E}_1 \hat{E}_2 - \hat{E}_3) - \hat{E}_1^2 \hat{E}_4 = 0,$$

Straightforward computation we get:

$$D_3 r^{*3} + D_2 r^{*2} + D_1 r^* + D_0 = 0, \quad (21)$$

where:

$$D_3 = k_{33}^{(r)} k_{31}^{(r)} [s_{12} + s_{24} - p_{12}] - (k_{13})^2 (k_{31}^{(r)})^2 k_{44},$$

$$D_2 = k_{33}^{(r)} k_{31}^{(r)} [s_{12} k_{44} + s_{12} s_{34}^{(0)} + p_{12}] - k_{13} k_{31}^{(r)} [s_{12} k_{31}^{(0)} + s_{24} k_{33}^{(r)}] - (k_{13})^2 k_{31}^{(r)} k_{31}^{(0)} k_{44},$$

$$D_1 = k_{13} k_{31}^{(r)} [s_{12} p_{34}^{(0)} + s_{24} p_{12}] + k_{13} k_{31}^{(0)} [s_{12} s_{24} - p_{12}] - (k_{13})^2 k_{31}^{(0)} k_{44},$$

$$D_0 = (s_{12} p_{34}^{(0)} + s_{34}^{(0)} p_{12}) (s_{12} s_{34}^{(0)} + p_{12}) - (s_{12} + s_{34}^{(0)})^2 p_{12} p_{34}^{(0)},$$

Clearly, $D_0 < 0$, and $D_j > 0$, $j = 1, 2, 3$, providing that under the local conditions, see [30], in addition to the conditions (18), (19) hold.

Observe that, when utilizing Descartes law of sign eq.(20), has a unique positive root r^* .

Since $\Delta_2 = 0$, the characteristic polynomial is given by eq. (21). It can be written as:

$$\left(\lambda^2 + \frac{\hat{E}_3}{\hat{E}_1}\right) \left(\lambda^2 + \hat{E}_1 \lambda + \frac{\Delta_1}{\hat{E}_1}\right) = 0, \text{ which has four roots:}$$

$$\lambda_{1,2} = \pm i \sqrt{\frac{\hat{E}_3}{\hat{E}_1}} \quad \text{and} \quad \lambda_{3,4} = \frac{1}{2} \left(-\hat{E}_1 \pm \sqrt{\hat{E}_1^2 - 4 \frac{\Delta_1}{\hat{E}_1}} \right).$$

Clearly, at $r = r^*$ there are two pure imaginary eigenvalues (λ_1 and λ_2) and two real and negative eigenvalues. Now for all those values of r in the neighborhood of r^* , In general, the roots of the following form:

$$\lambda_{1,2} = \varsigma_1(r) \pm i \varsigma_2(r) \quad , \quad \lambda_{3,4} = \frac{1}{2} \left(-\hat{E}_1 \pm \sqrt{\hat{E}_1^2 - 4 \frac{\Delta_1}{\hat{E}_1}} \right).$$

Clearly, $\text{Re}(\lambda_M(r)) \big|_{r=r^*} = \varsigma_1(r^*) = 0$, $M = 1, 2$, this implies that the first condition of the necessary and sufficient Hopf bifurcation is achieved at $r = r^*$.

Now, that the transversality condition is verified, we have to prove that:

$$\theta^*(r^*) \Psi^*(r^*) + \Gamma^*(r^*) \Phi^*(r^*) \neq 0,$$

Note that for $r = r^*$ we have $\varsigma_1 = 0$ and $\varsigma_2 = \sqrt{\frac{\hat{E}_3}{\hat{E}_1}}$, substituting the value of ς_2 gives the following simplifications:

$$\Psi^*(r^*) = -2E_3(r^*),$$

$$\Phi^*(r^*) = 2 \frac{\varsigma_2}{\hat{E}_1(r^*)} \left(\hat{E}_1(r^*) \hat{E}_2(r^*) - 2 \hat{E}_3(r^*) \right),$$

$$\Theta^*(r^*) = \hat{E}_4'(r^*) - \hat{E}_2'(r^*) \frac{\hat{E}_3(r^*)}{\hat{E}_1(r^*)},$$

$$\Gamma^*(d^*) = \varsigma_2 \left(\hat{E}_3'(r^*) - \hat{E}_1'(r^*) \frac{\hat{E}_3(r^*)}{\hat{E}_1(r^*)} \right),$$

where

$$\hat{E}'_1 = \frac{\partial \hat{E}_1}{\partial r^*} \Big|_{r=r^*} = -k_{33}^{(r)},$$

$$\hat{E}'_2 = \frac{\partial \hat{E}_2}{\partial r^*} \Big|_{r=r^*} = k_{33}^{(r)}(k_{44} + s_{12}) - k_{13}k_{31}^{(r)},$$

$$\hat{E}'_3 = \frac{\partial \hat{E}_3}{\partial r^*} \Big|_{r=r^*} = -k_{33}^{(r)}(s_{12}k_{44} + p_{12}) + k_{13}k_{31}^{(r)}s_{24},$$

$$\hat{E}'_4 = \frac{\partial \hat{E}_4}{\partial r^*} \Big|_{r=r^*} = k_{33}^{(r)}k_{44}p_{12} - k_{13}k_{31}^{(r)}p_{24}.$$

$$\Theta^*(r^*)\Psi^*(r^*) + \Gamma^*(r^*)\Phi^*(r^*) = 2E_3(r^*)\mathcal{H}(r^*) \neq 0,$$

$$\begin{aligned} \text{where: } \mathcal{H}(r^*) = & -\left(k_{33}^{(r)}k_{44}p_{12} - k_{13}k_{31}^{(r)}p_{24}\right) + \frac{\hat{E}_3(r^*)}{\hat{E}_1(r^*)} \left(k_{33}^{(r)}(k_{44} + s_{12}) - k_{13}k_{31}^{(r)}\right) \\ & + \frac{\hat{E}_1(r^*)\hat{E}_2(r^*) - 2\hat{E}_3(r^*)}{(\hat{E}_1(r^*))^2} \left(k_{13}k_{31}^{(r)}s_{24} - k_{33}^{(r)}(s_{12}k_{44} + p_{12})\right), \end{aligned}$$

Under conditions (19.2) – (19.6), the coefficient $\hat{E}_i(r^*)$ and the Hurwitz determinants are positive, ensuring local stability. Moreover, condition (18) guarantees that the transversality condition holds, i.e., $\mathcal{H}(r^*) \neq 0$. Hence, a (HB) occurs at $r = r^*$ around the (EP) E_{11} .

5. NUMERICAL SIMULATION

In the following section the system's (2) dynamic behavior has been investigated. To validate the findings of the study and the influence of the parameters on the dynamic model, calculations can be performed for a number of sets of parameters with various initial points. Figure (1) (a-b). It is observed that system (2) possesses a positive global equilibrium point corresponding to the fictitious parameter set defined in (22).

$$f^* = 0.02, \quad \alpha_1^* = 0.8, \quad \rho_1^* = 0.07, \quad \delta_1^* = 0.02, \quad S = 0.1, \quad k = 0.6, \quad \beta_1^* = 0.15,$$

$$d_1^* = 0.15, \quad \alpha_3^* = 0.15, \quad b_1^* = 0.4, \quad \delta_2^* = 0.02, \quad d_2^* = 0.08, \quad r = 1.4, \quad \alpha_4^* = 0.5,$$

$$b_2^* = 0.5, \quad \rho_2^* = 0.03, \quad \delta_3^* = 0.02, \quad \alpha_2^* = 0.35, \quad d_3^* = 0.15, \quad \delta_4^* = 0.05, \quad d_4^* = 0.2.$$

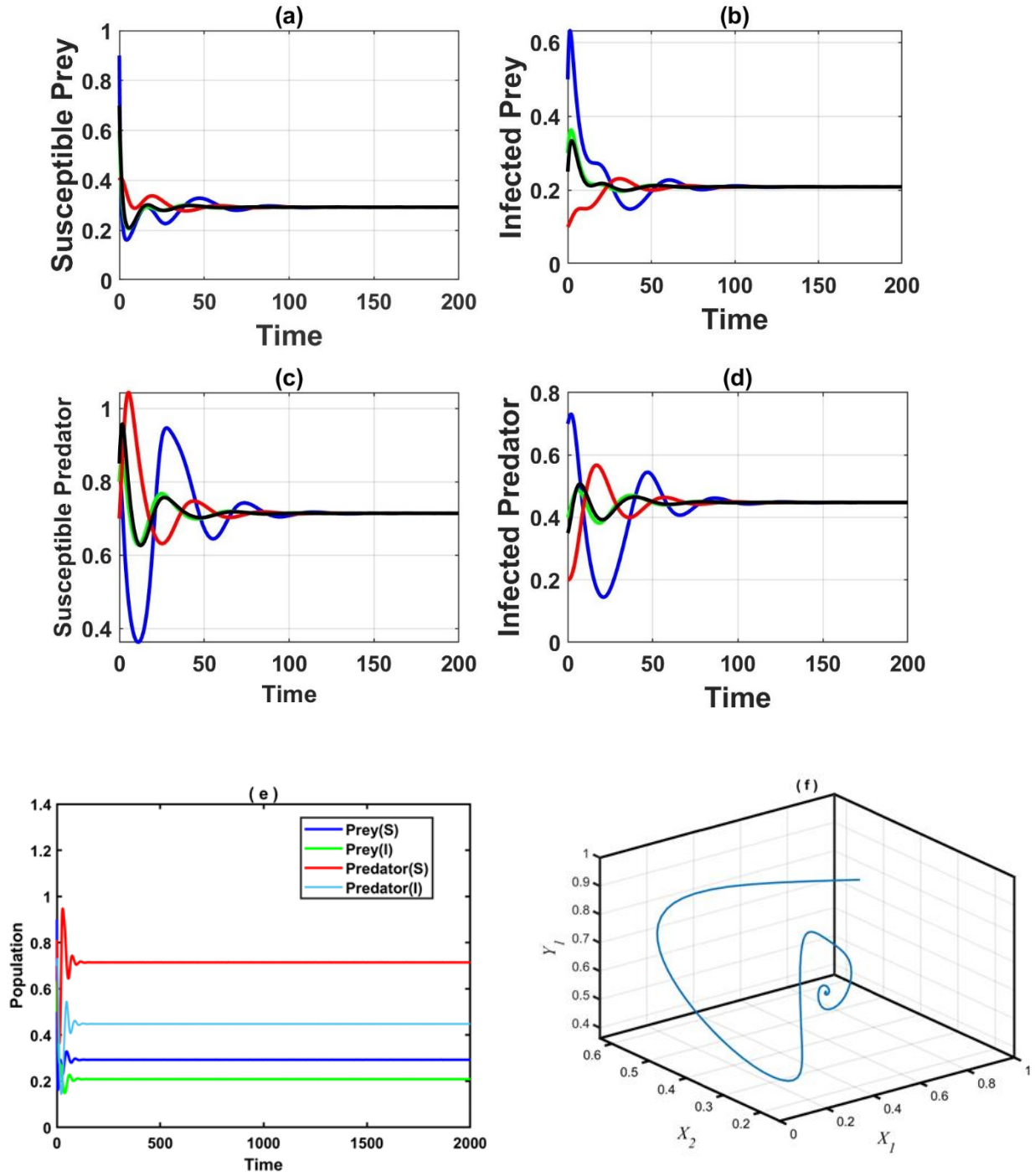


Figure 1(a – f): The trajectories of system (2) that started from four different initial points $(0.9, 0.5, 0.8, 0.7)$, $(0.6, 0.3, 0.8, 0.4)$, $(0.4, 0.1, 0.7, 0.2)$, and $(0.7, 0.25, 0.85, 0.35)$ for the data given in $t \in [0, 200]$, (a) trajectory of x_1 as a function of time, (b) trajectory of x_2 as a function of time, (c) trajectory of x_3 as a function of time, (d) trajectory of x_4 as a

function of time, (e) The solutions converge to the equilibrium point $E_{11} = (0.292, 0.208, 0.714, 0.447)$, (f) Solution of system (2).

Now, we investigate the influence of parameter values on the dynamical behavior of the system. by altering each parameter individually while keeping the other as specified in (22) and the results are presented in Table 2.

Table 2: The dynamical behaviour of system (2) at each parameter of the system

Parameter	Range	Stable Point	Behavior	The bifurcation point
f^*	$0.01 - 0.904$,	E_{11}	Stable	$f^* = 0.905$
	$0.905 - 1.248$,	E_{10}	Stable	$f^* = 1.248$
	≥ 1.248 ,	E_2	Stable	
α_1^*	≥ 0.1 ,	E_{11}	Stable	
α_2^*	$0.1 - 0.196$,	E_9	Stable	$\alpha_2^* = 0.197$
	$0.197 - 0.388$,	E_{11}	Stable	$\alpha_2^* = 0.389$
	≥ 0.389 ,	E_4	Stable	
α_3^*	$0.1 - 0.406$,	E_{11}	Stable	$\alpha_3^* = 0.407$
	$0.407 - 0.420$	E_{10}	Stable	$\alpha_3^* = 0.421$
	≥ 0.421 ,	E_2	Stable	
α_4^*	$0.1 - 0.281$,	E_4	stable	$\alpha_4^* = 0.282$
	$0.282 - 0.355$,		periodic	$\alpha_4^* = 0.356$
	≥ 0.356 ,	E_4	stable	
ρ_1^*	$0.01 - 0.5995$,	E_{11}	Stable	$\rho_1^* = 0.5996$
	$0.5996 - 0.977$,	E_{10}	Stable	$\rho_1^* = 0.978$
	≥ 0.978 ,	E_2	Stable	
ρ_2^*	$0.01 - 1.223$,	E_{11}	Stable	$\rho_2^* = 1.224$
	≥ 1.224	E_4	Stable	
δ_1^*	$0.001 - 0.970$,	E_{11}	Stable	$\delta_1^* = 0.971$
	≥ 0.971 ,	E_{10}	Stable	
δ_2^*	≥ 0.001 ,	E_{11}	Stable	
δ_3^*	$0.001 - 0.051$,	E_{11}	Stable	$\delta_3^* = 0.052$
	≥ 0.052 ,	E_4	Stable	
δ_4^*	$0.001 - 0.020$,	E_4	Stable	$\delta_4^* = 0.021$

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	$0.021 - 0.241,$	E_{11}	Stable	$\delta_4^* = 0.242$
	$\geq 0.242,$	E_9	Stable	
r	$0.1 - 1.214,$	E_4	Stable	$r = 1.215$
	$\geq 1.215,$	E_{11}	Stable	
b_1^*	$0.01 - 0.089,$	E_2	Stable	$b_1^* = 0.099$
	$\geq 0.099,$	E_{11}	Stable	
b_2^*	$0.01 - 0.259,$	E_4	Stable	$b_2^* = 0.260$
	$\geq 0.260,$	E_{11}	Stable	
S	$0.01 - 0.417,$	E_{11}	Stable	$S = 0.418$
	$0.418 - 0.546$	E_{10}	Stable	$S = 0.547$
	≥ 0.547	E_2	Stable	
k	≥ 0.01	E_{11}	Stable	
β_1^*	≥ 0.01	E_{11}	Stable	
d_1^*	$0.01 - 0.421,$	E_{11}	Stable	$d_1^* = 0.422$
	$0.422 - 0.593,$	E_{10}	Stable	$d_1^* = 0.594$
	$\geq 0.594,$	E_8	Stable	
d_2^*	$\geq 0.01,$	E_{11}	Stable	
d_3^*	$0.01 - 0.181,$	E_{11}	Stable	$d_2^* = 0.182$
	$\geq 0.182,$	E_4	Stable	
d_4^*	$0.01 - 0.169,$	E_4	Stable	$d_4^* = 0.170$
	$0.170 - 0.391,$	E_{11}	Stable	$d_4^* = 0.392$
	$\geq 0.392,$	E_9	Stable	

For the parameter f^* in the vicinity $0.01 \leq f^* < 0.904$, the solution gets closer to E_{11} , showing in Figure 2a, increasing the range further $0.905 \leq f^* < 1.247$, the solution gets closer to E_{10} , showing in Figure 2b, and at $f^* = 1.248$, the solution gets closer to E_2 , showing in Figure 2c.

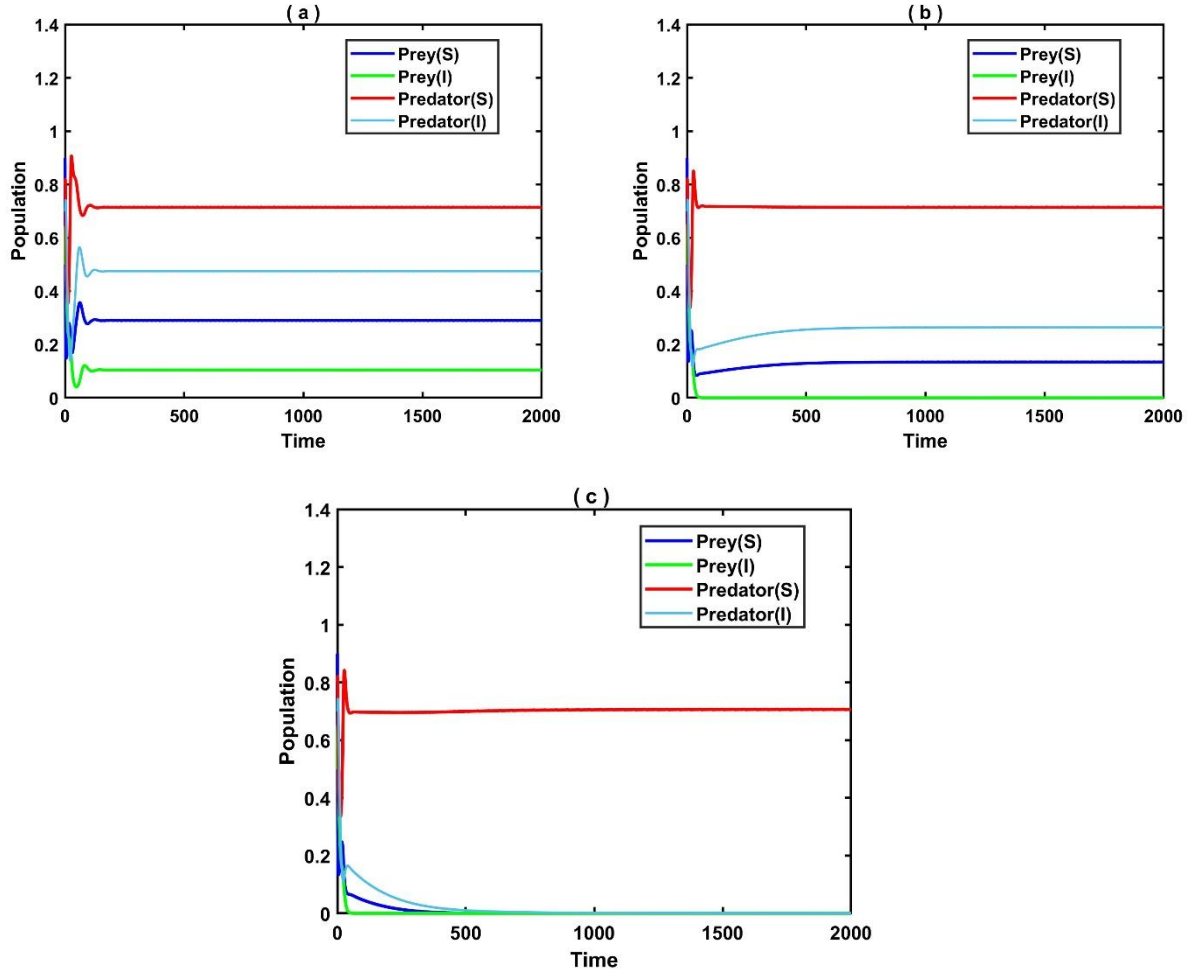


Figure 2. (a – b). (a) Time series of the system's solution (2) with $f^* = 0.5$, which approaches to $E_{11} = (0.290, 0.104, 0.714, 0.474)$ and (b) time series of the solution of system (2) with $f^* = 1.2$, which approaches to $E_{10} = (0.134, 0, 0.713, 0.264)$, (c) time series of the solution of system (2) with $f^* = 1.248$, which approaches to $E_2 = (0, 0, 0.706, 0)$.

The effect of changing the parameter α_2^* in the vicinity $0.1 \leq \alpha_2^* < 0.196$, the solution gets closer to

E_9 , showing in Figure 3(a), increasing the range further $0.197 \leq \alpha_2^* < 0.388$, the solution gets closer to E_{11} , showing in Figure 3(b), and at $\alpha_2^* = 0.389$, the solution gets closer to E_4 , showing in Figure 3(c).

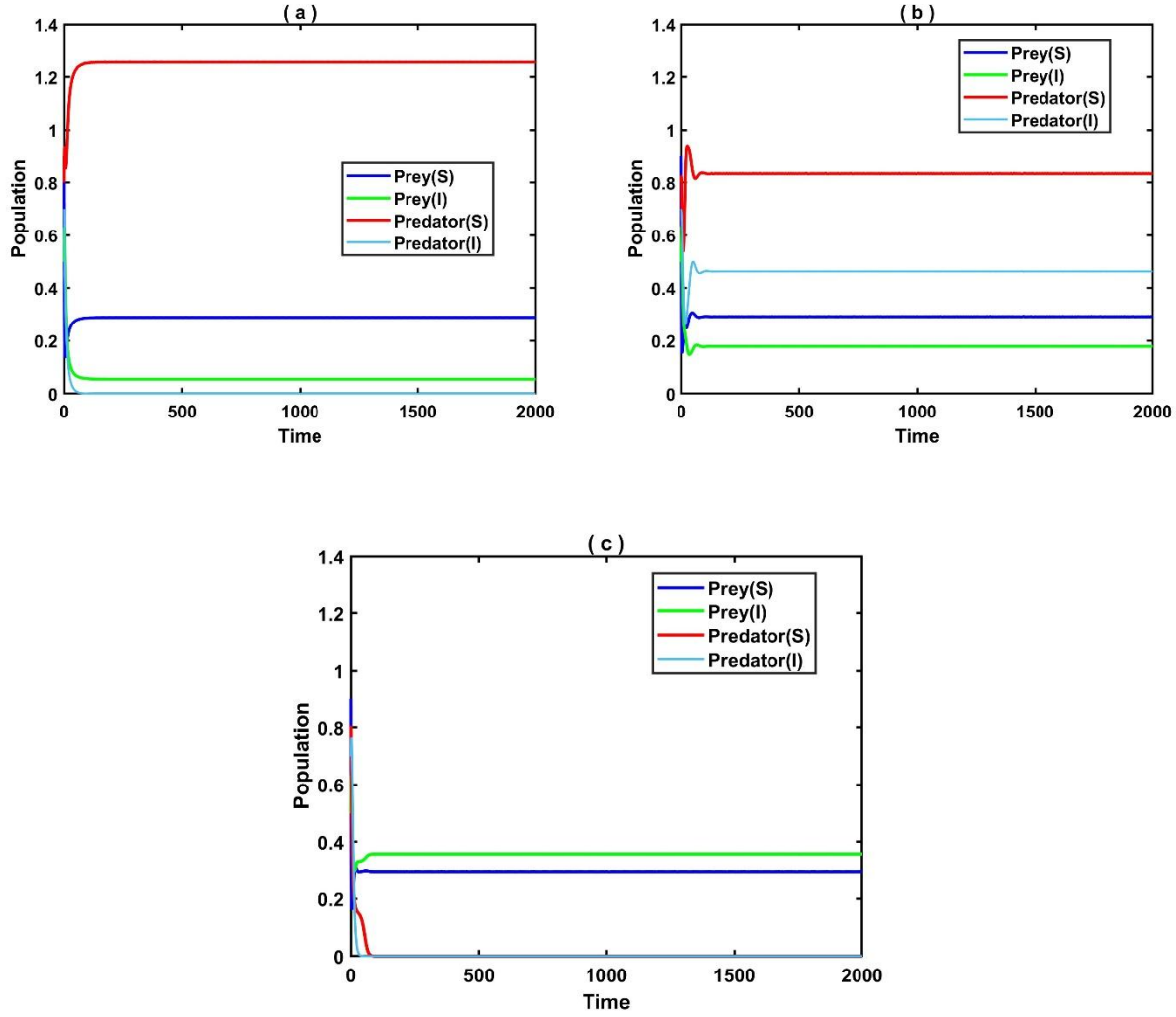


Figure 3. (a – b). (a) Time series of the system's solution (2) with $\alpha_2^* = 0.15$, which approaches to $E_9 = (0.288, 0.054, 1.256, 0)$ and (b) time series of the solution of system (2) with $\alpha_2^* = 0.3$, which approaches to $E_{11} = (0.291, 0.178, 0.833, 0.462)$, (c) time series of the solution of system (2) with $\alpha_2^* = 0.389$, which approaches to $E_4 = (0.296, 0.357, 0, 0)$. For the parameter α_3^* in the vicinity $0.1 \leq \alpha_3^* < 0.406$, the solution gets closer to E_{11} , showing in Figure 4(a), increasing the range further $0.407 \leq \alpha_3^* < 0.420$, the solution gets closer to E_{10} , showing in Figure 4(b), and at $\alpha_3^* = 0.421$, the solution gets closer to E_2 , showing in Figure 4(c).

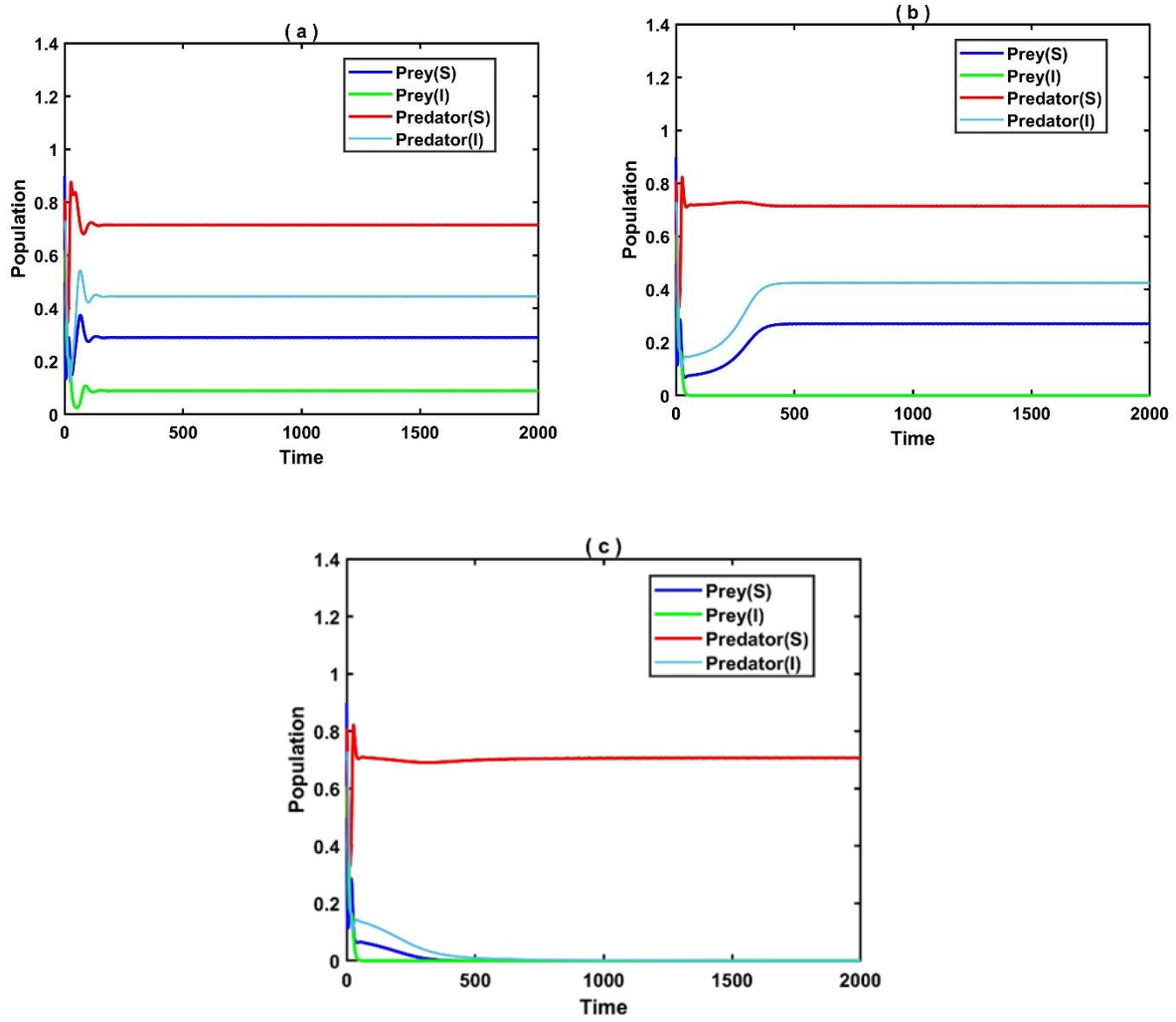


Figure 4. (a – b). (a) Time series of the system's solution (2) with $\alpha_3^* = 0.3$, which approaches to $E_{11} = (0.289, 0.890, 0.714, 0.444)$ and (b) time series of the solution of system (2) with $\alpha_3^* = 0.415$, which approaches to $E_{10} = (0.270, 0, 0.713, 0.424)$, (c) time series of the solution of system (2) with $\alpha_3^* = 0.421$, which approaches to $E_2 = (0, 0, 0.707, 0)$.

The effect of varying the parameter α_4^* is investigated as follows. For $0.1 \leq \alpha_4^* < 0.281$, the solution approaches the equilibrium point E_4 , as illustrated in Figure 5(a), when the parameter increases within the range $0.282 \leq \alpha_4^* < 0.355$, the system exhibits periodic behavior, as shown in Figure 5(b), and at $\alpha_4^* = 0.356$, the solution tends toward the equilibrium point E_{11} , as depicted in Figure 5(c).

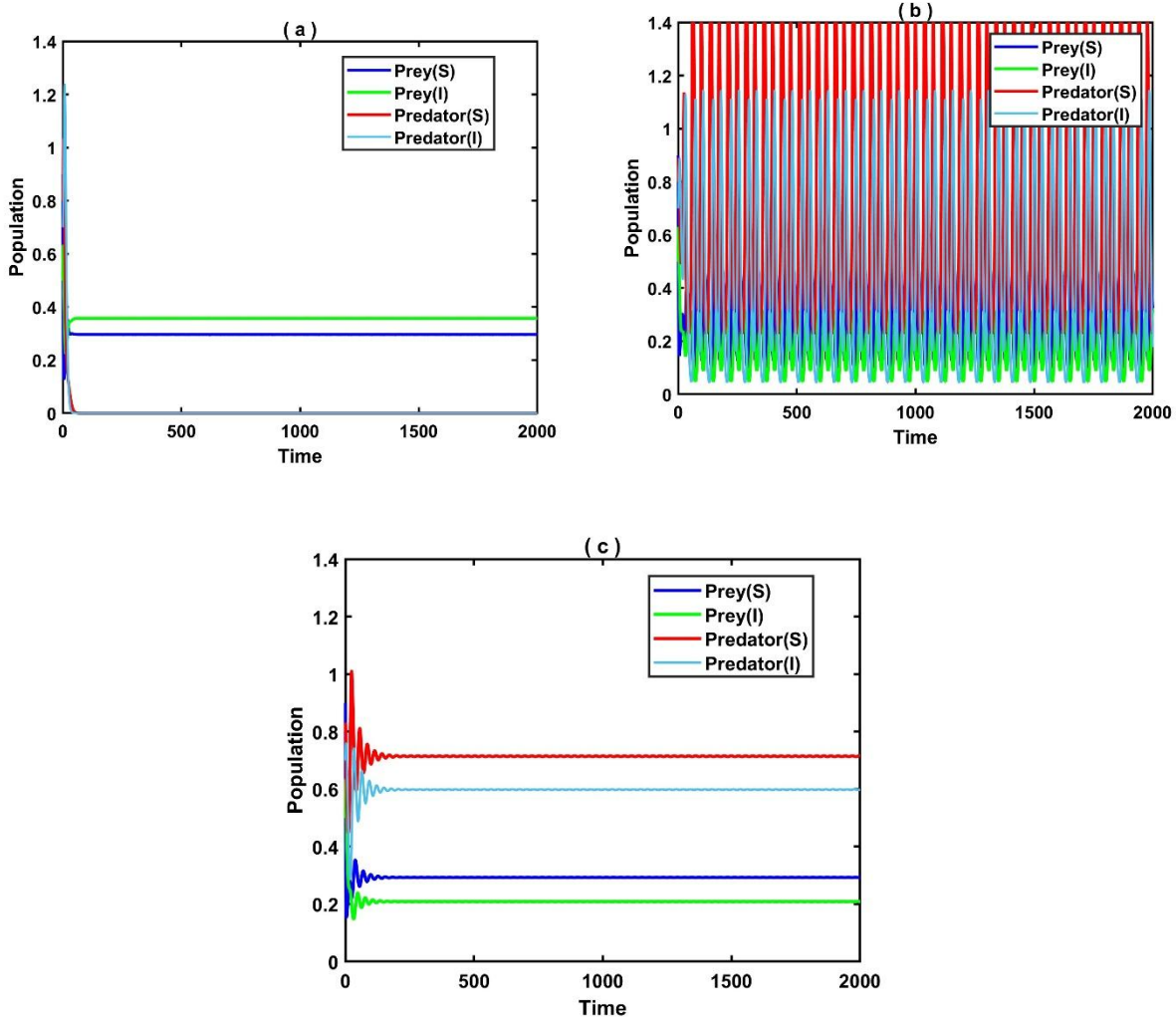


Figure 5. (a – b). (a) Time series of the system's solution (2) with $\alpha_4^* = 0.2$, which approaches to $E_4 = (0.296, 0.357, 0, 0)$ and (b) time series of the solution of system (2) with $\alpha_4^* = 0.285$, exhibits periodic behavior, (c) time series of the solution of system (2) with $\alpha_4^* = 0.356$, which approaches to $E_{11} = (0.292, 0.209, 0.713, 0.598)$.

The effect of varying the parameter ρ_1^* is investigated as follows. For $0.01 \leq \rho_1^* < 0.5995$, the solution approaches the equilibrium point E_{11} , as illustrated in Figure 6(a), when the parameter increases within the range $0.5996 \leq \rho_1^* < 0.977$, the solution approaches the equilibrium point E_{10} , as shown in Figure 6(b), and at $\rho_1^* = 0.978$, the solution tends toward the equilibrium point E_2 , as depicted in Figure 6(c).

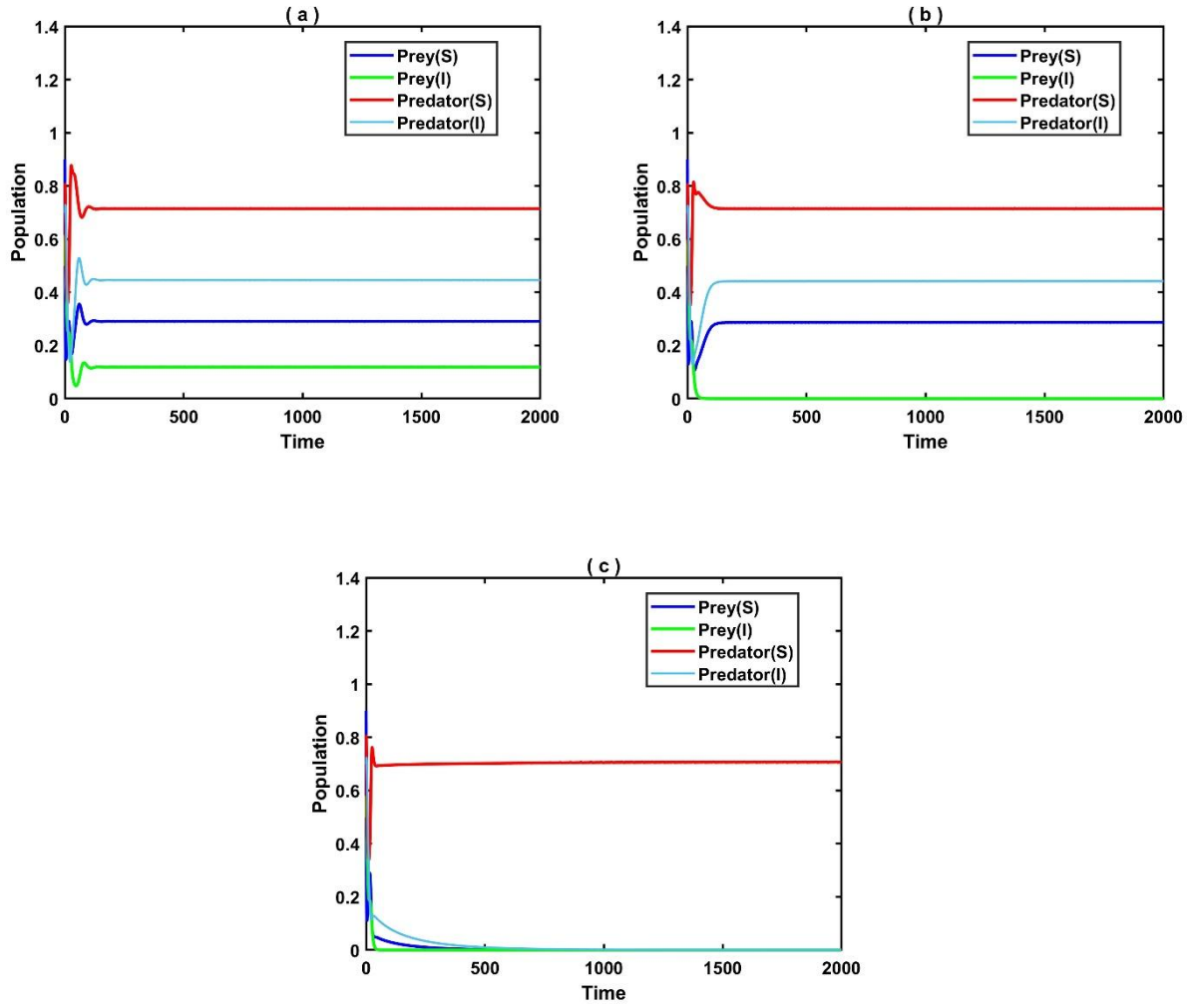


Figure 6. (a – b). (a) Time series of the system's solution (2) with $\rho_1^* = 0.3$, which approaches to $E_{11} = (0.290, 0.118, 0.714, 0.445)$ and (b) time series of the solution of system (2) with $\rho_1^* = 0.6$, which approaches to $E_{10} = (0.286, 0, 0.714, 0.441)$, (c) time series of the solution of system (2) with $\rho_1^* = 0.978$, which approaches to $E_2 = (0, 0, 0.706, 0)$.

For the parameter r in the vicinity $0.1 \leq r < 1.214$, the solution gets closer to E_4 , showing in Figure 7(a), and at $r = 1.215$, the solution gets closer to E_{11} , showing in Figure 7(b).

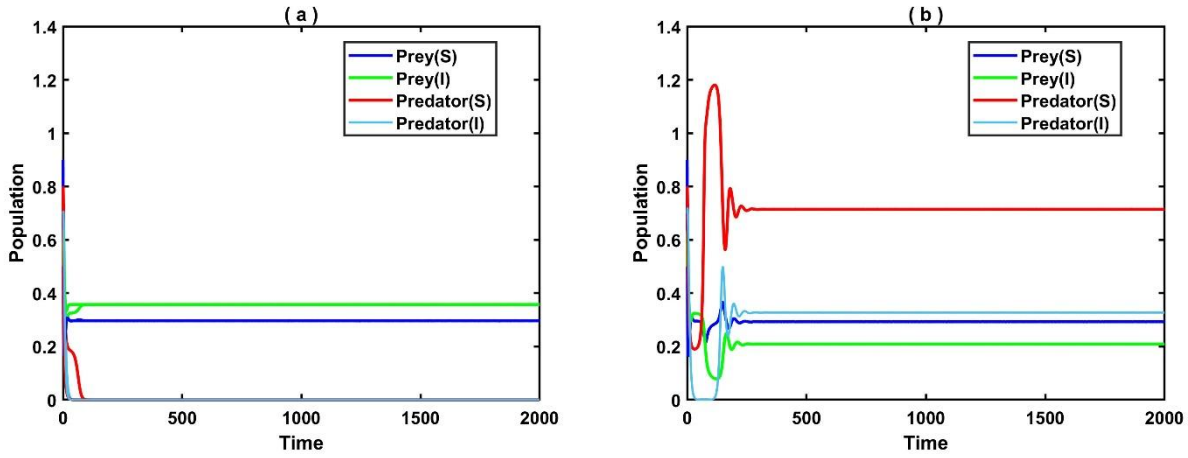


Figure 7. (a – b). (a) Time series of the system's solution (2) with $r = 0.3$, which approaches to $E_4 = (0.296, 0.357, 0, 0)$ and (b) time series of the solution of system (2) with $r = 1.215$, which approaches to $E_{11} = (0.292, 0.208, 0.714, 0.326)$.

The varying of the transfer rate of disease in susceptible prey, the natural death rates of susceptible predator (α_1^*, d_3^*). the solution goes to E_1 , as it is illustrated in Figure (8) with the typical values $\alpha_1^* = 0.25$, and $d_3^* = 0.3$.

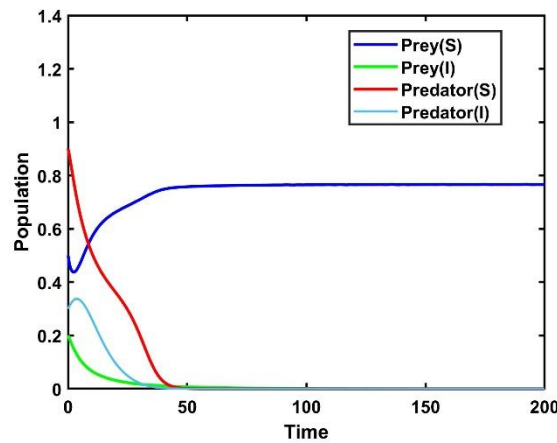


Figure 8. (T.S.) of solution which goes to $E_1 = (0.765, 0, 0, 0)$ when $\alpha_1^* = 0.25$ and $d_3^* = 0.3$.

The influence of the natural death rates of the susceptible prey and predator respectively and the harvesting rates of the prey (d_1^*, d_3^*, S). the solution goes to E_0 , as it is illustrated in Figure (9) with the typical values, $d_1^* = 0.5$, $d_3^* = 0.3$, $S = 0.5$.

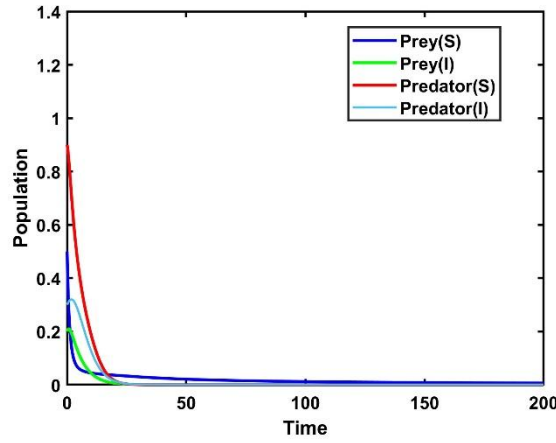


Figure 9. (T.S.) of solution which goes to $E_0 = (0, 0, 0, 0)$ when $d_1^* = 0.5$, $d_3^* = 0.3$, $S = 0.5$.

6. CONCLUSIONS AND DISCUSSION

This research examines a nonlinear eco-epidemiological system to ascertain the impact of various characteristics on its dynamic behavior. The results indicate that the system has intricate dynamics, with several equilibrium points and transitions contingent upon parameter values.

One equilibrium point, specifically E_{11} , is shown to maintain stability over a broad spectrum of parameters, suggesting that the system gravitates towards a stable coexistence state under various conditions.

However, bifurcation events and the system's susceptibility to parameter variations are highlighted by the fact that changes in some parameters might cause the system to move from one equilibrium to another.

Furthermore, the model exhibits oscillatory behavior by showing the formation of periodic solutions for particular parameter ranges. This is linked to the occurrence of a Hopf bifurcation, which reflects actual dynamics like recurring outbreaks or population cycles.

Overall, the results show that important parameters have a significant influence on the behavior of the system; certain parameters stabilize the system while others cause qualitative changes. This highlights how crucial parameter management is for forecasting and controlling system dynamics. Moreover, the system undergoes a Hopf bifurcation at the critical parameter value $r = r^*$ where the stability of the equilibrium E_{11} changes due to the crossing of a pair of complex conjugate eigenvalues through the imaginary axis. This bifurcation gives rise to periodic solutions, reflecting

oscillatory dynamics in the eco-epidemiological interactions. In conclusion, the suggested model offers a clear comprehension of how ecological and epidemiological elements interact. The findings demonstrate how bifurcation analysis may be used to explain system behavior and provide insightful information for further research. In order to improve the model's realism, future work may expand it by adding other elements like delays, spatial effects, or stochastic impacts.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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