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MODELING EXTREME EVENTS AND WAITING-TIME PHENOMENA: AN INVERSE LOG-SHIFTED GOMPERTZ STATISTICAL MODEL FOR PHYSICAL, ENVIRONMENTAL, AND BIOMEDICAL DATA

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Abstract. Statistical modeling of real-life data from hydrology, engineering, and medical sciences often requires flexible probability distributions that can capture skewness, heavy tails, and complex hazard rate patterns. Classical distributions often do not provide an adequate fit to such data because of their limited flexibility. A new probability model called the inverse log-shifted Gompertz distribution is proposed in this study by applying the inverse scheme

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to the log-shifted Gompertz distribution. Several statistical and reliability properties of the model are derived. The estimation of the parameters is carried out using fifteen different estimation methods, and a simulation study is conducted to assess the finite-sample performance of the estimators. Three real datasets involving annual maximum flood discharges, mechanical component failure times, and analgesic relief times are used to demonstrate the practical performance of the proposed model. Standard goodness-of-fit measures, such as the Akaike information criterion, Hannan–Quinn information criterion, root mean square error, Kolmogorov–Smirnov statistic with its p-value, and Anderson–Darling statistic with its p-value, are used to compare the models. The results show that the proposed distribution provides a better fit than several classical distributions, which demonstrates its usefulness in modeling complex data.

Keywords: inverse log-shifted Gompertz distribution; lifetime distribution; parameter estimation; simulation study; reliability analysis; goodness-of-fit

2020 AMS Subject Classification: 62E15; 60E05

1. INTRODUCTION

Flood events pose serious risks to lives, infrastructure, and economic stability. Very high river flows can damage farmlands, destroy buildings, and disrupt transportation networks. For this reason, accurate modeling of flood data is important for flood risk assessment, hydraulic design, and long-term water resource planning. Poor modeling of extreme floods can lead to unsafe designs and costly policy decisions. Failures of mechanical components are also a major concern in industrial systems. Unexpected breakdowns may cause production losses, safety hazards, and high maintenance costs. Proper modeling of failure-time data helps engineers assess system reliability, plan preventive maintenance, and improve design performance. Weak or unsuitable models may underestimate failure risks and reduce system reliability. Treatment response times are important indicators of patient outcomes in medical studies. For example, analgesic relief times provide useful information about drug effectiveness and patient recovery. Accurate statistical analysis of such data supports better clinical decisions and improved healthcare delivery, while poor modeling can produce misleading results.

These fields often produce data that are highly skewed and heavy-tailed. The same characteristics can also be found in finance, insurance, and actuarial studies. Probability distributions are often used to describe these features. Classical models commonly used because of their simplicity include the Gompertz, Weibull, gamma, and Rayleigh distributions. Nevertheless,

these distributions may not be sufficiently flexible to provide an adequate fit to the complex patterns observed in real datasets. Their limited flexibility has inspired the quest to formulate more flexible probability models through alternative construction methods, such as generator frameworks, transformation methods, and compounding techniques.

These methods of construction have yielded numerous extensions of known distributions. For example, Ref. [12] applied the alpha-power transformation to derive a modified Fréchet distribution that performed better than the classical Fréchet model for engineering data. Ref. [10] adopted the ratio transformation technique to propose the ratio transformation Rayleigh distribution and demonstrated its superior fit to engineering data. Ref. [11], using the Marshall–Olkin generator, generalized the Shanker distribution and applied the new model to public health and demographic data, where it performed better than both the original Shanker model and other competing models. Ref. [13] presented the power Gompertz distribution based on the power transformation technique and showed good results on bladder cancer data. Similarly, the transmuted power unit Lindley distribution, introduced in Ref. [14], performed well on economic and medical data.

Although these extended models offer greater flexibility, the methods used to create them usually add extra parameters to the original distribution. This can make the models more complex and parameter estimation more difficult. The inverse scheme provides a useful alternative by extending classical distributions without adding new parameters. This may simplify parameter estimation and the derivation of statistical properties. Inverting the baseline model may also produce heavier tails and more flexible hazard rate shapes. Several inverse distributions have performed well in practical applications. Among them are the inverse Weibull [15], inverse Muth [16], inverse xgamma [17], inverse Copoun [22], inverse Gompertz–Fréchet [21], inverse Pham [19], inverse Monsef [20], and inverse type II Topp–Leone half-logistic distributions [18].

The log-shifted Gompertz distribution is a simple and flexible model that has been used in applied studies. Despite these advantages, the distribution is defined on the bounded interval $0 < x < 1$. This restricted support may limit its use when modelling datasets that extend beyond

this range. One possible way to address this issue is through the use of the inverse transformation. The inverse scheme transforms the bounded support of the baseline distribution into the unbounded interval $x > 1$, while retaining the same parameters.

Motivated by this idea, the present study introduces a new probability model known as the inverse log-shifted Gompertz distribution (ILSGD). The model is developed by applying the inverse scheme to the log-shifted Gompertz distribution. Several mathematical properties of the ILSGD are derived. Fifteen classical methods are used to estimate the model parameters. A simulation study is conducted to assess the finite-sample performance of these estimators. The findings show that the methods generally perform satisfactorily.

The practical usefulness of the proposed model is demonstrated using three real datasets from hydrology, reliability engineering, and medical science. The ILSGD is compared with the Gompertz, inverse Gompertz, cosine Gompertz, Weibull, gamma, Rayleigh, and inverse Rayleigh distributions. Model performance is assessed using several goodness-of-fit measures. In all three applications, the proposed model provides a better fit than the competing distributions.

Contributions of the Study. The main contributions of this work are summarized as follows:

- A new inverse probability model, the inverse log-shifted Gompertz distribution, is introduced.
- Key statistical and reliability properties of the proposed model are derived.
- Fifteen classical estimation methods are employed for parameter estimation.
- A simulation study is conducted to assess the finite-sample performance of the estimators.
- The model is applied to real datasets from hydrology, engineering, and medical science.
- The proposed distribution outperforms several competing classical and inverse models.

The organization of the remaining sections of this study is illustrated in Figure 1.

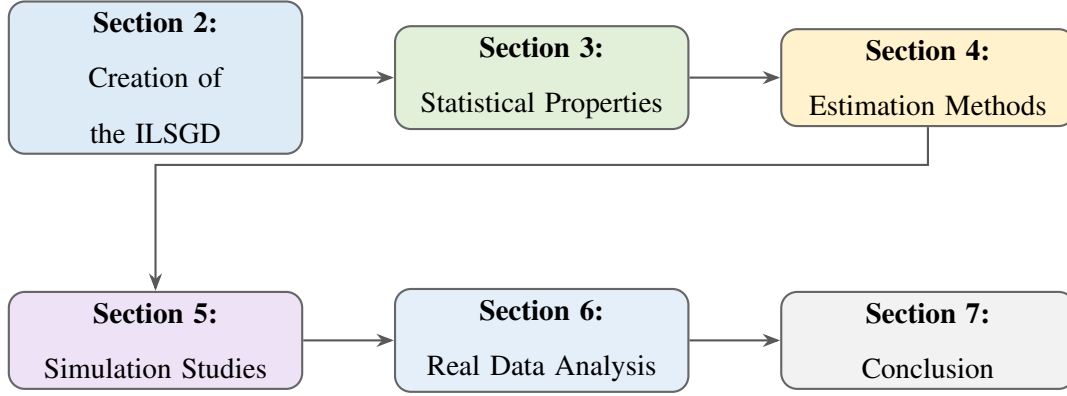


FIGURE 1. Flowchart of the remaining sections of the study.

2. MODEL FORMULATION

In 2020, Jodrá [23] derived a bounded distribution called the log-shifted Gompertz distribution (LSGD). Its cumulative distribution function (CDF) is given by

$$(1) \quad G(x) = 1 - \left(1 - x^\delta\right) e^{-\varepsilon x^\delta}; \quad 0 < x < 1, \delta > 0, \varepsilon > 0.$$

Using the transformation $X = Y^{-1}$, where Y is a random variable following the LSGD with CDF given in Equation (1), the resulting CDF and probability density function (PDF) of the inverse LSGD (ILSGD) are expressed as:

$$(2) \quad F(x) = \left(1 - x^{-\delta}\right) e^{-\varepsilon x^{-\delta}}; \quad x > 1, \delta > 0, \varepsilon > 0.$$

$$(3) \quad f(x) = \delta x^{-2\delta-1} e^{-\varepsilon x^{-\delta}} \left[(\varepsilon + 1)x^\delta - \varepsilon \right].$$

Some reliability measures of the proposed ILSGD are presented below.

Reliability Measures. The survival function gives the probability that a unit survives beyond a given time. For the ILSGD, the survival function is given by

$$S(x) = 1 - F(x) = 1 - \left(1 - x^{-\delta}\right) e^{-\varepsilon x^{-\delta}}$$

The hazard function represents the instantaneous failure rate at time x , given that the unit has survived up to that time. For the ILSGD, the hazard function is given by

$$HF = \frac{f(x)}{S(x)} = \frac{\delta x^{-2\delta-1} e^{-\epsilon x^{-\delta}} \left[(\epsilon + 1)x^{\delta} - \epsilon \right]}{1 - (1 - x^{-\delta}) e^{-\epsilon x^{-\delta}}}$$

The cumulative hazard function measures the total risk of failure accumulated up to time x . For the ILSGD, the cumulative hazard function is given by

$$CHF = -\log[S(x)] = -\log \left[1 - (1 - x^{-\delta}) e^{-\epsilon x^{-\delta}} \right]$$

The PDF and hazard function plots of the ILSGD are presented in Figures 2 and 3, respectively. The PDF plot shows that the model can be unimodal, right-skewed, and reversed J-shaped, making it suitable for modeling data from various fields. The hazard function plot also reveals that the model can have various shapes when the parameters are varied. In particular, the plots show decreasing and inverted bathtub-shaped hazard rates for the selected parameter values. This highlights the flexibility of the model.

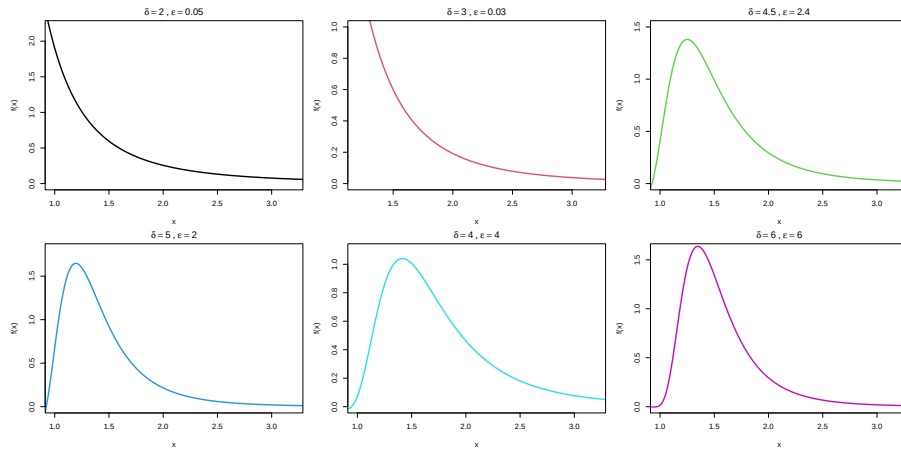


FIGURE 2. The PDF of the ILSGD.

3. STATISTICAL PROPERTIES

This section presents some important statistical properties of the ILSGD.

3.1. Quantile function. The quantile function is useful for generating random samples and studying the distributional behavior of the model. For the ILSGD, the quantile function is given by

$$QF = \left[\frac{\epsilon}{\epsilon - W(pe^{\epsilon}\epsilon)} \right]^{1/\delta}, \quad 0 < p < 1.$$

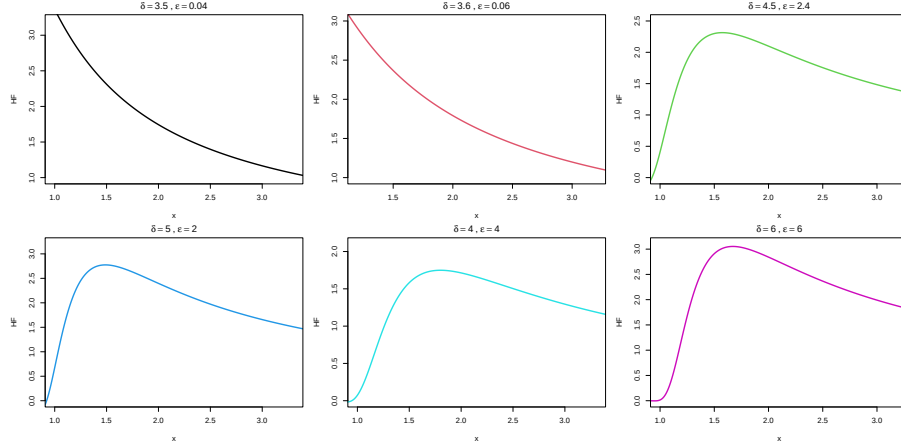


FIGURE 3. The hazard function of the ILSGD.

where $W(\cdot)$ denotes the Lambert W function.

3.2. Moments. Moments are used to describe key characteristics of a distribution such as location, dispersion, skewness, and kurtosis. The r th raw moment of the ILSGD is given by

$$\mu'_r = \frac{\delta e^{-\varepsilon} + (\delta \varepsilon + r) \left[\varepsilon^{\frac{r}{\delta}-1} \Gamma\left(1 - \frac{r}{\delta}\right) - E_{\frac{r}{\delta}}(\varepsilon) \right]}{\delta}, \quad r < \delta.$$

Here, $E_v(z) = \int_1^\infty e^{-tz} t^{-v} dt$ denotes the generalized exponential integral.

It should be noted that the existence of the r th moment requires $\delta > r$. Therefore, the mean, variance, skewness, and kurtosis are finite when $\delta > 1$, $\delta > 2$, $\delta > 3$, and $\delta > 4$, respectively.

The first four raw moments of the ILSGD are obtained as follows:

$$\begin{aligned} \mu'_1 &= \frac{\varepsilon^{\frac{1}{\delta}-1} \left(\delta^2 \left(\Gamma\left(2 - \frac{1}{\delta}, \varepsilon\right) - (\varepsilon + 1) \Gamma\left(\frac{\delta-1}{\delta}, \varepsilon\right) \right) - (\delta \varepsilon + 1) \Gamma\left(-\frac{1}{\delta}\right) \right)}{\delta^2}, \delta > 1 \\ \mu'_2 &= \frac{\varepsilon^{\frac{2}{\delta}-1} \left((\delta \varepsilon + 2) \Gamma\left(\frac{\delta-2}{\delta}, \varepsilon\right) + \delta \left(\Gamma\left(2 - \frac{2}{\delta}, \varepsilon\right) - (\varepsilon + 1) \Gamma\left(\frac{\delta-2}{\delta}, \varepsilon\right) \right) \right)}{\delta}, \delta > 2 \\ \mu'_3 &= \frac{\varepsilon^{\frac{3}{\delta}-1} \left((\delta \varepsilon + 3) \Gamma\left(\frac{\delta-3}{\delta}, \varepsilon\right) + \delta \left(\Gamma\left(2 - \frac{3}{\delta}, \varepsilon\right) - (\varepsilon + 1) \Gamma\left(\frac{\delta-3}{\delta}, \varepsilon\right) \right) \right)}{\delta}, \delta > 3 \\ \mu'_4 &= \frac{\varepsilon^{\frac{4}{\delta}-1} \left((\delta \varepsilon + 4) \Gamma\left(\frac{\delta-4}{\delta}, \varepsilon\right) + \delta \left(\Gamma\left(2 - \frac{4}{\delta}, \varepsilon\right) - (\varepsilon + 1) \Gamma\left(\frac{\delta-4}{\delta}, \varepsilon\right) \right) \right)}{\delta}, \delta > 4 \end{aligned}$$

where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$.

The mean (μ), variance (σ^2), skewness (S), and kurtosis (K) of the ILSGD are given, respectively, by:

$$\begin{aligned}\mu &= \mu'_1, \\ \sigma^2 &= \mu'_2 - (\mu'_1)^2, \\ S &= \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3}{\sigma^3}, \\ K &= \frac{\mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4}{\sigma^4}.\end{aligned}$$

Table 1 presents the first four raw moments and summary measures of the ILSGD for different values of δ and ε . For a fixed value of ε , increasing δ leads to a decrease in the mean, variance, skewness, and kurtosis. This indicates that the distribution becomes more concentrated around its center with lighter tails as δ increases. For a fixed value of δ , increasing ε increases the mean and variance, showing that the distribution becomes more spread out. However, the skewness and kurtosis decrease as ε increases, indicating a reduction in asymmetry and tail heaviness. All the parameter combinations considered in Table 1 produce positive skewness, indicating that the ILSGD is right-skewed for these parameter values. The large kurtosis values obtained for smaller values of δ indicate heavy-tailed behavior, whereas larger values of δ and ε produce more moderate kurtosis values. In general, these findings indicate that the ILSGD can accommodate a wide range of data patterns.

Table 1: First four raw moments and summary measures of the proposed model for different parameter values

δ	ε	μ'_1	μ'_2	μ'_3	μ'_4	Variance	Skewness	Kurtosis
4.1	0.5	1.40858	2.25231	4.77258	60.46813	0.26820	6.08001	675.39848
4.5	0.5	1.36086	2.04412	3.72489	12.55528	0.19219	4.98565	127.34559
5.0	0.5	1.31485	1.86381	3.01483	6.63101	0.13498	4.21885	62.65203
5.5	0.5	1.27923	1.73623	2.59466	4.68477	0.09981	3.74990	42.33580
6.0	0.5	1.25083	1.64129	2.31778	3.72489	0.07671	3.43206	32.64837
6.5	0.5	1.22767	1.56792	2.12194	3.15597	0.06074	3.20177	27.05230
7.0	0.5	1.20842	1.50954	1.97629	2.78075	0.04925	3.02688	23.43495
7.5	0.5	1.19217	1.46200	1.86381	2.51526	0.04073	2.88937	20.91627

Continued on next page

Table 1 – continued from previous page

δ	ε	μ'_1	μ'_2	μ'_3	μ'_4	Variance	Skewness	Kurtosis
4.1	1.0	1.48450	2.52606	5.76496	79.83597	0.32232	5.78218	620.23545
4.5	1.0	1.42700	2.26552	4.40544	16.03050	0.22918	4.72522	116.60177
5.0	1.0	1.37178	2.04149	3.49292	8.19855	0.15972	3.98340	57.19679
5.5	1.0	1.32916	1.88402	2.95802	5.65056	0.11736	3.52896	38.55840
6.0	1.0	1.29527	1.76746	2.60835	4.40544	0.08973	3.22047	29.67849
6.5	1.0	1.26770	1.67781	2.36270	3.67354	0.07074	2.99665	24.55283
7.0	1.0	1.24483	1.60675	2.18105	3.19436	0.05716	2.82647	21.24187
7.5	1.0	1.22555	1.54908	2.04149	2.85749	0.04711	2.69251	18.93799
4.1	1.5	1.55228	2.77829	6.71287	99.11751	0.36872	5.60742	589.78875
4.5	1.5	1.48591	2.46832	5.04896	19.43744	0.26040	4.57098	110.64471
5.0	1.5	1.42234	2.20327	3.94051	9.71243	0.18023	3.84265	54.15950
5.5	1.5	1.37340	2.01795	3.29554	6.57244	0.13171	3.39586	36.44904
6.0	1.5	1.33459	1.88137	2.87653	5.04896	0.10023	3.09221	28.01649
6.5	1.5	1.30306	1.77669	2.58373	4.15920	0.07872	2.87165	23.15186
7.0	1.5	1.27695	1.69399	2.36821	3.57998	0.06340	2.70379	20.01118
7.5	1.5	1.25496	1.62704	2.20327	3.17483	0.05211	2.57155	17.82686
4.1	2.0	1.61341	3.01260	7.62297	118.32385	0.40952	5.49861	571.55566
4.5	2.0	1.53890	2.65568	5.66119	22.78541	0.28746	4.47427	107.06587
5.0	2.0	1.46771	2.35192	4.36248	11.18038	0.19774	3.75376	52.32952
5.5	2.0	1.41303	2.14045	3.61143	7.45698	0.14379	3.31132	35.17559
6.0	2.0	1.36974	1.98518	3.12606	5.66119	0.10898	3.01034	27.01170
6.5	2.0	1.33463	1.86654	2.78838	4.61801	0.08530	2.79154	22.30403
7.0	2.0	1.30559	1.77305	2.54078	3.94214	0.06849	2.62490	19.26589
7.5	2.0	1.28117	1.69754	2.35192	3.47137	0.05616	2.49352	17.15366

3.3. Inverse Moments. Inverse moments are useful in reliability and lifetime analysis, especially in studying harmonic-type measures and failure behavior. The r th inverse moment of the

ILSGD is given by

$$\begin{aligned}\mu_{-r} &= \int_1^{\infty} x^{-r} f(x) dx = \int_1^{\infty} x^{-r} \delta x^{-2\delta-1} e^{-\epsilon x^{-\delta}} \left[(\epsilon + 1)x^{\delta} - \epsilon \right] dx \\ &= \frac{\epsilon^{-\frac{\delta+r}{\delta}} \left((\delta\epsilon - r)\Gamma\left(\frac{r+\delta}{\delta}\right) + \delta \left(\Gamma\left(\frac{r}{\delta} + 2, \epsilon\right) - (\epsilon + 1)\Gamma\left(\frac{r+\delta}{\delta}, \epsilon\right) \right) \right)}{\delta}.\end{aligned}$$

By setting $r = 1$ in the above expression, the first inverse moment of the ILSGD is obtained as follows:

$$\begin{aligned}\mu_{-1} &= \int_1^{\infty} x^{-1} f(x) dx = \int_1^{\infty} x^{-1} \delta x^{-2\delta-1} e^{-\epsilon x^{-\delta}} \left[(\epsilon + 1)x^{\delta} - \epsilon \right] dx \\ &= \frac{\epsilon^{-\frac{\delta+1}{\delta}} \left(\delta^2 \left(\Gamma\left(2 + \frac{1}{\delta}, \epsilon\right) - (\epsilon + 1)\Gamma\left(1 + \frac{1}{\delta}, \epsilon\right) \right) + (\delta\epsilon - 1)\Gamma\left(\frac{1}{\delta}\right) \right)}{\delta^2}\end{aligned}$$

3.4. Incomplete Moments. Incomplete moments are useful for studying truncated data and income or reliability measures below a specified threshold. The s th incomplete moment of the ILSGD is given by

$$\begin{aligned}I.M &= \int_1^t x^s f(x) dx = \int_1^t x^s \delta x^{-2\delta-1} e^{-\epsilon x^{-\delta}} \left[(\epsilon + 1)x^{\delta} - \epsilon \right] dx \\ &= e^{-\epsilon} + \frac{(\delta\epsilon + s) \left(t^{s-\delta} E_{\frac{s}{\delta}}\left(t^{-\delta}\epsilon\right) - E_{\frac{s}{\delta}}(\epsilon) \right)}{\delta} + t^{s-\delta} \left(-e^{\epsilon(-t^{-\delta})} \right)\end{aligned}$$

3.5. Conditional Moment. Conditional moments are useful for studying the behavior of a random variable beyond a given threshold and are often applied in reliability and survival analysis. The s th conditional moment of the ILSGD is given by

$$\begin{aligned}\text{C.Mo} &= \frac{1}{S(t)} \int_t^{\infty} x^s f(x) dx = \frac{1}{S(t)} \int_t^{\infty} x^s \delta x^{-2\delta-1} e^{-\epsilon x^{-\delta}} \left[(\epsilon + 1)x^{\delta} - \epsilon \right] dx \\ &= \frac{1}{S(t)} \frac{\left(\frac{t^{\delta}}{\epsilon} \right)^{-\frac{s}{\delta}} \left((\delta\epsilon + s)\Gamma\left(1 - \frac{s}{\delta}\right) \left(t^{\delta} \right)^{s/\delta} - \delta t^s \left((\epsilon + 1)\Gamma\left(1 - \frac{s}{\delta}, t^{-\delta}\epsilon\right) - \Gamma\left(2 - \frac{s}{\delta}, t^{-\delta}\epsilon\right) \right) \right)}{\delta\epsilon}\end{aligned}$$

3.6. Mean Inactivity Time. The mean inactivity time describes the expected elapsed time since failure, given that the failure has occurred before time t . For the ILSGD, the mean inactivity time is given by

$$\begin{aligned}MIT &= t - \frac{1}{F(t)} \int_1^t x f(x) dx \\ &= t - \frac{1}{F(t)} \int_1^t x \delta x^{-2\delta-1} e^{-\epsilon x^{-\delta}} \left[(\epsilon + 1)x^{\delta} - \epsilon \right] dx\end{aligned}$$

$$\begin{aligned}
&= t - \frac{\varepsilon^{\frac{1}{\delta}-1}}{F(t)} \left[\Gamma\left(2 - \frac{1}{\delta}, \varepsilon\right) - \Gamma\left(2 - \frac{1}{\delta}, \varepsilon t^{-\delta}\right) \right. \\
&\quad \left. - (\varepsilon + 1) \left\{ \Gamma\left(1 - \frac{1}{\delta}, \varepsilon\right) - \Gamma\left(1 - \frac{1}{\delta}, \varepsilon t^{-\delta}\right) \right\} \right].
\end{aligned}$$

3.7. Mean Residual Life Function. The mean residual life function represents the expected remaining lifetime of a unit that has survived up to time t . For the ILSGD, the mean residual life function is given by

$$\begin{aligned}
MRL(t) &= \frac{1}{S(t)} \int_t^\infty x f(x) dx - t \\
&= \frac{1}{S(t)} \int_t^\infty x \delta x^{-2\delta-1} e^{-\varepsilon x^{-\delta}} \left[(\varepsilon + 1)x^\delta - \varepsilon \right] dx - t \\
&= \frac{\varepsilon^{\frac{1}{\delta}-1}}{\delta S(t)} \left[(\delta \varepsilon + 1) \Gamma\left(1 - \frac{1}{\delta}\right) + \delta \Gamma\left(2 - \frac{1}{\delta}, \varepsilon t^{-\delta}\right) \right. \\
&\quad \left. - \delta (\varepsilon + 1) \Gamma\left(1 - \frac{1}{\delta}, \varepsilon t^{-\delta}\right) \right] - t, \quad \delta > 1.
\end{aligned}$$

3.8. Order Statistics. Let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ denote the order statistics from a random sample of size n drawn from the ILSGD, with CDF and PDF $F(x)$ and $f(x)$ given by Equations (2) and (3), respectively. The PDF of the s -th order statistic $X_{s:n}$ is given by

$$(4) \quad f_{X_{s:n}}(x) = \frac{n!}{(s-1)!(n-s)!} [F(x)]^{s-1} [1 - F(x)]^{n-s} f(x), \quad x > 1.$$

Substituting the expressions for $F(x)$ and $f(x)$ for the ILSGD, we obtain

$$\begin{aligned}
f_{X_{s:n}}(x) &= \frac{n!}{(s-1)!(n-s)!} \left[\left(1 - x^{-\delta}\right) e^{-\varepsilon x^{-\delta}} \right]^{s-1} \left[1 - \left(1 - x^{-\delta}\right) e^{-\varepsilon x^{-\delta}} \right]^{n-s} \\
&\quad \times \left[\delta x^{-2\delta-1} e^{-\varepsilon x^{-\delta}} \left((\varepsilon + 1)x^\delta - \varepsilon \right) \right], \quad x > 1.
\end{aligned}$$

Setting $s = 1$ gives the PDF of the minimum order statistic $X_{1:n}$, while setting $s = n$ gives the PDF of the maximum order statistic $X_{n:n}$.

4. ESTIMATION METHODS

In this section, the conventional methods used to estimate the parameters of the ILSGD are presented. These methods are based on maximizing or minimizing an objective function to obtain suitable parameter estimates.

4.1. Method of Maximum Likelihood (MLE). The log-likelihood function corresponding to the PDF defined in (3) is given by

$$\ell(\delta, \varepsilon) = n \log \delta - (2\delta + 1) \sum_{i=1}^n \log x_i - \varepsilon \sum_{i=1}^n x_i^{-\delta} + \sum_{i=1}^n \log \left[(\varepsilon + 1)x_i^{\delta} - \varepsilon \right].$$

The score functions are obtained by differentiating the log-likelihood function.

The first partial derivative with respect to δ is

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} - 2 \sum_{i=1}^n \log x_i + \varepsilon \sum_{i=1}^n x_i^{-\delta} \log x_i + \sum_{i=1}^n \frac{(\varepsilon + 1)x_i^{\delta} \log x_i}{(\varepsilon + 1)x_i^{\delta} - \varepsilon}.$$

The first partial derivative with respect to ε is

$$\frac{\partial \ell}{\partial \varepsilon} = - \sum_{i=1}^n x_i^{-\delta} + \sum_{i=1}^n \frac{x_i^{\delta} - 1}{(\varepsilon + 1)x_i^{\delta} - \varepsilon}.$$

The observed information matrix is defined as

$$I(\delta, \varepsilon) = - \begin{bmatrix} \frac{\partial^2 \ell}{\partial \delta^2} & \frac{\partial^2 \ell}{\partial \delta \partial \varepsilon} \\ \frac{\partial^2 \ell}{\partial \varepsilon \partial \delta} & \frac{\partial^2 \ell}{\partial \varepsilon^2} \end{bmatrix}.$$

The second-order derivatives are given by

$$\frac{\partial^2 \ell}{\partial \delta^2} = -\frac{n}{\delta^2} - \varepsilon \sum_{i=1}^n x_i^{-\delta} (\log x_i)^2 - \sum_{i=1}^n \frac{(\varepsilon + 1)\varepsilon x_i^{\delta} (\log x_i)^2}{[(\varepsilon + 1)x_i^{\delta} - \varepsilon]^2}.$$

$$\frac{\partial^2 \ell}{\partial \varepsilon^2} = - \sum_{i=1}^n \frac{(x_i^{\delta} - 1)^2}{[(\varepsilon + 1)x_i^{\delta} - \varepsilon]^2}.$$

$$\frac{\partial^2 \ell}{\partial \delta \partial \varepsilon} = \sum_{i=1}^n x_i^{-\delta} \log x_i + \sum_{i=1}^n \frac{x_i^{\delta} \log x_i}{[(\varepsilon + 1)x_i^{\delta} - \varepsilon]^2}.$$

Due to the complexity of the likelihood function, the observed information matrix is evaluated numerically. The asymptotic standard errors and confidence intervals of the estimators can be obtained using numerical approximations of the observed information matrix.

4.2. Method of Anderson-Darling (AD). The AD estimators for the ILSGD can be obtained by minimizing the following function:

$$A(x_i) = -n - \frac{1}{n} \sum_{i=1}^n (2i - 1) \left[\log \left(\left(1 - x_{i:n}^{-\delta} \right) e^{-\varepsilon x_{i:n}^{-\delta}} \right) + \log \left\{ 1 - \left(\left(1 - x_{n-i+1:n}^{-\delta} \right) e^{-\varepsilon x_{n-i+1:n}^{-\delta}} \right) \right\} \right].$$

4.3. Method of Cramér–von Mises (CVM). The CVM estimators of the proposed distribution are determined by minimizing the following function:

$$C(x_i) = \frac{1}{12n} + \sum_{i=1}^n \left[\left(1 - x_{i:n}^{-\delta}\right) e^{-\varepsilon x_{i:n}^{-\delta}} - \frac{2i-1}{2n} \right]^2.$$

4.4. Method of maximum product of spacings (MPS). The parameters of the ILSGD can be estimated using the MPS method by maximizing the following expression:

$$MPS = \frac{1}{n+1} \sum_{i=1}^{n+1} \log I_i,$$

where $I_i = \left(\left(1 - x_{i:n}^{-\delta}\right) e^{-\varepsilon x_{i:n}^{-\delta}} \right) - \left(\left(1 - x_{i-1:n}^{-\delta}\right) e^{-\varepsilon x_{i-1:n}^{-\delta}} \right)$.

4.5. Method of least squares. The parameters of the ILSGD can be estimated using this method by minimizing the following function:

$$LS = \sum_{i=1}^n \left[\left(1 - x_{i:n}^{-\delta}\right) e^{-\varepsilon x_{i:n}^{-\delta}} - \frac{i}{n+1} \right]^2.$$

4.6. Method of right-tail Anderson–Darling. The parameters of the ILSGD can be estimated using this method by minimizing the following function:

$$R(x_i) = \frac{n}{2} - 2 \sum_{i=1}^n \left(1 - x_{i:n}^{-\delta}\right) e^{-\varepsilon x_{i:n}^{-\delta}} - \frac{1}{n} \sum_{i=1}^n (2i-1) \log \left(1 - \left(1 - x_{n-i+1:n}^{-\delta}\right) e^{-\varepsilon x_{n-i+1:n}^{-\delta}} \right).$$

4.7. Method of weighted least squares. The parameters of the ILSGD can be estimated using this method by minimizing the following function:

$$W(x_i) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[\left(\left(1 - x_{i:n}^{-\delta}\right) e^{-\varepsilon x_{i:n}^{-\delta}} \right) - \frac{i}{n+1} \right]^2.$$

4.8. Method of left-tail Anderson–Darling. The parameters of the ILSGD can be estimated using this method by minimizing the following function:

$$L(x_i) = -\frac{3}{2}n + 2 \sum_{i=1}^n \left(1 - x_{i:n}^{-\delta}\right) e^{-\varepsilon x_{i:n}^{-\delta}} - \frac{1}{n} \sum_{i=1}^n (2i-1) \log \left(\left(1 - x_{i:n}^{-\delta}\right) e^{-\varepsilon x_{i:n}^{-\delta}} \right).$$

4.9. Method of minimum spacing absolute distance. The parameters of the ILSGD can be estimated using this method by minimizing the following function:

$$\zeta(x_i) = \sum_{i=1}^{n+1} \left| I_i - \frac{1}{n+1} \right|.$$

4.10. Method of minimum spacing absolute-log distance. The parameters of the ILSGD can be estimated using this method by minimizing the following function:

$$Y(x_i) = \sum_{i=1}^{n+1} \left| \log I_i - \log \frac{1}{n+1} \right|.$$

4.11. Method of Anderson Darling left tail second order. The parameters of the ILSGD can be estimated using this method by minimizing the following function:

$$LTS = 2 \sum_{i=1}^n \log \left(\left(1 - x_i^{-\delta} \right) e^{-\varepsilon x_i^{-\delta}} \right) + \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{\left(1 - x_i^{-\delta} \right) e^{-\varepsilon x_i^{-\delta}}}.$$

4.12. Kolmogorov–Smirnov Method. The parameters of the ILSGD can be estimated using this method by optimizing the following function:

$$KM = \max_{1 \leq i \leq n} \left[\frac{i}{n} - \left(1 - x_i^{-\delta} \right) e^{-\varepsilon x_i^{-\delta}}, \left(1 - x_i^{-\delta} \right) e^{-\varepsilon x_i^{-\delta}} - \frac{i-1}{n} \right].$$

4.13. Method of minimum spacing square distance. The parameters of the ILSGD can be estimated using this method by minimizing the following function:

$$\phi_{(x_i)} = \sum_{i=1}^{n+1} \left(I_i - \frac{1}{n+1} \right)^2.$$

4.14. Method of minimum spacing square-log distance. The parameters of the ILSGD can be estimated using this method by minimizing the following function:

$$\Psi_{(x_i)} = \sum_{i=1}^{n+1} \left(\log I_i - \log \frac{1}{n+1} \right)^2.$$

4.15. Method of minimum spacing Linex distance. The parameters of the ILSGD can be estimated using this method by minimizing the following function:

$$\Delta_{(x_i)} = \sum_{i=1}^{n+1} \left[e^{I_i - \frac{1}{n+1}} - \left(I_i - \frac{1}{n+1} \right) - 1 \right].$$

5. NUMERICAL SIMULATION

This section evaluates the performance of the estimation methods presented in Section 4 using randomly generated samples from the ILSGD. Different sample sizes and predetermined parameter values are considered to examine the accuracy of the estimators. Six measures are used to compare their performance:

- Average bias (BIAS), $|Bias(\hat{\mathbf{J}})| = \frac{1}{L} \sum_{i=1}^L |\hat{\mathbf{J}} - \mathbf{J}|.$

- Mean squared errors (MSE), $MSE = \frac{1}{L} \sum_{i=1}^L (\hat{\mathbf{J}} - \mathbf{J})^2$.
- Mean relative errors (MRE), $MRE = \frac{1}{L} \sum_{i=1}^L |\hat{\mathbf{J}} - \mathbf{J}| / \mathbf{J}$.
- Average absolute difference (D_{abs}), $D_{abs} = \frac{1}{nH} \sum_{i=1}^H \sum_{j=1}^n |F(x_{ij}; \mathbf{J}) - F(x_{ij}; \hat{\mathbf{J}})|$.
- Maximum absolute difference (D_{max}), $D_{max} = \frac{1}{H} \sum_{i=1}^H \max_{j=1, \dots, n} |F(x_{ij}; \mathbf{J}) - F(x_{ij}; \hat{\mathbf{J}})|$.
- Average scaled absolute error, $ASAE = \frac{1}{n} \sum_{i=1}^n \frac{|x_i - \hat{x}_i|}{x_n - x_1}$, $\mathbf{J} = (\delta, \varepsilon)$.

The main aim is to identify the best-performing method for estimating the parameters of the proposed model. We used numerous simulations to obtain reliable results. The outcomes of our simulation are presented in Tables 2 to 6. We also created graphs that illustrate the data in Table 2. Figures 4 to 9 illustrate these graphs. Each estimation method possesses the characteristic of consistency regarding the complete set of outcomes. The ranking method was employed to determine the most effective approach and to evaluate all alternatives. Table 7 presents the overall rankings of the methods. The results show that MLE performs best, followed by MPSE.

6. REAL DATA ANALYSIS

This section examines the practical performance of the inverse Log-shifted Gompertz distribution (ILSGD) using three real datasets from hydrology, reliability engineering, and medical studies. These datasets are right-skewed and contain extreme observations, which makes them suitable for assessing the flexibility of the proposed distribution. The datasets are described in the next subsection.

6.1. Data description. The first dataset, discussed by [1], contains the maximum annual flood discharges of the North Saskatchewan River at Edmonton over a period of 47 years. The observations are measured in thousands of cubic feet per second. The data are: 46.3, 40.4, 32.6, 32.68, 38.1, 50.33, 24.888, 106.6, 34.4, 74.1, 58.7, 40.0, 39.02, 65.597, 58.8, 185.56, 31.5, 61.2, 51.442, 30.2, 26.72, 44.73, 44.9, 61.74, 28.1, 25.46, 28.6, 84.1, 19.885, 121.97, 57.22, 23.7, 109.7, 44.02, 20.94, 65.44, 75.8, 30.38, 40.4, 27.5, 21.82, 35.7, 42.25, 35.347, 66.0, 39.2, 25.76.

The second dataset, discussed by [2], presents the failure times of 24 mechanical components. The observations are: 18.84, 14.1, 12.26, 22.57, 30.94, 14.97, 22.38, 17.12, 16.62, 10.67, 29.93, 22.85, 19.08, 27.98, 27.47, 51.56, 14.7, 49.56, 19.4, 25.43, 10.24, 18.51, 18.14, 36.02.

TABLE 2. Numerical values of simulation measures for $\delta = 1.25$, $\varepsilon = 2.75$.

n	Est.	MLE	ADE	CVME	MPSE	OLSA	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{\delta}$)	0.18815 ⁽¹⁾	0.20631 ⁽⁴⁾	0.22551 ⁽⁷⁾	0.20361 ⁽³⁾	0.23409 ⁽⁹⁾	0.20027 ⁽²⁾	0.23391 ⁽⁸⁾	0.26610 ⁽¹²⁾	0.23492 ⁽¹⁰⁾	0.29466 ⁽¹¹⁾	0.29433 ⁽¹⁴⁾	0.21987 ⁽⁶⁾	0.29405 ⁽¹³⁾	0.21223 ⁽⁵⁾	0.31895 ⁽¹⁵⁾
20	BIAS($\hat{\varepsilon}$)	0.82638 ⁽¹¹⁾	0.97765 ⁽³⁾	1.20260 ⁽¹⁴⁾	1.02344 ⁽⁶⁾	0.88770 ⁽²⁾	1.02254 ⁽⁵⁾	1.07014 ⁽⁹⁾	1.05132 ⁽⁸⁾	1.01622 ⁽⁴⁾	1.13004 ⁽¹¹⁾	1.15675 ⁽¹²⁾	1.12621 ⁽¹⁰⁾	1.17290 ⁽¹³⁾	1.02525 ⁽⁷⁾	1.25235 ⁽¹⁵⁾
20	MSE($\hat{\delta}$)	0.05645 ⁽¹¹⁾	0.07089 ⁽⁵⁾	0.08099 ⁽⁷⁾	0.06228 ⁽³⁾	0.08115 ⁽⁸⁾	0.06132 ⁽²⁾	0.08570 ⁽⁹⁾	0.11383 ⁽¹²⁾	0.09088 ⁽¹¹⁾	0.08639 ⁽¹⁰⁾	0.12854 ⁽¹⁴⁾	0.07910 ⁽⁶⁾	0.12269 ⁽¹³⁾	0.06747 ⁽⁴⁾	0.14638 ⁽¹⁵⁾
20	MSE($\hat{\varepsilon}$)	1.08557 ⁽¹⁾	1.37149 ⁽³⁾	1.95579 ⁽¹³⁾	1.44627 ⁽⁵⁾	1.15668 ⁽²⁾	1.44266 ⁽⁴⁾	1.60199 ⁽⁸⁾	1.65517 ⁽⁹⁾	1.52571 ⁽⁷⁾	1.80729 ⁽¹⁰⁾	1.88219 ⁽¹²⁾	1.80968 ⁽¹¹⁾	2.01588 ⁽¹⁴⁾	1.47465 ⁽⁶⁾	2.08370 ⁽¹⁵⁾
20	MRE($\hat{\delta}$)	0.15052 ⁽¹⁾	0.16504 ⁽⁴⁾	0.18041 ⁽⁷⁾	0.16289 ⁽³⁾	0.18727 ⁽⁹⁾	0.16022 ⁽²⁾	0.18712 ⁽⁸⁾	0.21288 ⁽¹²⁾	0.18793 ⁽¹⁰⁾	0.19972 ⁽¹¹⁾	0.23547 ⁽¹⁴⁾	0.17589 ⁽⁶⁾	0.23524 ⁽¹³⁾	0.16978 ⁽⁵⁾	0.25516 ⁽¹⁵⁾
20	MRE($\hat{\varepsilon}$)	0.30050 ⁽¹⁾	0.35551 ⁽³⁾	0.43731 ⁽¹⁴⁾	0.37216 ⁽⁶⁾	0.32280 ⁽²⁾	0.37183 ⁽⁵⁾	0.38914 ⁽⁹⁾	0.38230 ⁽⁸⁾	0.36953 ⁽⁴⁾	0.41092 ⁽¹¹⁾	0.42064 ⁽¹²⁾	0.40953 ⁽¹⁰⁾	0.42651 ⁽¹³⁾	0.37282 ⁽⁷⁾	0.45540 ⁽¹⁵⁾
20	D_{abs}	0.05764 ⁽⁵⁾	0.05176 ⁽¹⁾	0.05650 ⁽⁴⁾	0.05529 ⁽³⁾	0.06549 ⁽¹¹⁾	0.05241 ⁽²⁾	0.06073 ⁽⁸⁾	0.06350 ⁽⁹⁾	0.06950 ⁽¹²⁾	0.07035 ⁽¹⁴⁾	0.06359 ⁽¹⁰⁾	0.05887 ⁽⁶⁾	0.07010 ⁽¹³⁾	0.06064 ⁽⁷⁾	0.07467 ⁽¹⁵⁾
20	D_{max}	0.09070 ⁽⁴⁾	0.08244 ⁽¹⁾	0.09138 ⁽⁵⁾	0.08775 ⁽³⁾	0.10210 ⁽⁹⁾	0.08555 ⁽²⁾	0.09564 ⁽⁸⁾	0.10283 ⁽¹⁰⁾	0.10703 ⁽¹²⁾	0.11096 ⁽¹⁴⁾	0.10521 ⁽¹¹⁾	0.09519 ⁽⁷⁾	0.10888 ⁽¹³⁾	0.09454 ⁽⁶⁾	0.12111 ⁽¹⁵⁾
20	ASAE	0.04800 ⁽³⁾	0.04695 ⁽²⁾	0.05747 ⁽⁸⁾	0.04422 ⁽¹⁾	0.06816 ⁽¹²⁾	0.04841 ⁽⁴⁾	0.05002 ⁽⁵⁾	0.06743 ⁽¹¹⁾	0.05028 ⁽⁶⁾	0.05625 ⁽⁷⁾	0.13047 ⁽¹⁵⁾	0.05784 ⁽⁹⁾	0.07709 ⁽¹³⁾	0.05856 ⁽¹⁰⁾	0.08431 ⁽¹⁴⁾
20	$\Sigma Ranks$	18 ⁽¹⁾	26 ⁽²⁾	79 ⁽¹⁰⁾	33 ⁽⁴⁾	64 ⁽⁶⁾	28 ⁽³⁾	72 ⁽⁸⁾	91 ⁽¹¹⁾	76 ⁽⁹⁾	99 ⁽¹²⁾	114 ⁽¹³⁾	71 ⁽⁷⁾	118 ⁽¹⁴⁾	57 ⁽⁵⁾	134 ⁽¹⁵⁾
70	BIAS($\hat{\delta}$)	0.10176 ⁽¹⁾	0.11247 ⁽²⁾	0.15069 ⁽¹²⁾	0.12611 ⁽⁴⁾	0.11344 ⁽³⁾	0.12912 ⁽⁶⁾	0.12868 ⁽⁵⁾	0.12996 ⁽⁷⁾	0.14479 ⁽¹⁰⁾	0.13318 ⁽⁸⁾	0.20091 ⁽¹⁵⁾	0.13650 ⁽⁹⁾	0.16301 ⁽¹³⁾	0.14830 ⁽¹¹⁾	0.18643 ⁽¹⁴⁾
70	BIAS($\hat{\varepsilon}$)	0.53106 ⁽¹⁾	0.58929 ⁽²⁾	0.71735 ⁽¹¹⁾	0.62079 ⁽⁵⁾	0.60134 ⁽⁴⁾	0.79958 ⁽¹²⁾	0.71043 ⁽¹⁰⁾	0.60061 ⁽³⁾	0.66125 ⁽⁸⁾	0.70456 ⁽⁹⁾	0.80889 ⁽¹³⁾	0.65816 ⁽⁷⁾	0.84845 ⁽¹⁴⁾	0.62725 ⁽⁶⁾	0.89589 ⁽¹⁵⁾
70	MSE($\hat{\delta}$)	0.01535 ⁽¹⁾	0.01935 ⁽²⁾	0.03200 ⁽¹¹⁾	0.02306 ⁽⁴⁾	0.02165 ⁽³⁾	0.02568 ⁽⁶⁾	0.02784 ⁽⁷⁾	0.02446 ⁽⁵⁾	0.03335 ⁽¹²⁾	0.02843 ⁽⁸⁾	0.06444 ⁽¹⁵⁾	0.03191 ⁽⁹⁾	0.03957 ⁽¹³⁾	0.03197 ⁽¹⁰⁾	0.04945 ⁽¹⁴⁾
70	MSE($\hat{\varepsilon}$)	0.47796 ⁽¹⁾	0.58733 ⁽³⁾	0.80958 ⁽¹¹⁾	0.60572 ⁽⁵⁾	0.61966 ⁽⁶⁾	1.01163 ⁽¹²⁾	0.70222 ⁽¹²⁾	0.59499 ⁽⁴⁾	0.67757 ⁽⁷⁾	0.75640 ⁽¹⁰⁾	1.07391 ⁽¹⁴⁾	0.74223 ⁽⁹⁾	1.03648 ⁽¹³⁾	0.58368 ⁽²⁾	1.12137 ⁽¹⁵⁾
70	MRE($\hat{\delta}$)	0.08141 ⁽¹⁾	0.08998 ⁽²⁾	0.12055 ⁽¹²⁾	0.10089 ⁽⁴⁾	0.09075 ⁽³⁾	0.10329 ⁽⁶⁾	0.10294 ⁽⁵⁾	0.10397 ⁽⁷⁾	0.11583 ⁽¹⁰⁾	0.10654 ⁽⁸⁾	0.16073 ⁽¹⁵⁾	0.10920 ⁽⁹⁾	0.13041 ⁽¹³⁾	0.11861 ⁽¹¹⁾	0.14915 ⁽¹⁴⁾
70	MRE($\hat{\varepsilon}$)	0.19311 ⁽¹⁾	0.21429 ⁽²⁾	0.26085 ⁽¹¹⁾	0.22574 ⁽⁵⁾	0.21867 ⁽⁴⁾	0.29076 ⁽¹²⁾	0.25834 ⁽¹⁰⁾	0.21840 ⁽³⁾	0.24045 ⁽⁸⁾	0.25620 ⁽⁹⁾	0.29414 ⁽¹³⁾	0.23933 ⁽⁷⁾	0.30853 ⁽¹⁴⁾	0.22809 ⁽⁶⁾	0.32578 ⁽¹⁵⁾
70	D_{abs}	0.02824 ⁽¹⁾	0.02942 ⁽³⁾	0.03289 ⁽⁷⁾	0.02962 ⁽⁴⁾	0.03280 ⁽²⁾	0.03246 ⁽⁵⁾	0.03516 ⁽⁶⁾	0.03143 ⁽¹⁵⁾	0.03767 ⁽¹²⁾	0.03574 ⁽¹⁰⁾	0.04027 ⁽¹⁵⁾	0.03568 ⁽⁹⁾	0.03948 ⁽¹⁴⁾	0.03654 ⁽¹¹⁾	0.03865 ⁽¹³⁾
70	D_{max}	0.04559 ⁽¹⁾	0.04775 ⁽³⁾	0.05467 ⁽⁷⁾	0.04796 ⁽⁴⁾	0.04733 ⁽²⁾	0.05371 ⁽⁶⁾	0.05671 ⁽⁸⁾	0.05155 ⁽⁵⁾	0.06140 ⁽¹²⁾	0.05832 ⁽¹⁰⁾	0.06645 ⁽¹⁵⁾	0.05787 ⁽⁹⁾	0.06514 ⁽¹⁴⁾	0.05905 ⁽¹¹⁾	0.06496 ⁽¹³⁾
70	ASAE	0.01805 ⁽¹⁾	0.02078 ⁽⁹⁾	0.01936 ⁽⁵⁾	0.02072 ⁽⁷⁾	0.02444 ⁽¹¹⁾	0.01932 ⁽⁴⁾	0.01809 ⁽²⁾	0.02501 ⁽¹²⁾	0.02160 ⁽¹⁰⁾	0.01998 ⁽⁶⁾	0.04517 ⁽¹⁵⁾	0.02075 ⁽⁸⁾	0.02581 ⁽¹³⁾	0.01826 ⁽³⁾	0.02714 ⁽¹⁴⁾
70	$\Sigma Ranks$	9 ⁽¹⁾	28 ⁽²⁾	87 ⁽¹¹⁾	42 ⁽⁴⁾	38 ⁽³⁾	70 ⁽⁷⁾	63 ⁽⁶⁾	51 ⁽⁵⁾	89 ⁽¹²⁾	78 ⁽¹⁰⁾	130 ⁽¹⁵⁾	76 ⁽⁹⁾	121 ⁽¹³⁾	71 ⁽⁸⁾	127 ⁽¹⁴⁾
150	BIAS($\hat{\delta}$)	0.07347 ⁽¹⁾	0.08150 ⁽³⁾	0.09127 ⁽⁹⁾	0.07650 ⁽²⁾	0.09257 ⁽¹⁰⁾	0.09348 ⁽¹¹⁾	0.08361 ⁽⁴⁾	0.09025 ⁽¹²⁾	0.09400 ⁽¹⁰⁾	0.08894 ⁽⁶⁾	0.14098 ⁽¹⁵⁾	0.08724 ⁽⁵⁾	0.10919 ⁽¹⁴⁾	0.09111 ⁽⁸⁾	0.09835 ⁽¹³⁾
150	BIAS($\hat{\varepsilon}$)	0.41294 ⁽²⁾	0.41621 ⁽³⁾	0.46940 ⁽⁸⁾	0.40496 ⁽¹⁾	0.46960 ⁽⁹⁾	0.55850 ⁽¹⁴⁾	0.42322 ⁽⁵⁾	0.43766 ⁽⁷⁾	0.48534 ⁽¹²⁾	0.48439 ⁽¹¹⁾	0.64212 ⁽¹⁵⁾	0.48400 ⁽¹⁰⁾	0.54354 ⁽¹³⁾	0.42792 ⁽⁶⁾	0.41739 ⁽⁴⁾
150	MSE($\hat{\delta}$)	0.00841 ⁽¹⁾	0.01061 ⁽³⁾	0.01262 ⁽⁷⁾	0.00834 ⁽¹⁾	0.01324 ⁽⁹⁾	0.01355 ⁽¹¹⁾	0.01161 ⁽⁴⁾	0.01226 ⁽⁶⁾	0.01388 ⁽¹²⁾	0.01276 ⁽⁸⁾	0.03577 ⁽¹⁵⁾	0.01208 ⁽⁵⁾	0.01865 ⁽¹⁴⁾	0.01334 ⁽¹⁰⁾	0.01416 ⁽¹³⁾
150	MSE($\hat{\varepsilon}$)	0.27961 ⁽³⁾	0.26080 ⁽²⁾	0.34592 ⁽⁷⁾	0.25709 ⁽¹⁾	0.35659 ⁽⁹⁾	0.54440 ⁽¹⁴⁾	0.30073 ⁽⁵⁾	0.30092 ⁽⁶⁾	0.39257 ⁽¹¹⁾	0.40390 ⁽¹²⁾	0.64421 ⁽¹⁵⁾	0.37403 ⁽¹⁰⁾	0.45468 ⁽¹³⁾	0.35131 ⁽⁸⁾	0.29046 ⁽⁴⁾
150	MRE($\hat{\delta}$)	0.05877 ⁽¹⁾	0.06520 ⁽³⁾	0.07302 ⁽⁹⁾	0.06120 ⁽²⁾	0.07406 ⁽¹⁰⁾	0.07478 ⁽¹¹⁾	0.06689 ⁽⁴⁾	0.07220 ⁽⁷⁾	0.07520 ⁽¹²⁾	0.07115 ⁽⁶⁾	0.11278 ⁽¹⁵⁾	0.06980 ⁽⁵⁾	0.08735 ⁽¹⁴⁾	0.07289 ⁽⁸⁾	0.07868 ⁽¹³⁾
150	MRE($\hat{\varepsilon}$)	0.15016 ⁽²⁾	0.15135 ⁽³⁾	0.17069 ⁽⁸⁾	0.12476 ⁽¹⁾	0.17076 ⁽⁹⁾	0.20309 ⁽¹⁴⁾	0.15390 ⁽⁵⁾	0.15915 ⁽⁷⁾	0.17649 ⁽¹²⁾	0.17614 ⁽¹¹⁾	0.23350 ⁽¹⁵⁾	0.17600 ⁽¹⁰⁾	0.19765 ⁽¹³⁾	0.15561 ⁽⁶⁾	0.15178 ⁽⁴⁾
150	D_{abs}	0.02314 ⁽⁷⁾	0.02279 ⁽⁶⁾	0.02212 ⁽⁴⁾	0.01995 ⁽¹⁾	0.02206 ⁽³⁾	0.02206 ⁽³⁾	0.02236 ⁽⁵⁾	0.02258 ⁽¹⁵⁾	0.02513 ⁽¹²⁾	0.02330 ⁽⁹⁾	0.02783 ⁽¹⁴⁾	0.02396 ⁽¹⁰⁾	0.02895 ⁽¹⁵⁾	0.02190 ⁽²⁾	0.02530 ⁽¹³⁾
150	D_{max}	0.03700 ⁽⁶⁾	0.03655 ⁽⁴⁾	0.03607 ⁽³⁾	0.03276 ⁽¹⁾	0.03907 ⁽¹¹⁾	0.03687 ⁽⁵⁾	0.03759 ⁽⁸⁾	0.03701 ⁽⁷⁾	0.04112 ⁽¹³⁾	0.03808 ⁽⁹⁾	0.04695 ⁽¹⁵⁾	0.03896 ⁽¹⁰⁾	0.04642 ⁽¹⁴⁾	0.03571 ⁽²⁾	0.04027 ⁽¹²⁾
150	ASAE	0.00945 ⁽²⁾	0.01048 ⁽⁸⁾	0.01071 ⁽⁹⁾	0.00893 ⁽¹⁾	0.01027 ⁽⁷⁾	0.00977 ⁽⁴⁾	0.00984 ⁽⁵⁾	0.01289 ⁽¹²⁾	0.01023 ⁽⁶⁾	0.00961 ⁽³⁾	0.02737 ⁽¹⁵⁾	0.01076 ⁽¹⁰⁾	0.01455 ⁽¹⁴⁾	0.01133 ⁽¹¹⁾	0.01356 ⁽¹³⁾
150	$\Sigma Ranks$	26 ⁽²⁾	35 ⁽³⁾	61 ^(6.5)	11 ⁽¹⁾	85 ⁽¹⁰⁾	87 ⁽¹¹⁾	48 ⁽⁴⁾	64 ^(6.5)	102 ⁽¹³⁾	75 ^(8.5)	134 ⁽¹⁵⁾	75 ^(8.5)	124 ⁽¹⁴⁾	61 ⁽⁵⁾	89 ⁽¹²⁾
220	BIAS($\hat{\delta}$)	0.06649 ⁽⁵⁾	0.06210 ⁽²⁾	0.06592 ⁽⁴⁾	0.05479 ⁽¹⁾	0.07365 ⁽⁹⁾	0.07275 ⁽⁸⁾	0.06226 ⁽³⁾	0.07779 ⁽¹²⁾	0.07271 ⁽⁷⁾	0.07638 ⁽¹¹⁾	0.11643 ⁽¹⁵⁾	0.07443 ⁽¹⁰⁾	0.09471 ⁽¹⁴⁾	0.06848 ⁽⁶⁾	0.08246 ⁽¹³⁾
220	BIAS($\hat{\varepsilon}$)	0.39556 ⁽¹⁰⁾	0.35435 ⁽²⁾	0.36706 ⁽⁴⁾	0.34242 ⁽¹⁾	0.38192 ⁽⁷⁾	0.40935 ⁽¹²⁾	0.36095 ⁽⁵⁾	0.37373 ⁽⁶⁾	0.36957 ⁽⁵⁾	0.40755 ⁽¹¹⁾	0.48338 ⁽¹⁵⁾	0.43986 ⁽¹³⁾	0.47025 ⁽¹⁴⁾	0.38364 ⁽⁸⁾	0.39513 ⁽⁹⁾
220	MSE($\hat{\delta}$)	0.00714 ⁽⁵⁾	0.00582 ⁽²⁾	0.00674 ⁽⁴⁾	0.00490 ⁽¹⁾	0.00864 ⁽¹⁰⁾	0.00794 ⁽⁷⁾	0.00606 ⁽³⁾	0.00851 ⁽⁹⁾	0.00878 ⁽¹¹⁾	0.00886 ⁽¹²⁾	0.02162 ⁽¹⁵⁾	0.00818 ⁽⁸⁾	0.01384 ⁽¹⁴⁾	0.00716 ⁽⁶⁾	0.01049 ⁽¹³⁾
220	MSE($\hat{\varepsilon}$)	0.23574 ⁽⁹⁾	0.20113 ⁽¹⁾	0.22436 ⁽⁶⁾	0.20333 ⁽²⁾	0.22981 ⁽⁷⁾	0.26869 ⁽¹²⁾	0.21692 ⁽⁵⁾	0.21034 ⁽³⁾	0.21679 ⁽⁴⁾	0.23934 ⁽¹⁰⁾	0.39057 ⁽¹⁵⁾	0.29277 ⁽¹³⁾	0.34622 ⁽¹⁴⁾	0.23019 ⁽⁸⁾	0.24260 ⁽¹¹⁾
220	MRE($\hat{\delta}$)	0.05320 ⁽⁵⁾	0.04968 ⁽²⁾	0.05273 ⁽⁴⁾	0.04383 ⁽¹⁾	0.05892 ⁽⁹⁾	0.05820 ⁽⁸⁾	0.04981 ⁽³⁾	0.06223 ⁽¹²⁾	0.05817 ⁽⁷⁾	0.06111 ⁽¹¹⁾	0.09315 ⁽¹⁵⁾	0.05954 ⁽¹⁰⁾	0.07577 ⁽¹⁴⁾	0.05478 ⁽⁶⁾	0.06597 ⁽¹³⁾
220	MRE($\hat{\varepsilon}$)	0.14384 ⁽¹⁰⁾	0.12885 ⁽²⁾	0.13348 ⁽⁴⁾	0.12452 ⁽¹⁾	0.13888 ⁽⁷⁾	0.14886 ⁽¹²⁾	0.13126 ⁽³⁾	0.13590 ⁽⁶⁾	0.13439 ⁽⁵⁾	0.14820 ⁽¹¹⁾	0.17578 ⁽¹⁵⁾	0.15995 ⁽¹³⁾	0.17100 ⁽¹⁴⁾	0.13951 ⁽⁸⁾	0.14368 ⁽⁹⁾
220	D_{abs}	0.01790 ⁽⁶⁾	0.01701 ⁽¹⁾	0.01772 ⁽⁵⁾	0.01949 ⁽¹⁰⁾	0.01749 ⁽³⁾	0.01872 ⁽⁷⁾	0.01747 ⁽²⁾	0.01926 ⁽⁹⁾	0.01920 ⁽⁸⁾	0.01993 ⁽¹²⁾	0.02194 ⁽¹⁴⁾	0.01754 ⁽⁴⁾	0.02251 ⁽¹⁵⁾	0.01972 ⁽¹¹⁾	0.02134 ⁽¹³⁾
220	D_{max}	0.02930 ^(5.5)	0.02794 ⁽¹⁾	0.02919 ⁽⁴⁾	0.03058 ⁽⁷⁾	0.02902 ⁽³⁾	0.03078 ⁽⁸⁾	0.02849 ⁽²⁾	0.03145 ⁽¹⁰⁾	0.03106 ⁽⁹⁾	0.03244 ⁽¹²⁾	0.03680 ⁽¹⁴⁾	0.02930 ^(5.5)	0.03701 ⁽¹⁵⁾	0.03164 ⁽¹¹⁾	0.03392 ⁽¹³⁾
220	ASAE	0.00671 ⁽²⁾	0.00779 ⁽⁸⁾	0.00696 ⁽⁵⁾	0.00661 ⁽¹⁾	0.00809 ⁽¹¹⁾	0.00684 ⁽⁴⁾	0.00673 ⁽³⁾	0.00795 ⁽⁹⁾	0.00804 ⁽¹⁰⁾	0.00946 ⁽¹⁴⁾	0.01284 ⁽¹⁵⁾	0.00747 ⁽⁶⁾	0.00864 ⁽¹²⁾	0.00756 ⁽⁷⁾	0.00879 ⁽¹³⁾
220	$\Sigma Ranks$	57.5 ⁽⁵⁾	21 ⁽¹⁾	40 ⁽⁴⁾	25 ⁽²⁾	66 ^(6.5)	78 ⁽¹⁰⁾	27 ⁽³⁾	76 ⁽⁹⁾	66 ^(6.5)	104 ⁽¹²⁾	133 ⁽¹⁵⁾	82.5 ⁽¹¹⁾	126 ⁽¹⁴⁾	71 ⁽⁸⁾	107 ⁽¹³⁾
280	BIAS($\hat{\delta}$)	0.06097 ⁽⁵⁾	0.06159 ⁽⁶⁾	0.06484 ⁽⁸⁾	0.04919 ⁽¹⁾	0.07006 ⁽¹¹⁾	0.05858 ⁽³⁾	0.05157 ⁽²⁾	0.07014 ⁽¹²⁾	0.06361 ⁽⁷⁾	0.06718 ⁽¹⁰⁾	0.10565 ⁽¹⁵⁾	0.0603			

TABLE 3. Numerical values of simulation measures for $\delta = 0.25$, $\varepsilon = 0.75$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSDSE	MSLDE	MSLNDE
20	BIAS($\hat{\delta}$)	0.04265 ⁽¹⁾	0.04716 ⁽⁴⁾	0.05190 ⁽¹²⁾	0.04745 ⁽⁵⁾	0.05018 ⁽¹⁰⁾	0.04489 ⁽²⁾	0.04787 ⁽⁷⁾	0.05166 ⁽¹¹⁾	0.04892 ⁽⁸⁾	0.04942 ⁽⁹⁾	0.05854 ⁽¹⁵⁾	0.04767 ⁽⁶⁾	0.05489 ⁽¹⁴⁾	0.04618 ⁽³⁾	0.05193 ⁽¹³⁾
20	BIAS($\hat{\varepsilon}$)	0.35511 ⁽¹⁰⁾	0.36371 ⁽¹⁴⁾	0.35889 ⁽¹¹⁾	0.35199 ⁽⁸⁾	0.35165 ⁽⁶⁾	0.34050 ⁽¹⁾	0.35320 ⁽⁹⁾	0.36778 ⁽¹⁵⁾	0.34658 ⁽⁹⁾	0.34313 ⁽¹²⁾	0.34885 ⁽⁴⁾	0.35169 ⁽⁷⁾	0.35929 ⁽¹²⁾	0.36084 ⁽¹³⁾	0.35083 ⁽⁵⁾
20	MSE($\hat{\delta}$)	0.00314 ⁽¹⁾	0.00346 ⁽⁴⁾	0.00447 ⁽¹³⁾	0.00350 ⁽⁸⁾	0.00383 ⁽¹⁰⁾	0.00332 ⁽³⁾	0.00362 ⁽⁶⁾	0.00431 ⁽¹²⁾	0.00379 ⁽⁹⁾	0.00373 ⁽⁸⁾	0.00558 ⁽¹⁵⁾	0.00368 ⁽⁷⁾	0.00452 ⁽¹⁴⁾	0.00325 ⁽²⁾	0.00390 ⁽¹¹⁾
20	MSE($\hat{\varepsilon}$)	0.17049 ⁽⁹⁾	0.17894 ⁽¹⁴⁾	0.17317 ⁽¹¹⁾	0.17004 ⁽⁸⁾	0.16555 ⁽⁴⁾	0.15931 ⁽¹⁾	0.17392 ⁽¹²⁾	0.18146 ⁽¹⁵⁾	0.16513 ⁽³⁾	0.16175 ⁽²⁾	0.16792 ⁽⁷⁾	0.16765 ⁽⁶⁾	0.17401 ⁽¹³⁾	0.17153 ⁽¹⁰⁾	0.16596 ⁽⁵⁾
20	MRE($\hat{\delta}$)	0.17062 ⁽¹⁾	0.18864 ⁽⁴⁾	0.20759 ⁽¹²⁾	0.18979 ⁽⁸⁾	0.20070 ⁽¹⁰⁾	0.17955 ⁽²⁾	0.19149 ⁽⁷⁾	0.20665 ⁽¹¹⁾	0.19568 ⁽⁹⁾	0.19767 ⁽⁹⁾	0.23416 ⁽¹⁵⁾	0.19068 ⁽⁶⁾	0.21956 ⁽¹⁴⁾	0.18470 ⁽³⁾	0.20771 ⁽¹³⁾
20	MRE($\hat{\varepsilon}$)	0.47348 ⁽¹⁰⁾	0.48495 ⁽¹⁴⁾	0.47852 ⁽¹¹⁾	0.46932 ⁽⁸⁾	0.46886 ⁽⁶⁾	0.45400 ⁽¹⁾	0.47094 ⁽⁹⁾	0.49037 ⁽¹⁵⁾	0.46211 ⁽³⁾	0.45751 ⁽²⁾	0.46514 ⁽⁴⁾	0.46892 ⁽⁷⁾	0.47905 ⁽¹²⁾	0.48111 ⁽¹³⁾	0.46777 ⁽⁵⁾
20	D_{abs}	0.04583 ⁽¹⁾	0.04920 ⁽³⁾	0.05214 ⁽⁹⁾	0.05201 ⁽⁸⁾	0.05158 ⁽⁷⁾	0.04963 ⁽⁴⁾	0.05034 ⁽⁵⁾	0.05384 ⁽¹⁰⁾	0.05583 ⁽¹²⁾	0.05718 ⁽¹³⁾	0.05418 ⁽¹¹⁾	0.05042 ⁽⁶⁾	0.06008 ⁽¹⁵⁾	0.04842 ⁽²⁾	0.05744 ⁽¹⁴⁾
20	D_{max}	0.07355 ⁽¹⁾	0.07772 ⁽³⁾	0.08341 ⁽⁹⁾	0.07963 ⁽⁶⁾	0.08064 ⁽⁸⁾	0.07817 ⁽⁴⁾	0.07859 ⁽⁵⁾	0.08488 ⁽¹⁰⁾	0.08589 ⁽¹¹⁾	0.08713 ⁽¹³⁾	0.08638 ⁽¹²⁾	0.08009 ⁽⁷⁾	0.09225 ⁽¹⁵⁾	0.07610 ⁽²⁾	0.08776 ⁽¹⁴⁾
20	ASAE	0.10828 ⁽¹⁾	0.97665 ⁽⁸⁾	0.66987 ⁽⁶⁾	0.52731 ⁽⁵⁾	2.95561 ⁽¹²⁾	0.17920 ⁽²⁾	0.38240 ⁽³⁾	3.64363 ⁽¹³⁾	0.45481 ⁽⁴⁾	10.05164 ⁽¹⁵⁾	9.51637 ⁽¹⁴⁾	2.11132 ⁽⁹⁾	2.16418 ⁽¹⁰⁾	0.71799 ⁽⁷⁾	2.95001 ⁽¹¹⁾
20	$\Sigma Ranks$	35 ⁽²⁾	68 ⁽⁸⁾	94 ⁽¹²⁾	58 ⁽⁴⁾	73 ^(9,5)	20 ⁽¹⁾	63 ⁽⁷⁾	112 ⁽¹⁴⁾	61 ^(5,5)	73 ^(9,5)	97 ⁽¹³⁾	61 ^(5,5)	119 ⁽¹⁵⁾	55 ⁽³⁾	91 ⁽¹¹⁾
70	BIAS($\hat{\delta}$)	0.02887 ⁽¹⁾	0.02923 ⁽²⁾	0.03201 ⁽⁹⁾	0.03029 ⁽⁶⁾	0.03118 ⁽⁷⁾	0.02947 ⁽³⁾	0.02951 ⁽⁴⁾	0.03613 ⁽¹³⁾	0.03328 ⁽¹⁰⁾	0.03361 ⁽¹¹⁾	0.03880 ⁽¹⁵⁾	0.03140 ⁽⁸⁾	0.03690 ⁽¹⁴⁾	0.03021 ⁽⁵⁾	0.03610 ⁽¹²⁾
70	BIAS($\hat{\varepsilon}$)	0.28442 ⁽²⁾	0.30089 ⁽⁶⁾	0.30169 ⁽⁷⁾	0.30904 ⁽⁸⁾	0.31652 ⁽¹⁰⁾	0.31186 ⁽⁹⁾	0.28167 ⁽¹⁾	0.32941 ⁽¹²⁾	0.28872 ⁽³⁾	0.33192 ⁽¹⁴⁾	0.32015 ⁽¹¹⁾	0.29831 ⁽⁴⁾	0.33161 ⁽¹³⁾	0.30047 ⁽⁵⁾	0.34142 ⁽¹⁵⁾
70	MSE($\hat{\delta}$)	0.00132 ⁽¹⁾	0.00135 ⁽⁴⁾	0.00154 ⁽⁸⁾	0.00139 ⁽⁶⁾	0.00146 ⁽⁷⁾	0.00134 ^(2,5)	0.00134 ^(2,5)	0.00199 ⁽¹³⁾	0.00167 ⁽¹⁰⁾	0.00169 ⁽¹¹⁾	0.00232 ⁽¹⁵⁾	0.00157 ⁽⁹⁾	0.00207 ⁽¹⁴⁾	0.00138 ⁽⁵⁾	0.00191 ⁽¹²⁾
70	MSE($\hat{\varepsilon}$)	0.12121 ⁽²⁾	0.13112 ⁽⁵⁾	0.13482 ⁽⁷⁾	0.13670 ⁽⁸⁾	0.14058 ⁽¹⁰⁾	0.13841 ⁽⁹⁾	0.11724 ⁽¹⁾	0.15078 ⁽¹³⁾	0.12485 ⁽³⁾	0.15045 ⁽¹²⁾	0.14460 ⁽¹¹⁾	0.12901 ⁽⁴⁾	0.15284 ⁽¹⁴⁾	0.13124 ⁽⁶⁾	0.15913 ⁽¹⁵⁾
70	MRE($\hat{\delta}$)	0.11550 ⁽¹⁾	0.11693 ⁽²⁾	0.12803 ⁽⁹⁾	0.12115 ⁽⁶⁾	0.12474 ⁽⁷⁾	0.11788 ⁽³⁾	0.11804 ⁽⁴⁾	0.14451 ⁽¹³⁾	0.13311 ⁽¹⁰⁾	0.13445 ⁽¹¹⁾	0.15520 ⁽¹⁵⁾	0.12561 ⁽⁸⁾	0.14760 ⁽¹⁴⁾	0.12084 ⁽⁵⁾	0.14441 ⁽¹²⁾
70	MRE($\hat{\varepsilon}$)	0.37923 ⁽²⁾	0.40119 ⁽⁶⁾	0.40225 ⁽⁷⁾	0.41205 ⁽⁸⁾	0.42203 ⁽¹⁰⁾	0.41582 ⁽⁹⁾	0.37556 ⁽¹⁾	0.43921 ⁽¹²⁾	0.38496 ⁽³⁾	0.44256 ⁽¹⁴⁾	0.42686 ⁽¹¹⁾	0.39775 ⁽⁴⁾	0.44215 ⁽¹³⁾	0.40062 ⁽⁵⁾	0.45522 ⁽¹⁵⁾
70	D_{abs}	0.02732 ⁽¹⁾	0.02952 ⁽⁶⁾	0.02920 ⁽²⁾	0.03010 ⁽⁷⁾	0.03023 ⁽¹⁰⁾	0.02902 ⁽³⁾	0.03563 ⁽¹⁴⁾	0.03417 ⁽¹²⁾	0.03563 ⁽¹⁴⁾	0.03417 ⁽¹²⁾	0.03563 ⁽¹⁴⁾	0.03053 ⁽⁹⁾	0.03647 ⁽¹⁵⁾	0.02932 ⁽⁵⁾	0.03508 ⁽¹³⁾
70	D_{max}	0.04503 ⁽¹⁾	0.04825 ⁽⁶⁾	0.04817 ⁽⁵⁾	0.04953 ⁽⁸⁾	0.04756 ⁽³⁾	0.04700 ⁽²⁾	0.05101 ⁽¹⁰⁾	0.05705 ⁽¹³⁾	0.05518 ⁽¹²⁾	0.05296 ⁽¹¹⁾	0.04993 ⁽⁹⁾	0.05877 ⁽¹⁵⁾	0.04793 ⁽⁴⁾	0.05727 ⁽¹⁴⁾	0.05727 ⁽¹⁴⁾
70	ASAE	0.09294 ⁽¹⁾	0.29824 ⁽⁴⁾	0.60073 ⁽¹⁰⁾	0.33394 ⁽⁶⁾	0.93127 ⁽¹³⁾	0.16376 ⁽²⁾	0.26277 ⁽³⁾	0.31906 ⁽¹⁴⁾	0.31113 ⁽⁵⁾	0.84822 ⁽¹²⁾	4.37354 ⁽¹⁵⁾	0.51429 ⁽⁹⁾	0.50151 ⁽⁸⁾	0.34853 ⁽⁷⁾	0.61329 ⁽¹¹⁾
70	$\Sigma Ranks$	12 ⁽¹⁾	41 ⁽³⁾	66 ⁽⁸⁾	62 ⁽⁶⁾	80 ⁽¹⁰⁾	43 ^(4,5)	20 ^(5,2)	110 ⁽¹²⁾	71 ⁽⁹⁾	109 ⁽¹¹⁾	115 ⁽¹³⁾	64 ⁽⁷⁾	120 ⁽¹⁵⁾	47 ⁽⁵⁾	119 ⁽¹⁴⁾
150	BIAS($\hat{\delta}$)	0.02229 ⁽²⁾	0.02165 ⁽¹⁾	0.02257 ⁽⁷⁾	0.02281 ⁽³⁾	0.02529 ⁽⁸⁾	0.02309 ⁽⁴⁾	0.02386 ⁽⁵⁾	0.02546 ⁽⁹⁾	0.03722 ⁽¹⁵⁾	0.03722 ⁽¹⁵⁾	0.02565 ⁽¹⁰⁾	0.02885 ⁽¹³⁾	0.02390 ⁽⁶⁾	0.02903 ⁽¹⁴⁾	0.02903 ⁽¹⁴⁾
150	BIAS($\hat{\varepsilon}$)	0.24774 ⁽³⁾	0.24100 ⁽¹⁾	0.26448 ⁽⁷⁾	0.24624 ⁽²⁾	0.26516 ⁽⁸⁾	0.25224 ⁽⁴⁾	0.25799 ⁽⁵⁾	0.27217 ⁽¹¹⁾	0.28452 ⁽¹³⁾	0.25952 ⁽⁶⁾	0.31601 ⁽¹⁵⁾	0.26664 ⁽⁹⁾	0.27757 ⁽¹²⁾	0.26844 ⁽¹⁰⁾	0.29879 ⁽¹⁴⁾
150	MSE($\hat{\delta}$)	0.00077 ⁽²⁾	0.00076 ⁽¹⁾	0.00097 ⁽⁷⁾	0.00081 ^(3,5)	0.00099 ^(8,5)	0.00081 ^(3,5)	0.00089 ^(5,5)	0.00121 ⁽¹²⁾	0.00119 ⁽¹¹⁾	0.00099 ^(8,5)	0.00211 ⁽¹⁵⁾	0.00103 ⁽¹⁰⁾	0.00130 ^(13,5)	0.00089 ^(5,5)	0.00130 ^(13,5)
150	MSE($\hat{\varepsilon}$)	0.09306 ⁽³⁾	0.08738 ⁽¹⁾	0.10239 ⁽⁷⁾	0.09226 ⁽²⁾	0.10492 ⁽⁸⁾	0.09560 ⁽⁴⁾	0.10005 ⁽⁵⁾	0.11120 ⁽¹¹⁾	0.11785 ⁽¹³⁾	0.10174 ⁽⁶⁾	0.13975 ⁽¹⁵⁾	0.10507 ⁽⁹⁾	0.10737 ⁽¹⁰⁾	0.11350 ⁽¹²⁾	0.13211 ⁽¹⁴⁾
150	MRE($\hat{\delta}$)	0.08916 ⁽²⁾	0.08658 ⁽¹⁾	0.09829 ⁽⁷⁾	0.09123 ⁽³⁾	0.10117 ⁽⁸⁾	0.09235 ⁽⁴⁾	0.09543 ⁽⁵⁾	0.11059 ⁽¹¹⁾	0.11138 ⁽¹²⁾	0.10182 ⁽⁹⁾	0.14889 ⁽¹⁵⁾	0.10258 ⁽¹⁰⁾	0.11538 ⁽¹³⁾	0.09559 ⁽⁶⁾	0.11611 ⁽¹⁴⁾
150	MRE($\hat{\varepsilon}$)	0.33032 ⁽³⁾	0.32134 ⁽¹⁾	0.35265 ⁽⁷⁾	0.32832 ⁽²⁾	0.35355 ⁽⁸⁾	0.33632 ⁽⁴⁾	0.34399 ⁽⁵⁾	0.36289 ⁽¹¹⁾	0.37936 ⁽¹³⁾	0.34603 ⁽⁶⁾	0.42135 ⁽¹⁵⁾	0.35552 ⁽⁹⁾	0.37009 ⁽¹⁰⁾	0.39830 ⁽¹⁴⁾	0.39830 ⁽¹⁴⁾
150	D_{abs}	0.02157 ⁽⁶⁾	0.02070 ⁽²⁾	0.02066 ⁽¹⁾	0.02134 ⁽⁵⁾	0.02224 ⁽⁹⁾	0.02097 ⁽⁴⁾	0.02080 ⁽³⁾	0.02194 ⁽⁸⁾	0.02589 ⁽¹³⁾	0.02271 ⁽¹⁰⁾	0.02400 ⁽¹²⁾	0.02180 ⁽⁷⁾	0.02648 ⁽¹⁵⁾	0.02307 ⁽¹¹⁾	0.02611 ⁽¹⁴⁾
150	D_{max}	0.03566 ⁽⁴⁾	0.03441 ⁽¹⁾	0.03471 ⁽²⁾	0.03515 ⁽⁵⁾	0.03692 ⁽⁸⁾	0.03484 ⁽⁴⁾	0.03472 ⁽³⁾	0.03730 ⁽⁹⁾	0.04263 ⁽¹³⁾	0.03761 ⁽¹⁰⁾	0.04229 ⁽¹²⁾	0.03668 ⁽⁷⁾	0.04349 ⁽¹⁵⁾	0.03790 ⁽¹¹⁾	0.04326 ⁽¹⁴⁾
150	ASAE	0.05570 ⁽¹⁾	0.29590 ⁽⁴⁾	0.58040 ⁽¹³⁾	0.08489 ⁽²⁾	0.48280 ⁽¹²⁾	0.13904 ⁽⁵⁾	0.11038 ⁽³⁾	1.58227 ⁽¹⁴⁾	0.11435 ⁽⁴⁾	0.45810 ⁽¹²⁾	3.68440 ⁽¹⁵⁾	0.32229 ⁽¹⁰⁾	0.20427 ⁽⁶⁾	0.23387 ⁽⁷⁾	0.24452 ⁽⁸⁾
150	$\Sigma Ranks$	28 ⁽³⁾	18 ⁽¹⁾	58 ⁽⁶⁾	27 ^(5,2)	77 ^(9,5)	36 ^(4,5)	39 ^(5,5)	98 ⁽¹¹⁾	104 ⁽¹²⁾	75 ^(7,1)	129 ⁽¹⁵⁾	81 ⁽¹⁰⁾	111 ⁽¹³⁾	76 ^(5,8)	119 ⁽¹⁴⁾
220	BIAS($\hat{\delta}$)	0.01796 ⁽¹⁾	0.01800 ⁽²⁾	0.02026 ⁽⁶⁾	0.01831 ⁽³⁾	0.02197 ⁽⁹⁾	0.01909 ⁽⁴⁾	0.01928 ⁽⁵⁾	0.02331 ⁽¹²⁾	0.02247 ⁽¹⁰⁾	0.02250 ⁽¹¹⁾	0.03516 ⁽¹⁵⁾	0.02051 ⁽⁷⁾	0.02501 ⁽¹⁴⁾	0.02053 ⁽⁸⁾	0.02483 ⁽¹³⁾
220	BIAS($\hat{\varepsilon}$)	0.20304 ⁽²⁾	0.20397 ⁽⁴⁾	0.21143 ⁽⁶⁾	0.20645 ⁽⁵⁾	0.23430 ⁽¹⁰⁾	0.19840 ⁽¹⁾	0.20351 ⁽³⁾	0.22547 ⁽⁸⁾	0.23923 ⁽¹¹⁾	0.24355 ⁽¹²⁾	0.30053 ⁽¹⁵⁾	0.22276 ⁽⁷⁾	0.26151 ⁽¹⁴⁾	0.22940 ⁽⁹⁾	0.25279 ⁽¹³⁾
220	MSE($\hat{\delta}$)	0.00053 ⁽²⁾	0.00053 ⁽²⁾	0.00063 ⁽⁶⁾	0.00053 ⁽²⁾	0.00073 ⁽⁹⁾	0.00057 ⁽⁴⁾	0.00059 ⁽⁵⁾	0.00083 ⁽¹²⁾	0.00077 ⁽¹⁰⁾	8e-04 ⁽¹⁾	0.00187 ⁽¹⁵⁾	0.00065 ^(7,5)	0.00099 ⁽¹⁴⁾	0.00065 ^(7,5)	0.00091 ⁽¹³⁾
220	MSE($\hat{\varepsilon}$)	0.06280 ⁽²⁾	0.06397 ⁽³⁾	0.07202 ⁽⁶⁾	0.06426 ⁽⁴⁾	0.08260 ⁽¹⁰⁾	0.06109 ⁽¹⁾	0.06587 ⁽⁵⁾	0.07621 ⁽⁷⁾	0.08651 ⁽¹¹⁾	0.08746 ⁽¹²⁾	0.13029 ⁽¹⁵⁾	0.07652 ⁽⁸⁾	0.10211 ⁽¹⁴⁾	0.07771 ⁽⁹⁾	0.09530 ⁽¹³⁾
220	MRE($\hat{\delta}$)	0.07183 ⁽¹⁾	0.07198 ⁽²⁾	0.08103 ⁽⁶⁾	0.07323 ⁽³⁾	0.08789 ⁽⁹⁾	0.07638 ⁽⁴⁾	0.07711 ⁽⁵⁾	0.09325 ⁽¹²⁾	0.08988 ⁽¹⁰⁾	0.09001 ⁽¹¹⁾	0.14064 ⁽¹⁵⁾	0.08206 ⁽⁷⁾	0.10006 ⁽¹⁴⁾	0.08213 ⁽⁸⁾	0.09930 ⁽¹³⁾
220	MRE($\hat{\varepsilon}$)	0.27073 ⁽²⁾	0.27197 ⁽⁴⁾	0.28191 ⁽⁶⁾	0.27527 ⁽⁵⁾	0.31239 ⁽¹⁰⁾	0.26453 ⁽¹⁾	0.27134 ⁽³⁾	0.30062 ⁽⁸⁾	0.31897 ⁽¹¹⁾	0.32474 ⁽¹²⁾	0.40071 ⁽¹⁵⁾	0.29702 ⁽⁷⁾	0.34868 ⁽¹⁴⁾	0.30587 ⁽⁹⁾	0.33706 ⁽¹³⁾
220	D_{abs}	0.01795 ^(6,5)	0.01747 ⁽⁴⁾	0.01795 ^(6,5)	0.01706 ⁽²⁾	0.01840 ⁽⁸⁾	0.01707 ⁽³⁾	0.01688 ⁽¹⁾	0.01755 ⁽⁵⁾	0.02081 ⁽¹³⁾	0.02017 ⁽¹¹⁾	0.02035 ⁽¹²⁾	0.01845 ⁽⁹⁾	0.02284 ⁽¹⁵⁾	0.01869 ⁽¹⁰⁾	0.02274 ⁽¹⁴⁾
220	D_{max}	0.02954 ⁽⁵⁾	0.02886 ⁽⁴⁾	0.02982 ⁽⁶⁾	0.02810 ⁽¹⁾	0.03093 ⁽⁹⁾	0.02838 ⁽³⁾	0.02823 ⁽²⁾	0.02989 ⁽⁷⁾	0.03434 ⁽¹²⁾	0.03334 ⁽¹¹⁾	0.03630 ⁽¹³⁾	0.03075 ⁽⁸⁾	0.03780 ⁽¹⁵⁾	0.03105 ⁽¹⁰⁾	0.03743 ⁽¹⁴⁾
220	ASAE	0.04144 ⁽¹⁾	0.24410 ⁽¹⁰⁾	0.26695 ⁽¹¹⁾	0.05195 ⁽²⁾	0.30982 ⁽¹³⁾	0.12920 ⁽⁶⁾	0.08027 ⁽³⁾	0.09154 ⁽¹⁴⁾	0.11266 ⁽⁵⁾	0.15032 ⁽⁹⁾	3.50104 ⁽¹⁵⁾	0.29740 ⁽¹²⁾	0.14359 ⁽⁷⁾	0.08669 ⁽⁴⁾	0.15005 ⁽⁸⁾
220	$\Sigma Ranks$	22 ^(5,1)	35 ⁽⁵⁾	59 ^(5,6)	27 ^(2,5)	87 ⁽¹⁰⁾	27 ^(2,5)	32 ⁽⁴⁾	85 ⁽⁹⁾	93 ⁽¹¹⁾	100 ⁽¹²⁾	130 ⁽¹⁵⁾	72 ^(5,7)	121 ⁽¹⁴⁾	74 ^(5,8)	114 ⁽¹³⁾
280	BIAS($\hat{\delta}$)	0.01664 ^(3,5)	0.01648 ⁽²⁾	0.01884 ⁽⁸⁾	0.01664 ^(3,5)	0.01860 ⁽⁷⁾	0.01617 ⁽¹⁾	0.01687 ⁽⁵⁾	0.02105 ⁽¹²⁾	0.01993 ⁽¹⁰⁾	0.02004 ⁽¹¹⁾	0				

TABLE 4. Numerical values of simulation measures for $\delta = 0.9$, $\varepsilon = 1.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{\delta}$)	0.14909 ⁽²⁾	0.15174 ⁽³⁾	0.15299 ⁽⁴⁾	0.17307 ⁽⁹⁾	0.16159 ⁽⁶⁾	0.14694 ⁽¹¹⁾	0.15750 ⁽⁵⁾	0.17261 ⁽⁸⁾	0.17925 ⁽¹²⁾	0.17320 ⁽¹⁰⁾	0.19768 ⁽¹⁴⁾	0.17735 ⁽¹¹⁾	0.20465 ⁽¹⁵⁾	0.16495 ⁽⁷⁾	0.19454 ⁽¹³⁾
20	BIAS($\hat{\varepsilon}$)	0.61405 ⁽³⁾	0.61623 ⁽⁴⁾	0.60688 ⁽²⁾	0.62491 ⁽⁵⁾	0.60535 ⁽¹⁾	0.62952 ⁽⁷⁾	0.64150 ⁽⁹⁾	0.62851 ⁽⁶⁾	0.65767 ⁽¹³⁾	0.65249 ⁽¹¹⁾	0.63411 ⁽⁸⁾	0.66070 ⁽¹⁴⁾	0.68563 ⁽¹⁵⁾	0.65651 ⁽¹²⁾	0.65104 ⁽¹⁰⁾
20	MSE($\hat{\delta}$)	0.03760 ⁽⁴⁾	0.03635 ⁽²⁾	0.03804 ⁽⁵⁾	0.04758 ⁽¹⁰⁾	0.03877 ⁽⁶⁾	0.03472 ⁽¹¹⁾	0.03650 ⁽³⁾	0.04560 ⁽⁸⁾	0.05420 ⁽¹²⁾	0.04593 ⁽⁹⁾	0.06551 ⁽¹⁵⁾	0.05268 ⁽¹¹⁾	0.06019 ⁽¹⁴⁾	0.04119 ⁽⁷⁾	0.05483 ⁽¹³⁾
20	MSE($\hat{\varepsilon}$)	0.55654 ⁽⁶⁾	0.53936 ⁽³⁾	0.52334 ⁽²⁾	0.56333 ⁽⁷⁾	0.50986 ⁽¹⁾	0.58461 ⁽⁹⁾	0.58075 ⁽⁸⁾	0.54537 ⁽⁴⁾	0.60053 ⁽¹¹⁾	0.59710 ⁽¹⁰⁾	0.55526 ⁽⁵⁾	0.61910 ⁽¹³⁾	0.64725 ⁽¹⁵⁾	0.62201 ⁽¹⁴⁾	0.61099 ⁽¹²⁾
20	MRE($\hat{\delta}$)	0.16566 ⁽²⁾	0.16860 ⁽³⁾	0.16921 ⁽⁴⁾	0.19230 ⁽⁹⁾	0.17954 ⁽⁶⁾	0.16327 ⁽¹¹⁾	0.17500 ⁽⁵⁾	0.19179 ⁽⁸⁾	0.19917 ⁽¹²⁾	0.19245 ⁽¹⁰⁾	0.21964 ⁽¹⁴⁾	0.19706 ⁽¹¹⁾	0.22739 ⁽¹⁵⁾	0.18328 ⁽⁷⁾	0.21615 ⁽¹³⁾
20	MRE($\hat{\varepsilon}$)	0.40937 ⁽³⁾	0.41082 ⁽⁴⁾	0.40459 ⁽²⁾	0.41661 ⁽⁵⁾	0.40357 ⁽¹⁾	0.41968 ⁽⁷⁾	0.42767 ⁽⁹⁾	0.41901 ⁽⁶⁾	0.43845 ⁽¹³⁾	0.43499 ⁽¹¹⁾	0.42274 ⁽⁸⁾	0.44047 ⁽¹⁴⁾	0.45709 ⁽¹⁵⁾	0.43767 ⁽¹²⁾	0.43403 ⁽¹⁰⁾
20	D_{abs}	0.05270 ⁽³⁾	0.05535 ⁽⁶⁾	0.05227 ⁽²⁾	0.05737 ⁽⁸⁾	0.05279 ⁽⁴⁾	0.04952 ⁽¹⁾	0.05378 ⁽⁵⁾	0.05771 ⁽⁹⁾	0.06601 ⁽¹⁴⁾	0.06511 ⁽¹²⁾	0.06227 ⁽¹¹⁾	0.05644 ⁽⁷⁾	0.06638 ⁽¹⁵⁾	0.05969 ⁽¹⁰⁾	0.06560 ⁽¹³⁾
20	D_{max}	0.08541 ⁽⁵⁾	0.08687 ⁽⁶⁾	0.08407 ⁽³⁾	0.08784 ⁽⁷⁾	0.08330 ⁽²⁾	0.07947 ⁽¹⁾	0.08489 ⁽⁴⁾	0.09001 ⁽⁸⁾	0.10438 ⁽¹⁵⁾	0.09905 ⁽¹¹⁾	0.09987 ⁽¹²⁾	0.09112 ⁽⁹⁾	0.10258 ⁽¹⁴⁾	0.09265 ⁽¹⁰⁾	0.10158 ⁽¹³⁾
20	ASAE	0.05385 ⁽⁴⁾	0.05323 ⁽³⁾	0.05816 ⁽⁷⁾	0.05741 ⁽⁶⁾	0.08523 ⁽¹²⁾	0.04751 ⁽¹¹⁾	0.05913 ⁽⁸⁾	0.07405 ⁽¹¹⁾	0.06045 ⁽⁹⁾	0.05563 ⁽⁵⁾	0.14197 ⁽¹⁵⁾	0.06796 ⁽¹⁰⁾	0.09695 ⁽¹³⁾	0.05253 ⁽²⁾	0.11697 ⁽¹⁴⁾
20	$\Sigma Ranks$	32 ⁽³⁾	34 ⁽⁴⁾	31 ⁽²⁾	66 ⁽⁷⁾	39 ⁽⁵⁾	29 ⁽¹⁾	56 ⁽⁶⁾	68 ⁽⁸⁾	111 ^(13.5)	89 ⁽¹⁰⁾	102 ⁽¹²⁾	100 ⁽¹¹⁾	131 ⁽¹⁵⁾	81 ⁽⁹⁾	111 ^(13.5)
70	BIAS($\hat{\delta}$)	0.09890 ⁽⁴⁾	0.10571 ⁽⁵⁾	0.11164 ⁽⁹⁾	0.09824 ⁽³⁾	0.11404 ⁽¹¹⁾	0.10993 ⁽⁸⁾	0.09575 ⁽¹¹⁾	0.10958 ⁽⁷⁾	0.12399 ⁽¹³⁾	0.11357 ⁽¹⁰⁾	0.15056 ⁽¹⁵⁾	0.10754 ⁽⁶⁾	0.12925 ⁽¹⁴⁾	0.09669 ⁽²⁾	0.12146 ⁽¹²⁾
70	BIAS($\hat{\varepsilon}$)	0.43739 ⁽²⁾	0.40806 ⁽¹⁾	0.45641 ⁽⁶⁾	0.44307 ⁽⁴⁾	0.46734 ⁽⁹⁾	0.51926 ⁽¹¹⁾	0.46538 ⁽⁷⁾	0.46721 ⁽⁸⁾	0.53644 ⁽¹³⁾	0.52182 ⁽¹²⁾	0.53729 ⁽¹⁵⁾	0.43868 ⁽³⁾	0.53713 ⁽¹⁴⁾	0.47269 ⁽¹⁰⁾	0.44361 ⁽⁵⁾
70	MSE($\hat{\delta}$)	0.01524 ⁽³⁾	0.01616 ⁽⁴⁾	0.01844 ⁽⁹⁾	0.01618 ⁽⁵⁾	0.02068 ⁽¹⁰⁾	0.01825 ⁽⁸⁾	0.01477 ⁽²⁾	0.01745 ⁽⁶⁾	0.02258 ⁽¹³⁾	0.02086 ⁽¹¹⁾	0.03416 ⁽¹⁵⁾	0.01824 ⁽⁷⁾	0.02816 ⁽¹⁴⁾	0.01390 ⁽¹⁾	0.02218 ⁽¹²⁾
70	MSE($\hat{\varepsilon}$)	0.30601 ⁽³⁾	0.25808 ⁽¹⁾	0.30752 ⁽⁵⁾	0.28920 ⁽²⁾	0.32891 ⁽⁸⁾	0.40114 ⁽¹¹⁾	0.31885 ⁽⁶⁾	0.34818 ⁽¹⁰⁾	0.42672 ⁽¹⁴⁾	0.42176 ⁽¹³⁾	0.41330 ⁽¹²⁾	0.30658 ⁽⁴⁾	0.43226 ⁽¹⁵⁾	0.34591 ⁽⁹⁾	0.32399 ⁽⁷⁾
70	MRE($\hat{\delta}$)	0.10988 ⁽⁴⁾	0.11746 ⁽⁵⁾	0.12405 ⁽⁹⁾	0.10916 ⁽³⁾	0.12672 ⁽¹¹⁾	0.12215 ⁽⁸⁾	0.10936 ⁽¹⁾	0.12176 ⁽⁷⁾	0.13777 ⁽¹³⁾	0.12619 ⁽¹⁰⁾	0.16729 ⁽¹⁵⁾	0.11949 ⁽⁶⁾	0.14361 ⁽¹⁴⁾	0.10743 ⁽²⁾	0.13495 ⁽¹²⁾
70	MRE($\hat{\varepsilon}$)	0.29159 ⁽²⁾	0.27204 ⁽¹⁾	0.30428 ⁽⁶⁾	0.29538 ⁽⁴⁾	0.31156 ⁽⁹⁾	0.34617 ⁽¹¹⁾	0.31025 ⁽⁷⁾	0.31147 ⁽⁸⁾	0.35763 ⁽¹³⁾	0.34788 ⁽¹²⁾	0.35819 ⁽¹⁵⁾	0.29245 ⁽³⁾	0.35809 ⁽¹⁴⁾	0.31513 ⁽¹⁰⁾	0.29574 ⁽⁵⁾
70	D_{abs}	0.02828 ⁽³⁾	0.03088 ⁽⁵⁾	0.02997 ⁽²⁾	0.03282 ⁽¹⁾	0.03214 ⁽⁸⁾	0.03174 ⁽⁶⁾	0.03392 ⁽¹⁰⁾	0.03202 ⁽⁷⁾	0.03749 ⁽¹³⁾	0.03749 ⁽¹¹⁾	0.03508 ⁽¹⁵⁾	0.03202 ⁽⁴⁾	0.04148 ⁽¹⁵⁾	0.03360 ⁽⁹⁾	0.03519 ⁽¹²⁾
70	D_{max}	0.04732 ⁽²⁾	0.05121 ⁽⁵⁾	0.04965 ⁽³⁾	0.04633 ⁽¹⁾	0.05359 ⁽⁷⁾	0.05368 ⁽⁸⁾	0.05460 ⁽¹⁰⁾	0.05289 ⁽⁶⁾	0.06404 ⁽¹⁴⁾	0.06096 ⁽¹³⁾	0.05937 ⁽¹²⁾	0.05008 ⁽⁴⁾	0.06642 ⁽¹⁵⁾	0.05435 ⁽⁹⁾	0.05669 ⁽¹¹⁾
70	ASAE	0.01951 ⁽³⁾	0.01882 ⁽²⁾	0.02822 ⁽¹²⁾	0.02071 ⁽⁵⁾	0.02297 ⁽⁷⁾	0.01971 ⁽⁸⁾	0.02104 ⁽⁶⁾	0.02318 ⁽⁹⁾	0.01820 ⁽¹¹⁾	0.02750 ⁽¹¹⁾	0.04424 ⁽¹⁵⁾	0.02401 ⁽¹⁰⁾	0.02852 ⁽¹³⁾	0.02304 ⁽⁸⁾	0.03085 ⁽¹⁴⁾
70	$\Sigma Ranks$	25 ⁽¹⁾	29 ⁽³⁾	62 ⁽⁷⁾	28 ⁽²⁾	80 ⁽¹⁰⁾	75 ⁽⁹⁾	50 ⁽⁵⁾	68 ⁽⁸⁾	108 ⁽¹³⁾	105 ⁽¹²⁾	125 ⁽¹⁴⁾	47 ⁽⁴⁾	128 ⁽¹⁵⁾	60 ⁽⁶⁾	90 ⁽¹¹⁾
150	BIAS($\hat{\delta}$)	0.07361 ⁽⁵⁾	0.07131 ⁽⁴⁾	0.07537 ⁽⁷⁾	0.05806 ⁽¹¹⁾	0.07497 ⁽⁶⁾	0.06778 ⁽³⁾	0.06116 ⁽¹²⁾	0.07920 ⁽⁹⁾	0.08118 ⁽¹³⁾	0.08118 ⁽¹¹⁾	0.13138 ⁽¹⁵⁾	0.07674 ⁽⁸⁾	0.09443 ⁽¹⁴⁾	0.08112 ⁽¹⁰⁾	0.08422 ⁽¹²⁾
150	BIAS($\hat{\varepsilon}$)	0.32140 ⁽⁵⁾	0.32017 ⁽⁴⁾	0.30647 ⁽³⁾	0.27075 ⁽¹⁾	0.30319 ⁽⁷⁾	0.35283 ⁽¹⁰⁾	0.30168 ⁽²⁾	0.35472 ⁽¹¹⁾	0.35960 ⁽¹³⁾	0.35718 ⁽¹²⁾	0.47932 ⁽¹⁵⁾	0.33022 ⁽⁸⁾	0.42638 ⁽¹⁴⁾	0.33122 ⁽⁹⁾	0.32317 ⁽⁶⁾
150	MSE($\hat{\delta}$)	0.00930 ⁽⁷⁾	0.00742 ⁽⁴⁾	0.00858 ⁽⁵⁾	0.00559 ⁽¹⁾	0.00874 ⁽⁶⁾	0.00676 ⁽³⁾	0.00594 ⁽²⁾	0.00944 ⁽⁸⁾	0.01325 ⁽¹⁴⁾	0.01032 ⁽¹⁰⁾	0.02836 ⁽¹⁵⁾	0.00947 ⁽⁹⁾	0.01297 ⁽¹³⁾	0.01068 ⁽¹²⁾	0.01062 ⁽¹¹⁾
150	MSE($\hat{\varepsilon}$)	0.17185 ⁽⁷⁾	0.15924 ⁽⁵⁾	0.14025 ⁽²⁾	0.12216 ⁽¹⁾	0.18269 ⁽¹⁰⁾	0.18189 ⁽⁹⁾	0.15295 ^(3.5)	0.18132 ⁽⁸⁾	0.21081 ⁽¹³⁾	0.20522 ⁽¹²⁾	0.34162 ⁽¹⁵⁾	0.19440 ⁽¹¹⁾	0.27451 ⁽¹⁴⁾	0.16257 ⁽⁶⁾	0.15295 ^(3.5)
150	MRE($\hat{\delta}$)	0.08179 ⁽⁵⁾	0.07923 ⁽⁴⁾	0.08374 ⁽⁷⁾	0.06451 ⁽¹⁾	0.08330 ⁽⁶⁾	0.07531 ⁽³⁾	0.06795 ⁽²⁾	0.08800 ⁽⁹⁾	0.10354 ⁽¹³⁾	0.09020 ⁽¹¹⁾	0.14598 ⁽¹⁵⁾	0.08527 ⁽⁸⁾	0.10492 ⁽¹⁴⁾	0.09014 ⁽¹⁰⁾	0.09364 ⁽¹²⁾
150	MRE($\hat{\varepsilon}$)	0.21426 ⁽⁴⁾	0.21345 ⁽⁴⁾	0.20432 ⁽³⁾	0.18050 ⁽¹⁾	0.22012 ⁽⁷⁾	0.23522 ⁽¹⁰⁾	0.20112 ⁽²⁾	0.23648 ⁽¹¹⁾	0.23812 ⁽¹³⁾	0.23812 ⁽¹²⁾	0.31955 ⁽¹⁵⁾	0.22015 ⁽⁸⁾	0.28425 ⁽¹⁴⁾	0.22081 ⁽⁹⁾	0.21545 ⁽⁶⁾
150	D_{abs}	0.02021 ⁽³⁾	0.02023 ⁽⁴⁾	0.02173 ⁽⁶⁾	0.01798 ⁽¹⁾	0.02258 ⁽⁸⁾	0.02115 ⁽⁵⁾	0.01987 ⁽²⁾	0.02298 ⁽⁹⁾	0.02671 ⁽¹²⁾	0.02451 ⁽¹¹⁾	0.02714 ⁽¹³⁾	0.02245 ⁽⁷⁾	0.02893 ⁽¹⁵⁾	0.02413 ⁽¹⁰⁾	0.02805 ⁽¹⁴⁾
150	D_{max}	0.03402 ⁽⁴⁾	0.03394 ⁽³⁾	0.03580 ⁽⁶⁾	0.02967 ⁽¹⁾	0.03713 ⁽⁸⁾	0.03554 ⁽⁵⁾	0.03296 ⁽²⁾	0.03757 ⁽⁹⁾	0.04423 ⁽¹²⁾	0.04070 ⁽¹¹⁾	0.04733 ⁽¹⁴⁾	0.03675 ⁽⁷⁾	0.04814 ⁽¹⁵⁾	0.03954 ⁽¹⁰⁾	0.04551 ⁽¹³⁾
150	ASAE	0.01167 ⁽⁸⁾	0.01002 ⁽⁶⁾	0.01309 ⁽¹²⁾	0.00982 ⁽³⁾	0.01285 ⁽¹⁰⁾	0.00942 ⁽¹⁾	0.00985 ⁽⁵⁾	0.01373 ⁽¹³⁾	0.00984 ⁽⁴⁾	0.01041 ⁽⁷⁾	0.03942 ⁽¹⁵⁾	0.01262 ⁽⁹⁾	0.01378 ⁽¹⁴⁾	0.00977 ⁽²⁾	0.01297 ⁽¹¹⁾
150	$\Sigma Ranks$	49 ^(4.5)	38 ⁽³⁾	51 ⁽⁶⁾	11 ⁽¹⁾	68 ⁽⁷⁾	49 ^(4.5)	22 ^(5.2)	87 ⁽¹⁰⁾	107 ⁽¹³⁾	97 ⁽¹²⁾	132 ⁽¹⁵⁾	75 ⁽⁸⁾	127 ⁽¹⁴⁾	88 ^(5.1)	88 ^(5.1)
220	BIAS($\hat{\delta}$)	0.05100 ⁽²⁾	0.05127 ⁽³⁾	0.04995 ⁽¹¹⁾	0.05353 ⁽⁴⁾	0.06786 ⁽¹⁰⁾	0.06220 ⁽⁷⁾	0.05745 ⁽⁵⁾	0.06362 ⁽⁹⁾	0.07526 ⁽¹³⁾	0.06266 ⁽⁸⁾	0.11637 ⁽¹⁵⁾	0.07123 ⁽¹¹⁾	0.08094 ⁽¹⁴⁾	0.06180 ⁽⁶⁾	0.07318 ⁽¹²⁾
220	BIAS($\hat{\varepsilon}$)	0.22735 ⁽¹⁾	0.22883 ⁽²⁾	0.26948 ⁽⁷⁾	0.24147 ⁽³⁾	0.25553 ⁽⁴⁾	0.26940 ⁽⁶⁾	0.26628 ⁽⁵⁾	0.27101 ⁽⁸⁾	0.35061 ⁽¹⁴⁾	0.28469 ⁽¹⁰⁾	0.41569 ⁽¹⁵⁾	0.30665 ⁽¹²⁾	0.33924 ⁽¹³⁾	0.30009 ⁽¹⁾	0.28439 ⁽⁹⁾
220	MSE($\hat{\delta}$)	0.00442 ⁽³⁾	0.00407 ⁽¹⁾	0.00441 ⁽²⁾	0.00469 ⁽⁴⁾	0.00673 ⁽⁹⁾	0.00544 ⁽⁶⁾	0.00537 ⁽⁵⁾	0.00695 ⁽¹⁰⁾	0.00846 ⁽¹²⁾	0.00658 ⁽⁸⁾	0.02113 ⁽¹⁵⁾	0.00798 ⁽¹¹⁾	0.01017 ⁽¹⁴⁾	0.00649 ⁽⁷⁾	0.00884 ⁽¹³⁾
220	MSE($\hat{\varepsilon}$)	0.09216 ⁽³⁾	0.08014 ⁽¹⁾	0.10981 ⁽⁵⁾	0.08446 ⁽²⁾	0.11322 ⁽⁷⁾	0.10346 ⁽⁴⁾	0.11244 ⁽⁶⁾	0.12055 ⁽⁸⁾	0.17643 ⁽¹⁴⁾	0.13214 ⁽⁹⁾	0.26361 ⁽¹⁵⁾	0.15524 ⁽¹²⁾	0.17400 ⁽¹³⁾	0.14069 ⁽¹⁾	0.14121 ⁽¹¹⁾
220	MRE($\hat{\delta}$)	0.05667 ⁽²⁾	0.05697 ⁽³⁾	0.05550 ⁽¹⁾	0.05948 ⁽⁴⁾	0.07540 ⁽¹⁰⁾	0.06911 ⁽⁷⁾	0.06384 ⁽⁵⁾	0.07069 ⁽⁹⁾	0.08362 ⁽¹³⁾	0.06962 ⁽⁸⁾	0.12930 ⁽¹⁵⁾	0.07915 ⁽¹¹⁾	0.08994 ⁽¹⁴⁾	0.06867 ⁽⁶⁾	0.08132 ⁽¹²⁾
220	MRE($\hat{\varepsilon}$)	0.15156 ⁽¹⁾	0.15255 ⁽²⁾	0.17965 ⁽⁷⁾	0.16098 ⁽³⁾	0.17035 ⁽⁴⁾	0.17960 ⁽⁶⁾	0.17752 ⁽⁵⁾	0.18068 ⁽⁸⁾	0.23374 ⁽¹⁴⁾	0.18979 ⁽¹⁰⁾	0.27712 ⁽¹⁵⁾	0.20443 ⁽¹²⁾	0.22616 ⁽¹³⁾	0.20006 ⁽¹¹⁾	0.18959 ⁽⁹⁾
220	D_{abs}	0.01658 ⁽³⁾	0.01645 ⁽²⁾	0.01819 ⁽⁶⁾	0.01663 ⁽⁴⁾	0.01830 ⁽⁷⁾	0.01685 ⁽⁵⁾	0.01603 ⁽¹⁾	0.01858 ⁽⁸⁾	0.02225 ⁽¹³⁾	0.01908 ⁽⁹⁾	0.02185 ⁽¹²⁾	0.02023 ⁽¹⁰⁾	0.02372 ⁽¹⁴⁾	0.02078 ⁽¹⁾	0.02501 ⁽¹⁵⁾
220	D_{max}	0.02731 ⁽³⁾	0.02686 ⁽¹⁾	0.02950 ⁽⁶⁾	0.02745 ⁽⁴⁾	0.03082 ^(7.5)	0.02871 ⁽⁵⁾	0.02703 ⁽²⁾	0.03082 ^(7.5)	0.03691 ⁽¹²⁾	0.03163 ⁽⁹⁾	0.03856 ⁽¹³⁾	0.03370 ⁽¹⁰⁾	0.03911 ⁽¹⁴⁾	0.03433 ⁽¹¹⁾	0.04031 ⁽¹⁵⁾
220	ASAE	0.00647 ⁽¹⁾	0.00743 ⁽⁴⁾	0.00867 ⁽¹⁰⁾	0.00675 ⁽²⁾	0.00886 ⁽¹¹⁾	0.00741 ⁽³⁾	0.00768 ⁽⁶⁾	0.01179 ⁽¹⁴⁾	0.00786 ⁽⁸⁾	0.00759 ⁽⁵⁾	0.02619 ⁽¹⁵⁾	0.00854 ⁽⁹⁾	0.00916 ⁽¹²⁾	0.00770 ⁽⁷⁾	0.00958 ⁽¹³⁾
220	$\Sigma Ranks$	19 ^(1.5)	19 ^(1.5)	45 ⁽⁵⁾	30 ⁽³⁾	69 ^(5.7)	49 ⁽⁶⁾	40 ⁽⁴⁾	81 ^(5.10)	113 ⁽¹³⁾	76 ⁽⁸⁾	130 ⁽¹⁵⁾	98 ⁽¹¹⁾	121 ⁽¹⁴⁾	80 ⁽⁹⁾	109 ⁽¹²⁾
280	BIAS($\hat{\delta}$)	0.04945 ⁽⁶⁾	0.04942 ⁽⁵⁾	0.04718 ⁽²⁾	0.04595 ⁽¹⁾	0.05478 ⁽¹⁰⁾	0.05215 ⁽⁸⁾	0.04767 ⁽³⁾	0.04791 ⁽⁴⁾	0.06673 ⁽¹³⁾	0					

TABLE 5. Numerical values of simulation measures for $\delta = 2.0$, $\varepsilon = 0.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{\delta}$)	0.35635 ⁽¹⁾	0.40167 ⁽¹¹⁾	0.37927 ⁽⁵⁾	0.36287 ⁽²⁾	0.39136 ⁽⁸⁾	0.36886 ⁽³⁾	0.38162 ⁽⁶⁾	0.40783 ⁽¹²⁾	0.39866 ⁽⁹⁾	0.37550 ⁽⁴⁾	0.43941 ⁽¹⁴⁾	0.40121 ⁽¹⁰⁾	0.41209 ⁽¹³⁾	0.38277 ⁽⁷⁾	0.44055 ⁽¹⁵⁾
20	BIAS($\hat{\varepsilon}$)	0.25023 ⁽¹³⁾	0.24483 ⁽⁷⁾	0.24767 ⁽¹¹⁾	0.25538 ⁽¹⁴⁾	0.24638 ⁽⁹⁾	0.24721 ⁽¹⁰⁾	0.23585 ⁽⁴⁾	0.25010 ⁽¹²⁾	0.24305 ⁽⁶⁾	0.23686 ⁽⁵⁾	0.22740 ⁽¹⁾	0.23244 ⁽²⁾	0.24524 ⁽⁸⁾	0.25773 ⁽¹⁵⁾	0.23554 ⁽³⁾
20	MSE($\hat{\delta}$)	0.20890 ⁽²⁾	0.26051 ⁽¹⁾	0.24325 ⁽⁷⁾	0.20173 ⁽¹¹⁾	0.24343 ⁽⁸⁾	0.22129 ⁽³⁾	0.23139 ⁽⁵⁾	0.26838 ⁽¹²⁾	0.24732 ⁽⁹⁾	0.22265 ⁽⁴⁾	0.31177 ⁽¹⁵⁾	0.28488 ⁽¹⁴⁾	0.25346 ⁽¹⁰⁾	0.23391 ⁽⁶⁾	0.28289 ⁽¹³⁾
20	MSE($\hat{\varepsilon}$)	0.08313 ⁽¹¹⁾	0.08132 ⁽⁹⁾	0.08315 ⁽¹²⁾	0.08619 ⁽¹⁵⁾	0.08140 ⁽¹⁰⁾	0.08101 ⁽⁷⁾	0.07542 ⁽³⁾	0.08387 ⁽¹³⁾	0.07952 ⁽⁶⁾	0.07629 ⁽⁵⁾	0.07254 ⁽¹⁾	0.07487 ⁽²⁾	0.08118 ⁽⁸⁾	0.08574 ⁽¹⁴⁾	0.07582 ⁽⁴⁾
20	MRE($\hat{\delta}$)	0.17817 ⁽¹⁾	0.20083 ⁽¹¹⁾	0.18963 ⁽⁵⁾	0.18143 ⁽²⁾	0.19581 ⁽⁸⁾	0.18443 ⁽³⁾	0.19081 ⁽⁶⁾	0.20391 ⁽¹²⁾	0.19933 ⁽⁹⁾	0.18775 ⁽⁴⁾	0.21971 ⁽¹⁴⁾	0.20061 ⁽¹⁰⁾	0.20605 ⁽¹³⁾	0.19139 ⁽⁷⁾	0.22027 ⁽¹⁵⁾
20	MRE($\hat{\varepsilon}$)	0.50046 ⁽¹³⁾	0.48966 ⁽⁷⁾	0.49534 ⁽¹¹⁾	0.51077 ⁽¹⁴⁾	0.49275 ⁽⁹⁾	0.49442 ⁽¹⁰⁾	0.47170 ⁽⁴⁾	0.50020 ⁽¹²⁾	0.48610 ⁽⁶⁾	0.47373 ⁽⁵⁾	0.45480 ⁽¹⁾	0.46488 ⁽²⁾	0.49047 ⁽⁸⁾	0.51545 ⁽¹⁵⁾	0.47109 ⁽³⁾
20	D _{abs}	0.04423 ⁽¹⁾	0.05113 ⁽⁹⁾	0.04758 ⁽²⁾	0.05131 ⁽¹⁰⁾	0.05094 ⁽⁸⁾	0.04819 ⁽³⁾	0.04998 ⁽⁵⁾	0.05023 ⁽⁷⁾	0.05446 ⁽¹²⁾	0.05006 ⁽⁶⁾	0.05505 ⁽¹³⁾	0.05132 ⁽¹¹⁾	0.05771 ⁽¹⁴⁾	0.04936 ⁽⁴⁾	0.06163 ⁽¹⁵⁾
20	D _{max}	0.07052 ⁽¹⁾	0.07921 ⁽¹⁰⁾	0.07455 ⁽²⁾	0.07741 ⁽⁷⁾	0.07744 ⁽⁸⁾	0.07487 ⁽³⁾	0.07649 ⁽⁶⁾	0.07810 ⁽⁹⁾	0.08262 ⁽¹²⁾	0.07611 ⁽⁵⁾	0.08446 ⁽¹³⁾	0.08019 ⁽¹¹⁾	0.08659 ⁽¹⁴⁾	0.07590 ⁽⁴⁾	0.09251 ⁽¹⁵⁾
20	ASAE	0.04559 ⁽⁶⁾	0.04400 ^(3,5)	0.04697 ⁽⁷⁾	0.04170 ⁽²⁾	0.04989 ⁽¹⁰⁾	0.04167 ⁽¹⁾	0.04429 ⁽⁵⁾	0.05251 ⁽¹¹⁾	0.04860 ⁽⁹⁾	0.04400 ^(3,5)	0.06530 ⁽¹³⁾	0.05302 ⁽¹²⁾	0.07048 ⁽¹⁴⁾	0.04836 ⁽⁸⁾	0.07099 ⁽¹⁵⁾
20	Σ Ranks	49 ⁽⁴⁾	78.5 ⁽¹⁰⁾	62 ⁽⁵⁾	67 ⁽⁶⁾	78 ^(8,5)	43 ⁽²⁾	44 ⁽³⁾	100 ⁽¹⁴⁾	78 ^(8,5)	41.5 ⁽¹⁾	85 ⁽¹²⁾	74 ⁽⁷⁾	102 ⁽¹⁵⁾	80 ⁽¹¹⁾	98 ⁽¹³⁾
70	BIAS($\hat{\delta}$)	0.22531 ⁽²⁾	0.23468 ⁽⁵⁾	0.25439 ⁽¹⁰⁾	0.22455 ⁽¹⁾	0.24647 ⁽⁸⁾	0.22679 ⁽³⁾	0.23722 ⁽⁶⁾	0.25860 ⁽¹¹⁾	0.26157 ⁽¹²⁾	0.23375 ⁽⁴⁾	0.30642 ⁽¹⁵⁾	0.25051 ⁽⁹⁾	0.28540 ⁽¹⁴⁾	0.23907 ⁽⁷⁾	0.27404 ⁽¹³⁾
70	BIAS($\hat{\varepsilon}$)	0.21375 ⁽²⁾	0.22920 ⁽¹⁾	0.23101 ⁽¹²⁾	0.21915 ⁽³⁾	0.22584 ⁽¹⁰⁾	0.22405 ⁽⁹⁾	0.22266 ⁽⁶⁾	0.22438 ⁽⁸⁾	0.23773 ⁽¹⁵⁾	0.21236 ⁽¹⁾	0.22298 ⁽⁷⁾	0.22006 ⁽⁴⁾	0.23761 ⁽¹⁴⁾	0.22203 ⁽⁵⁾	0.23280 ⁽¹³⁾
70	MSE($\hat{\delta}$)	0.07890 ⁽³⁾	0.09148 ⁽⁷⁾	0.10074 ⁽¹⁰⁾	0.07861 ⁽²⁾	0.09300 ⁽⁸⁾	0.07726 ⁽¹⁾	0.09039 ⁽⁶⁾	0.10511 ⁽¹¹⁾	0.09302 ⁽¹²⁾	0.08304 ⁽⁴⁾	0.14677 ⁽¹⁵⁾	0.09846 ⁽⁹⁾	0.12541 ⁽¹⁴⁾	0.08909 ⁽⁵⁾	0.11247 ⁽¹³⁾
70	MSE($\hat{\varepsilon}$)	0.06432 ⁽¹⁾	0.07070 ^(9,5)	0.07277 ⁽¹²⁾	0.06618 ⁽³⁾	0.07070 ^(9,5)	0.06971 ⁽⁶⁾	0.06904 ⁽⁵⁾	0.07122 ⁽¹¹⁾	0.07651 ⁽¹⁵⁾	0.06479 ⁽²⁾	0.07063 ⁽⁸⁾	0.06890 ⁽⁴⁾	0.07573 ⁽¹⁴⁾	0.07045 ⁽⁷⁾	0.07413 ⁽¹³⁾
70	MRE($\hat{\delta}$)	0.11265 ⁽²⁾	0.11734 ⁽⁵⁾	0.12719 ⁽¹⁰⁾	0.11228 ⁽¹⁾	0.12324 ⁽⁸⁾	0.11339 ⁽³⁾	0.11861 ⁽⁶⁾	0.12930 ⁽¹¹⁾	0.13079 ⁽¹²⁾	0.11688 ⁽¹⁾	0.15321 ⁽¹⁵⁾	0.12526 ⁽⁹⁾	0.14270 ⁽¹⁴⁾	0.11954 ⁽⁷⁾	0.13702 ⁽¹³⁾
70	MRE($\hat{\varepsilon}$)	0.42751 ⁽²⁾	0.45841 ⁽¹¹⁾	0.46202 ⁽¹²⁾	0.43830 ⁽³⁾	0.45168 ⁽¹⁰⁾	0.44990 ⁽⁶⁾	0.44532 ⁽⁶⁾	0.44877 ⁽⁸⁾	0.47545 ⁽¹⁵⁾	0.44272 ⁽¹⁾	0.44597 ⁽⁷⁾	0.44011 ⁽⁴⁾	0.47522 ⁽¹⁴⁾	0.44405 ⁽⁵⁾	0.46560 ⁽¹³⁾
70	D _{abs}	0.02847 ^(8,5)	0.02759 ⁽³⁾	0.02847 ^(8,5)	0.02728 ⁽²⁾	0.02759 ⁽³⁾	0.02612 ⁽¹⁾	0.02779 ⁽⁴⁾	0.02784 ⁽¹⁰⁾	0.03262 ⁽¹³⁾	0.02848 ⁽¹⁾	0.03067 ⁽¹²⁾	0.02841 ⁽⁷⁾	0.03397 ⁽¹⁵⁾	0.02892 ⁽¹¹⁾	0.03379 ⁽¹⁴⁾
70	D _{max}	0.04577 ⁽³⁾	0.04474 ⁽³⁾	0.04646 ⁽¹¹⁾	0.04369 ⁽²⁾	0.04381 ⁽⁵⁾	0.04253 ⁽¹⁾	0.04506 ⁽⁴⁾	0.04562 ⁽⁷⁾	0.05198 ⁽¹³⁾	0.04542 ⁽⁶⁾	0.05043 ⁽¹²⁾	0.04627 ⁽¹⁰⁾	0.05410 ⁽¹⁵⁾	0.04626 ⁽⁹⁾	0.05365 ⁽¹⁵⁾
70	ASAE	0.01707 ⁽⁸⁾	0.01815 ⁽⁵⁾	0.01977 ⁽⁸⁾	0.01697 ⁽²⁾	0.02068 ⁽¹⁰⁾	0.01689 ⁽¹⁾	0.01797 ⁽⁴⁾	0.02155 ⁽¹¹⁾	0.02057 ⁽⁹⁾	0.01892 ⁽⁷⁾	0.02648 ⁽¹⁵⁾	0.02165 ⁽¹²⁾	0.02519 ⁽¹³⁾	0.02187 ⁽⁶⁾	0.02529 ⁽¹⁴⁾
70	Σ Ranks	31.5 ⁽²⁾	59.5 ⁽⁶⁾	93.5 ⁽¹¹⁾	19 ⁽¹⁾	74.5 ⁽⁹⁾	34 ⁽³⁾	47 ⁽⁵⁾	83 ⁽¹⁰⁾	116 ⁽¹³⁾	39 ⁽⁴⁾	106 ⁽¹²⁾	68 ⁽⁸⁾	127 ⁽¹⁵⁾	62 ⁽⁷⁾	120 ⁽¹⁴⁾
150	BIAS($\hat{\delta}$)	0.17377 ⁽²⁾	0.17755 ⁽³⁾	0.19924 ⁽¹⁰⁾	0.16707 ⁽¹⁾	0.18005 ⁽⁵⁾	0.17838 ⁽⁴⁾	0.18250 ⁽⁶⁾	0.20339 ⁽¹¹⁾	0.19845 ⁽⁹⁾	0.19106 ⁽⁸⁾	0.26263 ⁽¹⁵⁾	0.21655 ⁽¹²⁾	0.22457 ⁽¹³⁾	0.18964 ⁽⁷⁾	0.22393 ⁽¹³⁾
150	BIAS($\hat{\varepsilon}$)	0.19639 ⁽³⁾	0.19548 ⁽²⁾	0.20142 ⁽⁶⁾	0.19321 ⁽¹⁾	0.19835 ⁽⁵⁾	0.19716 ⁽⁴⁾	0.20400 ⁽⁷⁾	0.20420 ⁽⁸⁾	0.20597 ⁽⁹⁾	0.21114 ⁽¹³⁾	0.22204 ⁽¹⁴⁾	0.21069 ⁽¹²⁾	0.20907 ⁽¹¹⁾	0.20612 ⁽¹⁰⁾	0.22516 ⁽¹⁵⁾
150	MSE($\hat{\delta}$)	0.04620 ⁽²⁾	0.04736 ⁽⁴⁾	0.06022 ⁽⁹⁾	0.04292 ⁽¹⁾	0.05043 ⁽⁵⁾	0.04649 ⁽³⁾	0.05174 ⁽⁶⁾	0.06379 ⁽¹¹⁾	0.06353 ⁽¹⁰⁾	0.05629 ⁽⁸⁾	0.10038 ⁽¹⁵⁾	0.07015 ⁽¹²⁾	0.07805 ⁽¹⁴⁾	0.05496 ⁽⁷⁾	0.07645 ⁽¹³⁾
150	MSE($\hat{\varepsilon}$)	0.05550 ⁽³⁾	0.05523 ⁽²⁾	0.06035 ⁽⁸⁾	0.05396 ⁽¹⁾	0.05671 ⁽⁵⁾	0.05602 ⁽⁴⁾	0.05998 ⁽⁷⁾	0.05986 ⁽⁶⁾	0.06123 ⁽⁹⁾	0.06336 ⁽¹¹⁾	0.06981 ⁽¹⁴⁾	0.06364 ⁽¹²⁾	0.06424 ⁽¹³⁾	0.06176 ⁽¹⁰⁾	0.07165 ⁽¹⁵⁾
150	MRE($\hat{\delta}$)	0.08694 ⁽²⁾	0.08877 ⁽³⁾	0.09962 ⁽¹⁰⁾	0.08354 ⁽¹⁾	0.09003 ⁽⁵⁾	0.08919 ⁽⁴⁾	0.09125 ⁽⁶⁾	0.10169 ⁽¹¹⁾	0.09922 ⁽⁹⁾	0.09553 ⁽⁸⁾	0.13131 ⁽¹⁵⁾	0.10827 ⁽¹²⁾	0.11229 ⁽¹⁴⁾	0.09482 ⁽⁷⁾	0.11197 ⁽¹³⁾
150	MRE($\hat{\varepsilon}$)	0.39279 ⁽³⁾	0.39096 ⁽²⁾	0.40284 ⁽¹⁾	0.38643 ⁽¹⁾	0.39670 ⁽⁵⁾	0.39432 ⁽⁴⁾	0.40799 ⁽⁷⁾	0.40841 ⁽⁸⁾	0.41195 ⁽⁹⁾	0.42227 ⁽¹³⁾	0.44409 ⁽¹⁴⁾	0.42138 ⁽¹²⁾	0.41814 ⁽¹¹⁾	0.41224 ⁽¹⁰⁾	0.45032 ⁽¹⁵⁾
150	D _{abs}	0.01959 ⁽⁴⁾	0.01958 ⁽³⁾	0.02066 ⁽⁸⁾	0.01926 ⁽¹⁾	0.02055 ⁽⁷⁾	0.01938 ⁽²⁾	0.01988 ⁽⁵⁾	0.02041 ⁽⁶⁾	0.02361 ⁽¹³⁾	0.02172 ⁽¹⁰⁾	0.02250 ⁽¹¹⁾	0.02117 ⁽⁹⁾	0.02531 ⁽¹⁵⁾	0.02257 ⁽¹²⁾	0.02510 ⁽¹³⁾
150	D _{max}	0.03231 ⁽⁴⁾	0.03207 ⁽³⁾	0.03424 ⁽⁸⁾	0.03161 ⁽¹⁾	0.03352 ⁽⁶⁾	0.03195 ⁽²⁾	0.03294 ⁽⁵⁾	0.03395 ⁽⁷⁾	0.03260 ⁽¹³⁾	0.03537 ^(9,5)	0.03803 ⁽¹²⁾	0.03537 ^(9,5)	0.04093 ⁽¹⁵⁾	0.03662 ⁽¹¹⁾	0.04089 ⁽¹⁴⁾
150	ASAE	0.00901 ⁽¹⁾	0.00991 ⁽⁴⁾	0.01077 ⁽⁹⁾	0.00955 ⁽³⁾	0.01084 ⁽¹⁰⁾	0.00926 ⁽²⁾	0.01006 ⁽⁵⁾	0.01177 ⁽¹¹⁾	0.01073 ⁽⁸⁾	0.01055 ⁽⁷⁾	0.01479 ⁽¹⁵⁾	0.01208 ⁽¹²⁾	0.01367 ⁽¹⁴⁾	0.01025 ⁽⁶⁾	0.01366 ⁽¹³⁾
150	Σ Ranks	24 ⁽²⁾	26 ⁽³⁾	74 ⁽⁷⁾	11 ⁽¹⁾	53 ⁽⁵⁾	29 ⁽⁴⁾	54 ⁽⁶⁾	79 ⁽⁸⁾	89 ⁽¹¹⁾	87.5 ⁽¹⁰⁾	125 ^(14,5)	102.5 ⁽¹²⁾	121 ⁽¹³⁾	80 ⁽⁹⁾	125 ^(14,5)
220	BIAS($\hat{\delta}$)	0.15027 ⁽³⁾	0.14913 ⁽²⁾	0.16524 ⁽⁸⁾	0.15469 ⁽⁴⁾	0.17283 ⁽⁹⁾	0.15879 ⁽⁵⁾	0.14379 ⁽¹⁾	0.18473 ⁽¹²⁾	0.18383 ⁽¹¹⁾	0.16335 ⁽⁷⁾	0.24919 ⁽¹⁵⁾	0.17891 ⁽¹⁰⁾	0.19134 ⁽¹⁴⁾	0.16279 ⁽⁶⁾	0.19118 ⁽¹³⁾
220	BIAS($\hat{\varepsilon}$)	0.17558 ⁽²⁾	0.16975 ⁽¹⁾	0.18944 ⁽⁸⁾	0.17612 ⁽³⁾	0.19413 ⁽¹⁰⁾	0.18463 ⁽⁶⁾	0.17614 ⁽⁴⁾	0.18691 ⁽⁷⁾	0.20570 ⁽¹³⁾	0.19921 ⁽¹¹⁾	0.22077 ⁽¹⁵⁾	0.20065 ⁽¹²⁾	0.20646 ⁽¹⁴⁾	0.18178 ⁽⁵⁾	0.19219 ⁽⁹⁾
220	MSE($\hat{\delta}$)	0.03580 ⁽³⁾	0.03557 ⁽²⁾	0.04258 ⁽⁷⁾	0.03768 ⁽⁴⁾	0.04615 ⁽⁹⁾	0.03990 ⁽⁶⁾	0.03230 ⁽¹⁾	0.05191 ⁽¹¹⁾	0.05195 ⁽¹²⁾	0.04278 ⁽⁸⁾	0.09189 ⁽¹⁵⁾	0.04842 ⁽¹⁰⁾	0.05725 ⁽¹⁴⁾	0.03986 ⁽⁵⁾	0.05492 ⁽¹³⁾
220	MSE($\hat{\varepsilon}$)	0.04733 ⁽⁴⁾	0.04395 ⁽¹⁾	0.05358 ⁽⁸⁾	0.04557 ⁽²⁾	0.05523 ⁽¹⁰⁾	0.05145 ⁽⁷⁾	0.04672 ⁽³⁾	0.05114 ⁽⁶⁾	0.06105 ⁽¹³⁾	0.05526 ⁽¹¹⁾	0.06828 ⁽¹⁵⁾	0.05696 ⁽¹²⁾	0.06129 ⁽¹⁴⁾	0.04968 ⁽⁵⁾	0.05436 ⁽⁹⁾
220	MRE($\hat{\delta}$)	0.07513 ⁽³⁾	0.07457 ⁽²⁾	0.08262 ⁽⁸⁾	0.07735 ⁽⁴⁾	0.07939 ⁽⁵⁾	0.07190 ⁽¹⁾	0.09237 ⁽¹²⁾	0.09191 ⁽¹¹⁾	0.08168 ⁽⁷⁾	0.12460 ⁽¹⁵⁾	0.08946 ⁽¹⁰⁾	0.09567 ⁽¹⁴⁾	0.08140 ⁽⁶⁾	0.09559 ⁽¹³⁾	0.09559 ⁽¹³⁾
220	MRE($\hat{\varepsilon}$)	0.35117 ⁽²⁾	0.33950 ⁽¹⁾	0.37888 ⁽⁸⁾	0.35223 ⁽³⁾	0.38827 ⁽¹⁰⁾	0.36927 ⁽⁶⁾	0.35228 ⁽⁴⁾	0.37382 ⁽⁷⁾	0.41139 ⁽¹³⁾	0.39843 ⁽¹⁾	0.44153 ⁽¹⁵⁾	0.40130 ⁽¹²⁾	0.41292 ⁽¹⁴⁾	0.36355 ⁽⁵⁾	0.38437 ⁽⁹⁾
220	D _{abs}	0.01680 ⁽⁵⁾	0.01610 ⁽¹⁾	0.01748 ⁽⁸⁾	0.01649 ⁽²⁾	0.01758 ⁽⁹⁾	0.01652 ⁽³⁾	0.01654 ⁽⁴⁾	0.01723 ⁽⁷⁾	0.02102 ⁽¹⁴⁾	0.01935 ⁽¹²⁾	0.01892 ⁽¹¹⁾	0.01705 ⁽⁶⁾	0.02134 ⁽¹⁵⁾	0.01855 ⁽¹⁰⁾	0.02095 ⁽¹³⁾
220	D _{max}	0.02767 ⁽⁵⁾	0.02652 ⁽¹⁾	0.02891 ^(7,5)	0.02729 ⁽³⁾	0.02925 ⁽⁹⁾	0.02751 ⁽⁴⁾	0.02721 ⁽²⁾	0.02891 ^(7,5)	0.03433 ⁽¹³⁾	0.03179 ⁽¹¹⁾	0.03303 ⁽¹²⁾	0.02875 ⁽⁶⁾	0.03478 ⁽¹⁵⁾	0.03017 ⁽¹⁰⁾	0.03443 ⁽¹⁴⁾
220	ASAE	0.00680 ⁽²⁾	0.00733 ⁽⁴⁾	0.00827 ⁽⁹⁾	0.00670 ⁽¹⁾	0.00821 ⁽⁸⁾	0.00697 ⁽³⁾	0.00740 ⁽⁵⁾	0.00934 ⁽¹²⁾	0.00858 ⁽¹⁰⁾	0.00773 ⁽⁷⁾	0.01149 ⁽¹⁵⁾	0.00872 ⁽¹¹⁾	0.00980 ⁽¹³⁾	0.00746 ⁽⁶⁾	0.00983 ⁽¹⁴⁾
220	Σ Ranks	29 ⁽⁴⁾	15 ⁽¹⁾	71.5 ⁽⁷⁾	26 ⁽³⁾	83 ⁽⁹⁾	45 ⁽⁵⁾	25 ⁽²⁾	81.5 ⁽⁸⁾	110 ⁽¹³⁾	85 ⁽¹⁰⁾	128 ⁽¹⁵⁾	89 ⁽¹¹⁾	127 ⁽¹⁴⁾	58 ⁽⁶⁾	107 ⁽¹²⁾
280	BIAS($\hat{\delta}$)	0.12902 ⁽¹⁾	0.14786 ⁽⁵⁾	0.15844 ⁽⁸⁾	0.14185 ⁽⁴⁾	0.16321 ⁽⁹⁾	0.13694 ⁽²⁾	0.14165 ⁽³⁾	0.16058 ⁽⁹⁾	0.16719 ⁽¹¹⁾	0.15551 ⁽⁷⁾	0.23758 ⁽¹⁵⁾	0.17115 ⁽¹²⁾	0.18400 ⁽¹⁴⁾	0.15107 ⁽⁶⁾	0.17608 ⁽¹³⁾
280</																

TABLE 6. Numerical values of simulation measures for $\delta = 3.5$, $\varepsilon = 0.65$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{\delta}$)	0.63115 ⁽⁴⁾	0.61519 ⁽¹⁾	0.72897 ⁽¹²⁾	0.62964 ⁽³⁾	0.68775 ⁽⁷⁾	0.62645 ⁽²⁾	0.69243 ⁽⁸⁾	0.69278 ⁽⁹⁾	0.70242 ⁽¹¹⁾	0.69712 ⁽¹⁰⁾	0.79031 ⁽¹⁴⁾	0.66207 ⁽⁶⁾	0.76098 ⁽¹³⁾	0.65345 ⁽⁵⁾	0.80845 ⁽¹⁵⁾
20	BIAS($\hat{\varepsilon}$)	0.30407 ⁽³⁾	0.32087 ⁽¹³⁾	0.31153 ⁽⁸⁾	0.30071 ⁽¹¹⁾	0.31259 ⁽⁹⁾	0.30481 ⁽⁴⁾	0.32580 ⁽¹⁸⁾	0.30301 ⁽²⁾	0.31453 ⁽¹⁰⁾	0.31485 ⁽¹¹⁾	0.30680 ⁽⁷⁾	0.30576 ⁽⁵⁾	0.32438 ⁽¹⁴⁾	0.30654 ⁽⁶⁾	0.31772 ⁽¹²⁾
20	MSE($\hat{\delta}$)	0.67695 ⁽⁴⁾	0.58487 ⁽¹⁾	0.86645 ⁽¹³⁾	0.63201 ⁽²⁾	0.72466 ⁽⁶⁾	0.65775 ⁽³⁾	0.74668 ⁽⁷⁾	0.74932 ⁽⁸⁾	0.79411 ⁽¹¹⁾	0.76189 ⁽¹⁰⁾	1.07481 ⁽¹⁵⁾	0.75797 ⁽⁹⁾	0.83763 ⁽¹²⁾	0.69576 ⁽⁵⁾	0.94419 ⁽¹⁴⁾
20	MSE($\hat{\varepsilon}$)	0.12829 ⁽⁴⁾	0.14138 ⁽¹⁴⁾	0.13098 ⁽⁷⁾	0.12574 ⁽²⁾	0.13159 ⁽⁹⁾	0.12671 ⁽³⁾	0.14173 ⁽¹⁵⁾	0.12405 ⁽¹⁾	0.13428 ⁽¹²⁾	0.13297 ⁽¹⁰⁾	0.12888 ⁽⁶⁾	0.12868 ⁽⁵⁾	0.13894 ⁽¹³⁾	0.13156 ⁽⁸⁾	0.13347 ⁽¹¹⁾
20	MRE($\hat{\delta}$)	0.18033 ⁽⁴⁾	0.17577 ⁽¹⁾	0.20828 ⁽¹²⁾	0.17990 ⁽³⁾	0.19650 ⁽⁷⁾	0.17899 ⁽²⁾	0.19784 ⁽⁸⁾	0.19794 ⁽⁹⁾	0.20069 ⁽¹¹⁾	0.19918 ⁽¹⁰⁾	0.22580 ⁽¹⁴⁾	0.18916 ⁽⁶⁾	0.21742 ⁽¹³⁾	0.18670 ⁽⁵⁾	0.23099 ⁽¹⁵⁾
20	MRE($\hat{\varepsilon}$)	0.46781 ⁽³⁾	0.49364 ⁽¹³⁾	0.47928 ⁽⁸⁾	0.46264 ⁽¹¹⁾	0.48091 ⁽⁹⁾	0.46894 ⁽⁴⁾	0.50123 ⁽¹⁵⁾	0.46617 ⁽²⁾	0.48390 ⁽¹⁰⁾	0.48439 ⁽¹¹⁾	0.47200 ⁽⁷⁾	0.47040 ⁽⁵⁾	0.49905 ⁽¹⁴⁾	0.47160 ⁽⁶⁾	0.48880 ⁽¹²⁾
20	D_{abs}	0.04709 ⁽²⁾	0.04671 ⁽¹⁾	0.05215 ⁽⁸⁾	0.05243 ⁽¹⁰⁾	0.04881 ⁽⁴⁾	0.04897 ⁽⁵⁾	0.05210 ⁽⁷⁾	0.05230 ⁽⁹⁾	0.05932 ⁽¹³⁾	0.05485 ⁽¹²⁾	0.05418 ⁽¹¹⁾	0.04875 ⁽³⁾	0.06136 ⁽¹⁵⁾	0.05089 ⁽⁶⁾	0.06002 ⁽¹⁴⁾
20	D_{max}	0.07534 ⁽²⁾	0.07328 ⁽¹⁾	0.08262 ⁽¹⁰⁾	0.07921 ⁽⁷⁾	0.07561 ⁽³⁾	0.07674 ⁽⁴⁾	0.08072 ⁽⁸⁾	0.08176 ⁽⁹⁾	0.09074 ⁽¹³⁾	0.08462 ⁽¹¹⁾	0.08469 ⁽¹²⁾	0.07716 ⁽⁵⁾	0.09285 ⁽¹⁵⁾	0.07869 ⁽⁶⁾	0.09187 ⁽¹⁴⁾
20	ASAE	0.04546 ⁽⁶⁾	0.04234 ⁽³⁾	0.04472 ⁽⁵⁾	0.04228 ⁽²⁾	0.04549 ⁽⁷⁾	0.03973 ⁽¹⁾	0.04267 ⁽⁴⁾	0.04858 ⁽⁹⁾	0.05311 ⁽¹²⁾	0.04822 ⁽⁸⁾	0.06088 ⁽¹³⁾	0.05105 ⁽¹¹⁾	0.06545 ⁽¹⁴⁾	0.04934 ⁽¹⁰⁾	0.06652 ⁽¹⁵⁾
20	$\Sigma Ranks$	32 ⁽³⁾	48 ⁽⁴⁾	83 ⁽⁹⁾	31 ⁽²⁾	61 ⁽⁸⁾	28 ⁽¹⁾	87 ⁽¹⁰⁾	58 ⁽⁷⁾	103 ⁽¹³⁾	93 ⁽¹¹⁾	99 ⁽¹²⁾	55 ⁽⁵⁾	123 ⁽¹⁵⁾	57 ⁽⁶⁾	122 ⁽¹⁴⁾
70	BIAS($\hat{\delta}$)	0.41307 ⁽⁵⁾	0.40177 ⁽¹⁾	0.45246 ⁽⁸⁾	0.40585 ⁽²⁾	0.45718 ⁽¹¹⁾	0.40818 ⁽⁴⁾	0.41330 ⁽⁶⁾	0.49404 ⁽¹²⁾	0.45289 ⁽⁹⁾	0.45117 ⁽⁷⁾	0.56052 ⁽¹⁵⁾	0.45503 ⁽¹⁰⁾	0.51439 ⁽¹⁴⁾	0.40749 ⁽³⁾	0.50524 ⁽¹³⁾
70	BIAS($\hat{\varepsilon}$)	0.28022 ⁽⁵⁾	0.26811 ⁽³⁾	0.29270 ⁽¹²⁾	0.26652 ⁽²⁾	0.28605 ⁽⁷⁾	0.28366 ⁽⁶⁾	0.25972 ⁽¹⁾	0.28944 ⁽⁹⁾	0.27298 ⁽⁴⁾	0.29109 ⁽¹¹⁾	0.30158 ⁽¹⁴⁾	0.29441 ⁽¹³⁾	0.30420 ⁽¹⁵⁾	0.28741 ⁽⁸⁾	0.29032 ⁽¹⁰⁾
70	MSE($\hat{\delta}$)	0.27368 ⁽⁶⁾	0.24033 ⁽¹⁾	0.32046 ⁽¹⁰⁾	0.24834 ⁽²⁾	0.31585 ⁽⁹⁾	0.25382 ⁽³⁾	0.27022 ⁽⁵⁾	0.37981 ⁽¹³⁾	0.31189 ⁽⁸⁾	0.29663 ⁽⁷⁾	0.47438 ⁽¹⁵⁾	0.32390 ⁽¹¹⁾	0.39320 ⁽¹⁴⁾	0.25619 ⁽⁴⁾	0.37779 ⁽¹²⁾
70	MSE($\hat{\varepsilon}$)	0.11015 ⁽⁵⁾	0.10188 ⁽³⁾	0.11667 ⁽⁹⁾	0.10104 ⁽²⁾	0.11150 ⁽⁶⁾	0.11156 ⁽⁷⁾	0.09544 ⁽¹⁾	0.11701 ⁽¹¹⁾	0.10605 ⁽⁴⁾	0.11611 ⁽⁸⁾	0.12472 ⁽¹⁴⁾	0.11958 ⁽¹²⁾	0.12593 ⁽¹⁵⁾	0.11693 ⁽¹⁰⁾	0.12066 ⁽¹³⁾
70	MRE($\hat{\delta}$)	0.11802 ⁽⁵⁾	0.11479 ⁽¹⁾	0.12928 ⁽⁸⁾	0.11596 ⁽²⁾	0.13062 ⁽¹¹⁾	0.11662 ⁽⁴⁾	0.11662 ⁽⁴⁾	0.1415 ⁽¹²⁾	0.10640 ⁽⁹⁾	0.12890 ⁽⁷⁾	0.16015 ⁽¹⁵⁾	0.13001 ⁽¹⁰⁾	0.14697 ⁽¹⁴⁾	0.11643 ⁽³⁾	0.14435 ⁽¹³⁾
70	MRE($\hat{\varepsilon}$)	0.43110 ⁽⁵⁾	0.41247 ⁽³⁾	0.45030 ⁽¹²⁾	0.41003 ⁽²⁾	0.44008 ⁽⁷⁾	0.43640 ⁽⁶⁾	0.39957 ⁽¹⁾	0.44530 ⁽⁹⁾	0.441998 ⁽⁴⁾	0.44783 ⁽¹¹⁾	0.46397 ⁽¹⁴⁾	0.45293 ⁽¹³⁾	0.46800 ⁽¹⁵⁾	0.44217 ⁽⁸⁾	0.44665 ⁽¹⁰⁾
70	D_{abs}	0.02745 ⁽¹⁾	0.02802 ⁽²⁾	0.02993 ⁽⁷⁾	0.02872 ⁽⁴⁾	0.03022 ⁽¹⁰⁾	0.02891 ⁽⁶⁾	0.02832 ⁽³⁾	0.02935 ⁽⁹⁾	0.03512 ⁽¹⁴⁾	0.03082 ⁽¹¹⁾	0.03110 ⁽¹²⁾	0.02884 ⁽⁵⁾	0.03402 ⁽¹³⁾	0.02952 ⁽¹⁵⁾	0.03575 ⁽¹⁵⁾
70	D_{max}	0.04514 ⁽¹⁾	0.04557 ⁽²⁾	0.04820 ⁽⁸⁾	0.04615 ⁽⁴⁾	0.04935 ⁽⁷⁾	0.04705 ⁽⁵⁾	0.04495 ⁽³⁾	0.04948 ⁽¹⁰⁾	0.05578 ⁽¹⁴⁾	0.04988 ⁽¹¹⁾	0.05233 ⁽¹²⁾	0.04775 ⁽⁶⁾	0.05528 ⁽¹³⁾	0.04789 ⁽⁷⁾	0.05703 ⁽¹⁵⁾
70	ASAE	0.01819 ^(4.5)	0.01819 ^(4.5)	0.01928 ⁽⁶⁾	0.01738 ⁽²⁾	0.01962 ⁽⁸⁾	0.01705 ⁽¹⁾	0.01786 ⁽³⁾	0.02141 ⁽¹¹⁾	0.02204 ⁽¹²⁾	0.02014 ⁽⁹⁾	0.02775 ⁽¹⁵⁾	0.02093 ⁽¹⁰⁾	0.02495 ⁽¹³⁾	0.01944 ⁽⁷⁾	0.02719 ⁽¹⁴⁾
70	$\Sigma Ranks$	37.5 ⁽⁴⁾	20.5 ⁽¹⁾	80 ⁽⁹⁾	22 ⁽²⁾	78 ^(7.5)	42 ⁽⁵⁾	29 ⁽³⁾	96 ⁽¹²⁾	78 ^(7.5)	82 ⁽¹⁰⁾	126 ^(14.5)	90 ⁽¹¹⁾	126 ^(14.5)	58 ⁽⁶⁾	115 ⁽¹³⁾
150	BIAS($\hat{\delta}$)	0.30247 ⁽¹⁾	0.31126 ⁽²⁾	0.33917 ⁽⁸⁾	0.31089 ⁽⁴⁾	0.36655 ⁽⁹⁾	0.32156 ⁽⁵⁾	0.31683 ⁽³⁾	0.38927 ⁽¹²⁾	0.36949 ⁽¹⁰⁾	0.33266 ⁽⁷⁾	0.47879 ⁽¹⁵⁾	0.37146 ⁽¹¹⁾	0.42347 ⁽¹⁴⁾	0.33044 ⁽⁶⁾	0.41839 ⁽¹³⁾
150	BIAS($\hat{\varepsilon}$)	0.22832 ⁽³⁾	0.22943 ⁽⁴⁾	0.23997 ⁽⁶⁾	0.22512 ⁽¹⁾	0.24559 ⁽⁸⁾	0.22791 ⁽²⁾	0.23094 ⁽⁵⁾	0.25021 ⁽¹⁾	0.24940 ⁽¹⁰⁾	0.24276 ⁽⁷⁾	0.28299 ⁽¹⁵⁾	0.25244 ⁽¹²⁾	0.27162 ⁽¹³⁾	0.24753 ⁽⁹⁾	0.28228 ⁽¹⁴⁾
150	MSE($\hat{\delta}$)	0.14152 ⁽³⁾	0.14886 ⁽²⁾	0.17429 ⁽⁸⁾	0.15171 ⁽³⁾	0.19958 ⁽⁹⁾	0.15721 ⁽⁵⁾	0.15233 ⁽⁴⁾	0.22869 ⁽¹²⁾	0.21159 ⁽¹¹⁾	0.16654 ⁽⁷⁾	0.34526 ⁽¹⁵⁾	0.21041 ⁽¹⁰⁾	0.26842 ⁽¹⁴⁾	0.16622 ⁽⁶⁾	0.25778 ⁽¹³⁾
150	MSE($\hat{\varepsilon}$)	0.07799 ⁽³⁾	0.07920 ⁽⁵⁾	0.08584 ⁽⁶⁾	0.07566 ⁽²⁾	0.08623 ⁽⁷⁾	0.07827 ⁽⁴⁾	0.07496 ⁽¹⁾	0.08933 ⁽⁹⁾	0.09097 ⁽¹¹⁾	0.08684 ⁽⁸⁾	0.11151 ⁽¹⁵⁾	0.09342 ⁽¹²⁾	0.10402 ⁽¹³⁾	0.08995 ⁽¹⁰⁾	0.11116 ⁽¹⁴⁾
150	MRE($\hat{\delta}$)	0.08642 ⁽¹⁾	0.08893 ⁽²⁾	0.09691 ⁽⁸⁾	0.09140 ⁽⁴⁾	0.10473 ⁽⁹⁾	0.09187 ⁽⁵⁾	0.09052 ⁽³⁾	0.11122 ⁽¹²⁾	0.10570 ⁽¹⁰⁾	0.09505 ⁽⁷⁾	0.13680 ⁽¹⁵⁾	0.10613 ⁽¹¹⁾	0.12099 ⁽¹⁴⁾	0.09441 ⁽⁶⁾	0.11954 ⁽¹³⁾
150	MRE($\hat{\varepsilon}$)	0.35127 ⁽³⁾	0.35298 ⁽⁴⁾	0.36918 ⁽⁶⁾	0.34634 ⁽¹⁾	0.37783 ⁽⁸⁾	0.35063 ⁽²⁾	0.35530 ⁽⁵⁾	0.38494 ⁽¹¹⁾	0.38369 ⁽¹⁰⁾	0.37347 ⁽⁷⁾	0.43537 ⁽¹⁵⁾	0.38837 ⁽¹²⁾	0.41788 ⁽¹³⁾	0.38081 ⁽⁹⁾	0.43428 ⁽¹⁴⁾
150	D_{abs}	0.02101 ⁽⁴⁾	0.02042 ⁽¹⁾	0.02110 ^(5.5)	0.02067 ^(2.5)	0.02110 ^(5.5)	0.02112 ⁽⁷⁾	0.02067 ^(2.5)	0.02170 ⁽⁹⁾	0.02491 ⁽¹⁾	0.02258 ⁽¹⁰⁾	0.02374 ⁽¹²⁾	0.02127 ⁽⁸⁾	0.02615 ⁽¹⁵⁾	0.02275 ⁽¹¹⁾	0.02518 ⁽¹³⁾
150	D_{max}	0.03454 ⁽⁴⁾	0.03389 ⁽²⁾	0.03504 ⁽⁶⁾	0.03385 ⁽¹⁾	0.03545 ⁽⁷⁾	0.03494 ⁽⁵⁾	0.03428 ⁽³⁾	0.03659 ⁽⁹⁾	0.04071 ⁽¹²⁾	0.03694 ⁽¹⁰⁾	0.04096 ⁽¹³⁾	0.03572 ⁽⁸⁾	0.04340 ⁽¹⁵⁾	0.03732 ⁽¹¹⁾	0.04191 ⁽¹⁴⁾
150	ASAE	0.01017 ⁽³⁾	0.01062 ⁽⁵⁾	0.01139 ⁽⁷⁾	0.00971 ⁽¹⁾	0.01126 ⁽⁶⁾	0.00975 ⁽²⁾	0.01043 ⁽⁵⁾	0.01275 ⁽¹¹⁾	0.01294 ⁽¹²⁾	0.01195 ⁽⁸⁾	0.01669 ⁽¹⁵⁾	0.01210 ⁽¹⁰⁾	0.01514 ⁽¹⁴⁾	0.01198 ⁽⁹⁾	0.01434 ⁽¹³⁾
150	$\Sigma Ranks$	23 ⁽²⁾	26 ⁽³⁾	60.5 ⁽⁶⁾	20.5 ⁽¹⁾	68.5 ⁽⁷⁾	37 ⁽⁵⁾	30.5 ⁽⁴⁾	96 ⁽¹¹⁾	99 ⁽¹²⁾	71 ⁽⁸⁾	130 ⁽¹⁵⁾	94 ⁽¹⁰⁾	125 ⁽¹⁴⁾	77 ⁽⁹⁾	122 ⁽¹³⁾
220	BIAS($\hat{\delta}$)	0.25484 ⁽²⁾	0.26711 ⁽⁴⁾	0.28389 ⁽⁶⁾	0.26108 ⁽³⁾	0.30146 ⁽⁸⁾	0.24943 ⁽¹⁾	0.26869 ⁽⁵⁾	0.32804 ⁽¹²⁾	0.31468 ⁽¹⁰⁾	0.28532 ⁽⁷⁾	0.45263 ⁽¹⁵⁾	0.32669 ⁽¹¹⁾	0.36405 ⁽¹⁴⁾	0.30986 ⁽⁹⁾	0.34736 ⁽¹³⁾
220	BIAS($\hat{\varepsilon}$)	0.19064 ⁽²⁾	0.19425 ⁽⁵⁾	0.20666 ⁽⁷⁾	0.18685 ⁽¹⁾	0.20535 ⁽⁶⁾	0.19288 ⁽³⁾	0.19320 ⁽⁴⁾	0.23071 ⁽¹¹⁾	0.23281 ⁽¹²⁾	0.22506 ⁽⁹⁾	0.26571 ⁽¹⁵⁾	0.22703 ⁽¹⁰⁾	0.24481 ⁽¹⁴⁾	0.22389 ⁽⁸⁾	0.24345 ⁽¹³⁾
220	MSE($\hat{\delta}$)	0.10320 ⁽²⁾	0.11285 ⁽⁵⁾	0.13268 ⁽⁷⁾	0.10364 ⁽³⁾	0.13784 ⁽⁸⁾	0.09873 ⁽¹⁾	0.11069 ⁽⁴⁾	0.16797 ⁽¹²⁾	0.15459 ⁽¹⁰⁾	0.13168 ⁽⁶⁾	0.30722 ⁽¹⁵⁾	0.15876 ⁽¹¹⁾	0.19566 ⁽¹⁴⁾	0.15026 ⁽⁹⁾	0.18327 ⁽¹³⁾
220	MSE($\hat{\varepsilon}$)	0.05584 ⁽¹⁾	0.05814 ⁽⁴⁾	0.06628 ⁽⁷⁾	0.05599 ⁽²⁾	0.06477 ⁽⁶⁾	0.05863 ⁽⁵⁾	0.05774 ⁽³⁾	0.07736 ⁽¹¹⁾	0.08011 ⁽¹²⁾	0.07587 ⁽⁹⁾	0.10294 ⁽¹⁵⁾	0.07703 ⁽¹⁰⁾	0.08682 ⁽¹⁴⁾	0.07497 ⁽⁸⁾	0.08665 ⁽¹³⁾
220	MRE($\hat{\delta}$)	0.07281 ⁽²⁾	0.07632 ⁽⁴⁾	0.08111 ⁽⁶⁾	0.07460 ⁽³⁾	0.08613 ⁽⁸⁾	0.07127 ⁽¹⁾	0.07677 ⁽⁵⁾	0.09373 ⁽¹²⁾	0.08991 ⁽¹⁰⁾	0.08152 ⁽⁷⁾	0.12932 ⁽¹⁵⁾	0.09334 ⁽¹¹⁾	0.10401 ⁽¹⁴⁾	0.08853 ⁽⁹⁾	0.09925 ⁽¹³⁾
220	MRE($\hat{\varepsilon}$)	0.29329 ⁽²⁾	0.29885 ⁽⁵⁾	0.31794 ⁽⁷⁾	0.28747 ⁽¹⁾	0.31593 ⁽⁶⁾	0.29674 ⁽³⁾	0.29723 ⁽⁴⁾	0.35493 ⁽¹¹⁾	0.35817 ⁽¹²⁾	0.34625 ⁽⁹⁾	0.40879 ⁽¹⁵⁾	0.34927 ⁽¹⁰⁾	0.37662 ⁽¹⁴⁾	0.34444 ⁽⁸⁾	0.37454 ⁽¹³⁾
220	D_{abs}	0.01707 ⁽³⁾	0.01724 ^(5.5)	0.01724 ^(5.5)	0.01644 ⁽¹⁾	0.01714 ⁽⁴⁾	0.01731 ⁽⁷⁾	0.01675 ⁽²⁾	0.01846 ⁽⁸⁾	0.02075 ⁽¹³⁾	0.02067 ⁽¹²⁾	0.02017 ⁽¹¹⁾	0.01895 ⁽⁹⁾	0.02145 ⁽¹⁴⁾	0.01911 ⁽¹⁰⁾	0.02198 ⁽¹⁵⁾
220	D_{max}	0.02836 ⁽³⁾	0.02855 ⁽⁵⁾	0.02875 ⁽⁷⁾	0.02713 ⁽¹⁾	0.02863 ⁽⁶⁾	0.02837 ⁽⁴⁾	0.02792 ⁽²⁾	0.03111 ⁽⁸⁾	0.03408 ⁽¹²⁾	0.03373 ⁽¹¹⁾	0.03518 ⁽¹³⁾	0.03191 ⁽¹⁰⁾	0.03590 ⁽¹⁴⁾	0.03167 ⁽⁹⁾	0.03634 ⁽¹³⁾
220	ASAE	0.00753 ⁽²⁾	0.00806 ⁽⁵⁾	0.00840 ⁽⁶⁾	0.00755 ⁽³⁾	0.00868 ⁽⁸⁾	0.00748 ⁽¹⁾	0.00784 ⁽⁴⁾	0.00934 ⁽¹¹⁾	0.00977 ⁽¹²⁾	0.00915 ⁽⁹⁾	0.01303 ⁽¹⁵⁾	0.00930 ⁽¹⁰⁾	0.01137 ⁽¹⁴⁾	0.00859 ⁽⁷⁾	0.01123 ⁽¹³⁾
220	$\Sigma Ranks$	19 ⁽²⁾	42.5 ⁽⁵⁾	58.5 ⁽⁶⁾	18 ⁽¹⁾	60 ⁽⁷⁾	26 ⁽³⁾	33 ⁽⁴⁾	96 ⁽¹¹⁾	103 ⁽¹²⁾	79 ⁽⁹⁾	129 ⁽¹⁵⁾	92 ⁽¹⁰⁾	126 ⁽¹⁴⁾	77 ⁽⁸⁾	121 ⁽¹³⁾
280	BIAS($\hat{\delta}$)	0.23187 ⁽¹⁾	0.24588 ⁽³⁾	0.27636 ⁽⁹⁾	0.25232 ⁽⁵⁾	0.27127 ⁽⁷⁾	0.23610 ⁽²⁾	0.24946 ⁽⁴⁾	0.31816 ⁽¹³⁾	0.29282 ⁽¹¹⁾ </						

TABLE 7. Partial and overall ranks for all estimation methods of our proposed model.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
$\delta = 1.25, \varepsilon = 2.75$	20	1.0	2.0	10.0	4.0	6.0	3.0	8.0	11.0	9.0	12.0	13.0	7.0	14.0	5.0	15.0
	70	1.0	2.0	11.0	4.0	3.0	7.0	6.0	5.0	12.0	10.0	15.0	9.0	13.0	8.0	14.0
	150	2.0	3.0	6.5	1.0	10.0	11.0	4.0	6.5	13.0	8.5	15.0	8.5	14.0	5.0	12.0
	220	5.0	1.0	4.0	2.0	6.5	10.0	3.0	9.0	6.5	12.0	15.0	11.0	14.0	8.0	13.0
	280	3.0	4.0	8.0	1.0	11.0	5.5	2.0	10.0	9.0	12.0	15.0	5.5	14.0	7.0	13.0
	350	4.0	3.0	7.5	1.0	6.0	7.5	2.0	11.0	10.0	14.0	15.0	9.0	12.5	5.0	12.5
$\delta = 0.25, \varepsilon = 0.75$	20	2.0	8.0	12.0	4.0	9.5	1.0	7.0	14.0	5.5	9.5	13.0	5.5	15.0	3.0	11.0
	70	1.0	3.0	8.0	6.0	10.0	4.0	2.0	12.0	9.0	11.0	13.0	7.0	15.0	5.0	14.0
	150	3.0	1.0	6.0	2.0	9.0	4.0	5.0	11.0	12.0	7.0	15.0	10.0	13.0	8.0	14.0
	220	1.0	5.0	6.0	2.5	10.0	2.5	4.0	9.0	11.0	12.0	15.0	7.0	14.0	8.0	13.0
	280	4.0	1.0	8.0	2.0	6.0	3.0	5.0	11.0	12.0	10.0	15.0	9.0	13.5	7.0	13.5
	350	1.0	3.0	7.0	2.0	9.0	4.0	5.0	11.0	12.0	8.0	15.0	10.0	13.0	6.0	14.0
$\delta = 0.9, \varepsilon = 1.5$	20	3.0	4.0	2.0	7.0	5.0	1.0	6.0	8.0	13.5	10.0	12.0	11.0	15.0	9.0	13.5
	70	1.0	3.0	7.0	2.0	10.0	9.0	5.0	8.0	13.0	12.0	14.0	4.0	15.0	6.0	11.0
	150	4.5	3.0	6.0	1.0	7.0	4.5	2.0	10.0	13.0	12.0	15.0	8.0	14.0	9.0	11.0
	220	1.5	1.5	5.0	3.0	7.0	6.0	4.0	10.0	13.0	8.0	15.0	11.0	14.0	9.0	12.0
	280	2.0	3.0	5.0	1.0	11.0	7.0	4.0	8.0	12.0	9.0	15.0	10.0	14.0	6.0	13.0
	350	2.0	1.0	7.0	5.0	12.0	6.0	3.0	8.0	11.0	10.0	15.0	9.0	13.5	4.0	13.5
$\delta = 2.0, \varepsilon = 0.5$	20	4.0	10.0	5.0	6.0	8.5	2.0	3.0	14.0	8.5	1.0	12.0	7.0	15.0	11.0	13.0
	70	2.0	6.0	11.0	1.0	9.0	3.0	5.0	10.0	13.0	4.0	12.0	8.0	15.0	7.0	14.0
	150	2.0	3.0	7.0	1.0	5.0	4.0	6.0	8.0	11.0	10.0	14.5	12.0	13.0	9.0	14.5
	220	4.0	1.0	7.0	3.0	9.0	5.0	2.0	8.0	13.0	10.0	15.0	11.0	14.0	6.0	12.0
	280	1.0	4.0	8.0	2.0	9.0	5.0	3.0	7.0	12.0	10.0	15.0	11.0	14.0	6.0	13.0
	350	2.0	5.0	6.0	1.0	7.0	3.0	4.0	10.0	11.0	9.0	15.0	12.0	14.0	8.0	13.0
$\delta = 3.5, \varepsilon = 0.65$	20	3.0	4.0	9.0	2.0	8.0	1.0	10.0	7.0	13.0	11.0	12.0	5.0	15.0	6.0	14.0
	70	4.0	1.0	9.0	2.0	7.5	5.0	3.0	12.0	7.5	10.0	14.5	11.0	14.5	6.0	13.0
	150	2.0	3.0	6.0	1.0	7.0	5.0	4.0	11.0	12.0	8.0	15.0	10.0	14.0	9.0	13.0
	220	2.0	5.0	6.0	1.0	7.0	3.0	4.0	11.0	12.0	9.0	15.0	10.0	14.0	8.0	13.0
	280	1.0	4.0	7.0	5.0	6.0	3.0	2.0	11.0	12.0	10.0	15.0	9.0	14.0	8.0	13.0
	350	1.0	3.0	6.5	2.0	8.0	5.0	4.0	11.0	12.0	10.0	15.0	9.0	13.0	6.5	14.0
Σ Ranks		70.0	100.5	213.5	77.5	239.0	140.0	127.0	292.5	333.5	289.0	430.0	266.5	420.0	208.5	392.5
Overall Rank		1	3	7	2	8	5	4	11	12	10	15	9	14	6	13

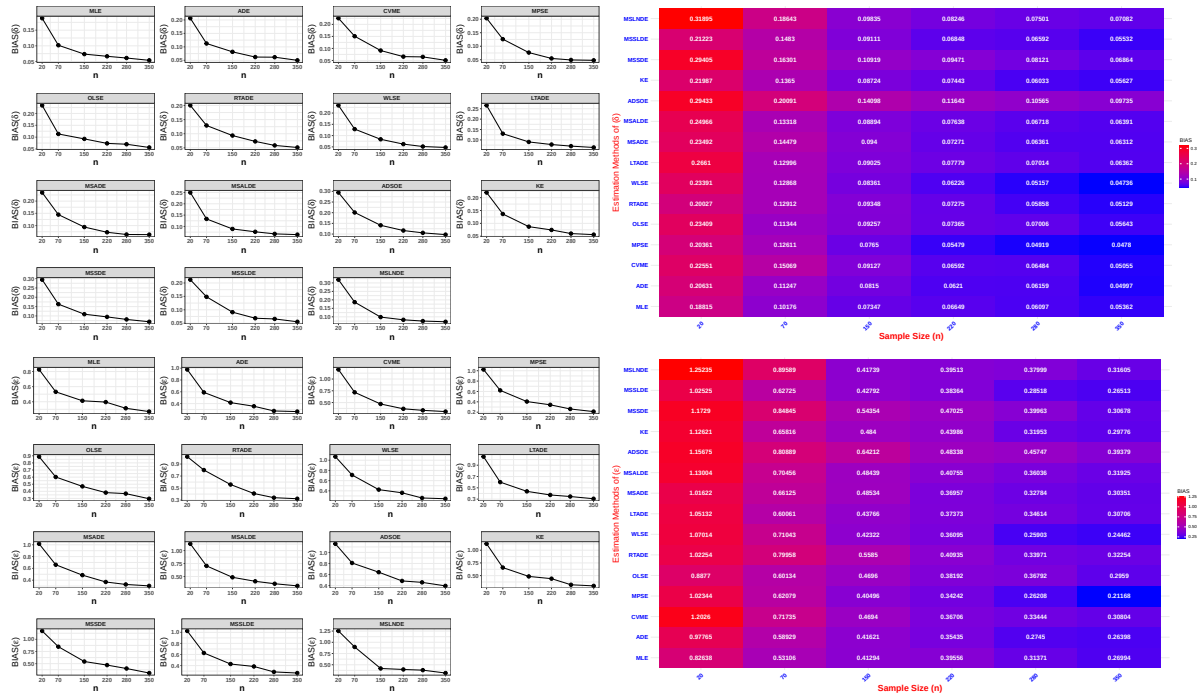


FIGURE 4. A graphical representation for BIAS values presented in Table 2.

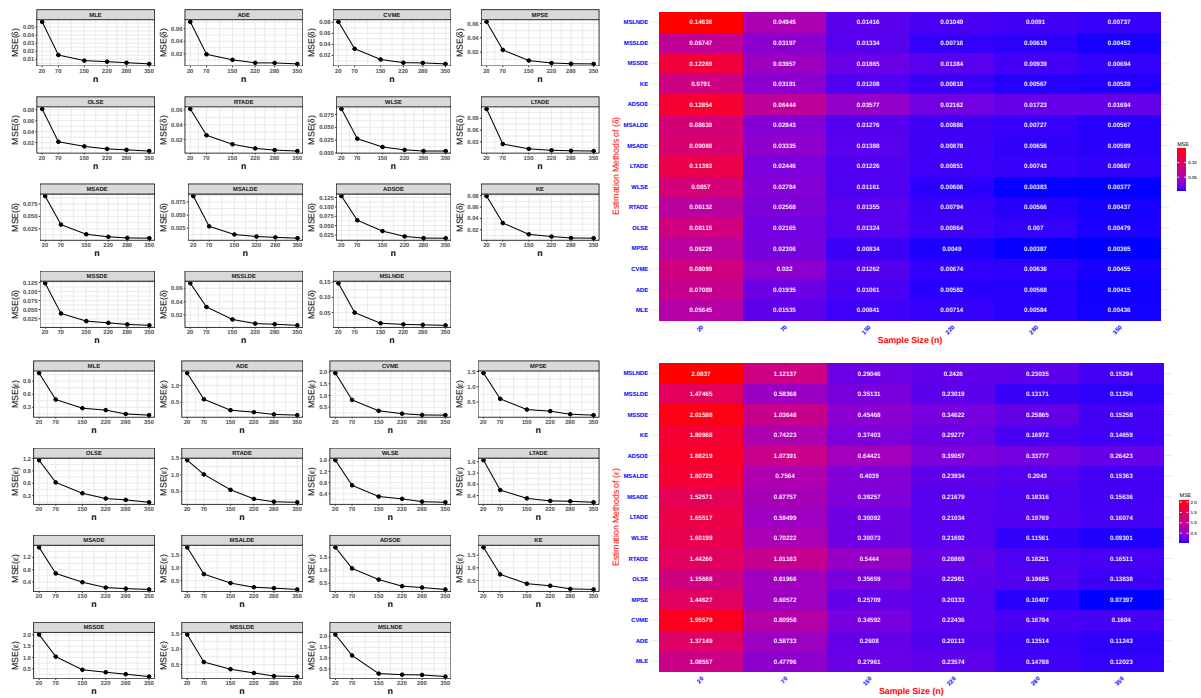


FIGURE 5. A graphical representation for MSE values presented in Table 2.

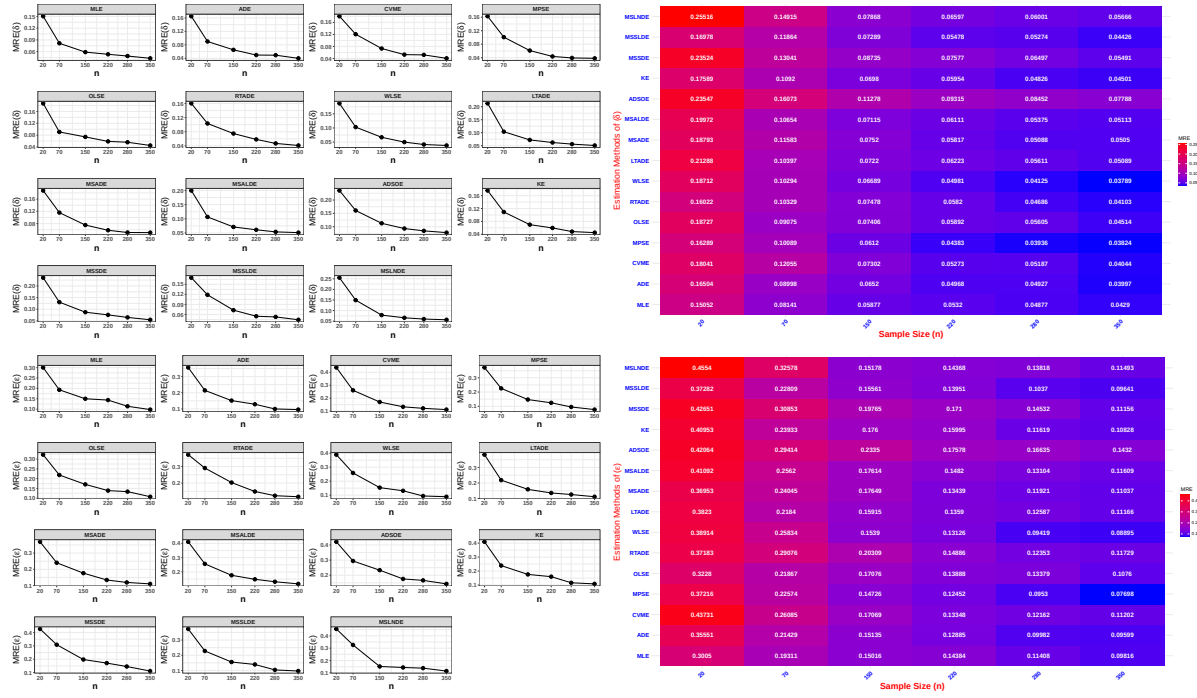


FIGURE 6. A graphical representation for MRE values presented in Table 2.

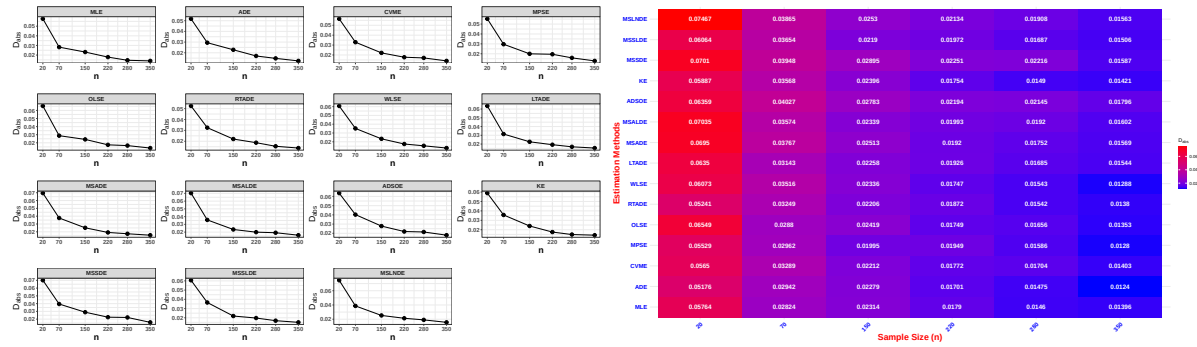


FIGURE 7. A graphical representation for D_{abs} values presented in Table 2.

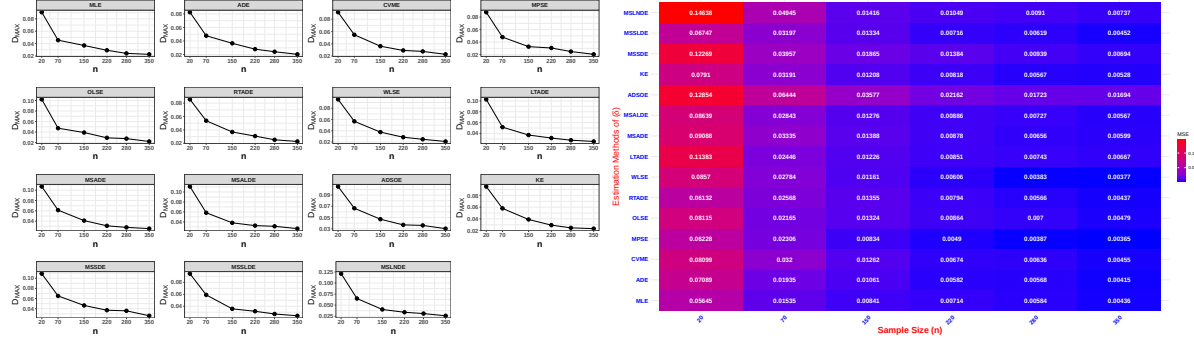
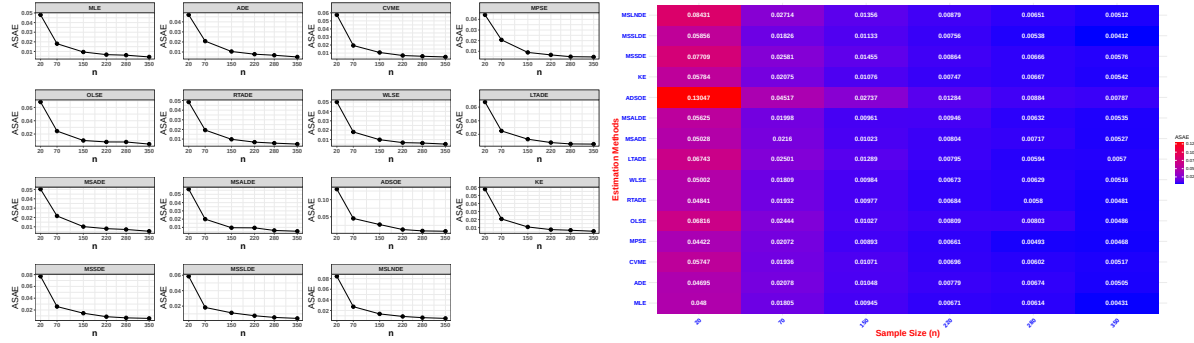
FIGURE 8. A graphical representation for D_{max} values presented in Table 2.

FIGURE 9. A graphical representation for ASAE values presented in Table 2.

The third dataset, recently analyzed by [3], contains the relief times (in minutes) of 20 patients receiving an analgesic. The values are: 3.0, 1.5, 1.7, 4.1, 1.4, 1.8, 1.4, 1.2, 2.0, 1.3, 1.9, 1.8, 1.7, 2.2, 1.7, 1.6, 2.3, 2.7, 1.1, 1.6.

6.2. Competing models. The performance of the proposed model is compared with several existing distributions. The cumulative distribution functions of the competing models are given below:

- Inverse Gompertz distribution (IGD) [4]:

$$F(x; \alpha, \beta) = e^{-\frac{\alpha}{\beta} \left(e^{\frac{\beta}{x}} - 1 \right)}.$$

- Cosine Gompertz distribution (CGD) [5]:

$$F(x) = 1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c} (e^{cx} - 1)} \right) \right\}.$$

- Gompertz distribution (GD) [6]:

$$F(x) = 1 - e^{-\frac{d}{c}(e^{cx}-1)}.$$

- Gamma distribution (GaMD):

$$F(x) = 1 - \frac{\Gamma\left(\alpha, \frac{x}{\beta}\right)}{\Gamma(\alpha)}.$$

- Weibull distribution (WD) [7]:

$$F(x) = 1 - e^{-\left(\frac{x}{\lambda}\right)^\kappa}.$$

- Rayleigh distribution (RD) [8]:

$$F(x) = 1 - \exp\left(-\frac{x^2}{2\zeta^2}\right).$$

- Inverse Rayleigh distribution (IRD) [9]:

$$F(x) = \exp\left(-\left(\frac{\sigma}{x}\right)^2\right).$$

6.3. Exploratory analysis. Table 8 presents the descriptive statistics of the three datasets. All datasets exhibit strong right skewness, with skewness values exceeding one, and leptokurtic behavior, as the kurtosis values are greater than three.

These features are further supported by Figure 10. The histograms and kernel density plots show clear right-skewed shapes for all datasets. The box plots reveal the presence of extreme values for the three datasets.

TABLE 8. Descriptive statistics for the three datasets.

Measure	Data I	Data II	Data III
n	47	24	20
Minimum	19.89	10.24	1.1
Mean	50.00	22.97	1.9
Median	40.40	19.24	1.7
Maximum	185.56	51.56	4.1
Kurtosis	9.52	4.3599	5.9241
Skewness	2.30	1.3454	1.7198

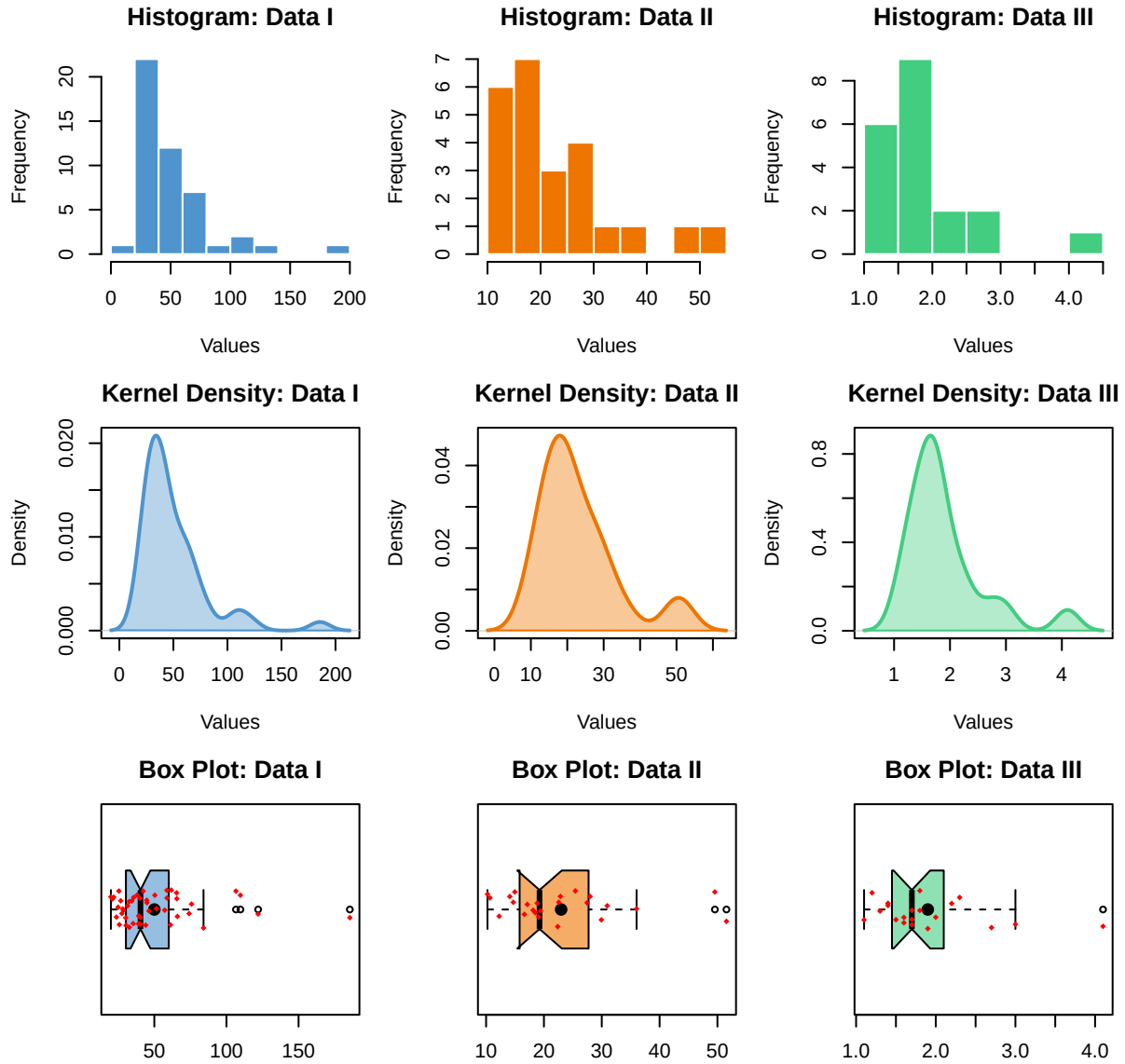


FIGURE 10. Nonparametric exploratory plots of the three datasets, including histograms, kernel density estimates, and box plots.

6.4. Model fitting and comparison. For model comparison, several standard criteria are considered. These include the negative log-likelihood (NLL), Akaike information criterion (AIC), corrected AIC (AICc), Hannan–Quinn information criterion (HQIC), Kolmogorov–Smirnov (KS), Anderson–Darling (AD), and Cramér–von Mises (CVM) statistics, along with their associated p-values. Root mean square error (RMSE) and coefficient of determination (COD) are also analyzed. In the case of NLL, AIC, AICc, HQIC, KS, AD, CVM, and RMSE, lower values

represent a better fit, while higher COD values indicate closer agreement between the fitted and empirical probabilities. In the case of the KS, AD, and CVM tests, the higher the p-value, the better the fitted model agrees with the observed data. These metrics provide a fair evaluation of the overall fit in the comparison of the competing models.

The maximum likelihood estimates, standard errors, negative log-likelihood values, and information criteria are provided in Table 9. In all three datasets, the ILSGD provides the lowest values of NLL compared to the competing models. The same trend is observed in the information criteria. In Dataset I, the ILSGD has the lowest scores in all three criteria. The same outcomes are observed in Datasets II and III. These findings indicate that the ILSGD outperforms the competing distributions in fitting the considered datasets.

The goodness-of-fit statistics are reported in Table 10. For all three datasets, the ILSGD records the smallest KS, AD, CVM, and RMSE values. At the same time, it yields the highest COD values. The corresponding p-values for the KS, AD, and CVM tests are also the largest. These results confirm that the ILSGD provides the closest agreement with the observed data.

6.5. Graphical assessment. The fitted probability density functions in Figures 11–13 show that the ILSGD follows the shape of the histograms more closely than the competing models.

The empirical and fitted cumulative distribution functions in Figures 14–16 further support this observation, as the ILSGD curve closely follows the empirical CDF across most of the data range, compared to the competing models. Finally, the P–P plots in Figures 17–19 indicate that the points corresponding to the ILSGD lie closest to the diagonal line, showing a stronger agreement between the theoretical and empirical probabilities compared to the other models.

6.6. Summary and implications. Overall, the numerical results and graphical evidence consistently show that the ILSGD outperforms the competing models across all three datasets. The model achieves the lowest information criteria, the best goodness-of-fit statistics, and the closest agreement in the graphical assessments. These findings highlight the ability of the ILSGD to effectively model right-skewed data. This makes it a reliable choice for applications in hydrology, reliability engineering, and medical studies, where accurate modeling of asymmetric data is essential.

TABLE 9. Parameter estimates, standard errors, negative log-likelihood values, and information criteria for the competing models fitted to the three right-skewed datasets.

Model	$\widehat{E}_1$	$SE(\widehat{E}_1)$	$\widehat{E}_2$	$SE(\widehat{E}_2)$	NLL	AIC	AICc	HQIC
Data I								
ILSGD($\hat{\delta}, \hat{\varepsilon}$)	2.5240	0.0427	7767.0017	5.9318	208.0740	420.1480	420.4200	421.5400
IGD($\hat{\alpha}, \hat{\beta}$)	8.7226	1.6045	75.1176	4.3450	210.3244	424.6489	424.9216	426.0413
CGD($\hat{c}, \hat{d}$)	0.0058	0.0032	0.0218	0.0035	218.4881	440.9762	441.2490	442.3687
GD($\hat{c}, \hat{d}$)	0.0107	0.0033	0.0129	0.0028	226.5270	457.0540	457.3267	458.4464
GaMD($\hat{\beta}, \hat{\alpha}$)	0.0773	0.0162	3.8654	0.7582	214.4539	432.9078	433.1805	434.3003
WD($\hat{\kappa}, \hat{\lambda}$)	1.7959	0.1767	56.7176	4.3988	219.0020	442.0040	442.2767	443.3965
RD($\hat{\zeta}$)	41.4688	3.4246	-	-	219.6309	441.2619	441.3508	441.9581
IRD($\hat{\sigma}$)	36.0875	2.4216	-	-	209.9130	421.8270	421.9160	422.5230
Data II								
ILSGD($\hat{\delta}, \hat{\varepsilon}$)	2.7348	0.0750	2383.1914	8.3889	86.0418	176.0836	176.6551	176.7087
IGD($\hat{\alpha}, \hat{\beta}$)	4.0694	2.1805	38.8591	9.8967	88.2103	180.4207	180.9921	181.0457
CGD($\hat{c}, \hat{d}$)	0.0373	0.0127	0.0323	0.0088	89.3765	182.7531	183.3245	183.3781
GD($\hat{c}, \hat{d}$)	0.0561	0.0143	0.0158	0.0061	92.5368	189.0737	189.6452	189.6988
GaMD($\hat{\beta}, \hat{\alpha}$)	0.2477	0.07189	5.6914	1.5781	86.9446	177.8892	178.4607	178.5143
WD($\hat{\kappa}, \hat{\lambda}$)	2.3096	0.3356	26.0285	2.3695	88.8910	181.7820	182.3534	182.4070
RD($\hat{\zeta}$)	17.8672	1.7885	-	-	89.3341	180.6683	180.8501	180.9808
IRD($\hat{\sigma}$)	18.0063	1.8758	-	-	87.7741	177.5481	177.7299	177.8607
Data III								
ILSGD($\hat{\delta}, \hat{\varepsilon}$)	3.8834	0.7385	4.4799	2.0185	15.3734	34.7469	35.4528	35.1356
IGD($\hat{\alpha}, \hat{\beta}$)	0.1104	0.0770	6.1450	1.1401	16.3915	36.7830	37.4889	37.1718
CGD($\hat{c}, \hat{d}$)	0.6682	0.1793	0.2967	0.0943	21.5351	47.0701	47.7760	47.4589
GD($\hat{c}, \hat{d}$)	0.8943	0.2012	0.1454	0.0644	24.5901	53.1801	53.8861	53.5690
GaMD($\hat{\beta}, \hat{\alpha}$)	5.0900	1.6750	9.6705	3.1094	17.8186	39.6371	40.3431	40.0260
WD($\hat{\kappa}, \hat{\lambda}$)	2.7870	0.4273	2.1299	0.1820	20.5864	45.1728	45.8786	45.5616
RD($\hat{\zeta}$)	1.4282	0.1596	-	-	22.4788	46.9576	47.1798	47.1520
IRD($\hat{\sigma}$)	1.6617	0.1858	-	-	21.1825	44.3650	44.5872	44.5594

TABLE 10. Goodness-of-fit statistics, including KS, AD, CVM, RMSE, and the coefficient of determination, for the competing models fitted to the three datasets

Model	KS	KS-pv	AD	AD-pv	CVM	CVM-pv	RMSE	COD
Data I								
ILSGD	0.0715	0.9700	0.1360	0.9990	0.0209	0.9960	0.0227	0.9941
IGD	0.0835	0.8982	0.4686	0.7781	0.0606	0.8124	0.0330	0.9855
CGD	0.1626	0.1667	1.8836	0.1068	0.2805	0.1535	0.0723	0.9023
GD	0.2480	0.0062	3.5811	0.0141	0.6027	0.0216	0.1079	0.7017
GaMD	0.1312	0.3935	1.0519	0.3307	0.1661	0.3450	0.0577	0.9562
WD	0.1412	0.3058	1.7577	0.1256	0.2695	0.1651	0.0717	0.9146
RD	0.1745	0.1143	2.0665	0.0847	0.3651	0.0892	0.0856	0.8917
IRD	0.1095	0.6260	0.6718	0.5820	0.1041	0.5670	0.0436	0.9726
Data II								
ILSGD	0.0892	0.9820	0.2215	0.9834	0.0299	0.9786	0.0325	0.9868
IGD	0.1252	0.8021	0.6199	0.6276	0.0874	0.6546	0.0528	0.9575
CGD	0.1372	0.7068	0.8530	0.4430	0.1237	0.4835	0.0637	0.9334
GD	0.1969	0.2722	1.4560	0.1874	0.2263	0.2223	0.0878	0.8448
GaMD	0.1404	0.6799	0.4031	0.8443	0.0568	0.8383	0.0464	0.9730
WD	0.1438	0.6515	0.7474	0.5190	0.1085	0.5485	0.0609	0.9448
RD	0.1515	0.5881	0.8582	0.4396	0.1219	0.4907	0.0619	0.9329
IRD	0.1623	0.5013	0.7908	0.4863	0.1296	0.4611	0.0649	0.9264
Data III								
ILSGD	0.1074	0.9751	0.1657	0.9972	0.0287	0.9825	0.0340	0.9853
IGD	0.1412	0.8200	0.3230	0.9191	0.0527	0.8652	0.0443	0.9727
CGD	0.1868	0.4881	1.2786	0.2391	0.2173	0.2370	0.0937	0.8253
GD	0.2383	0.2063	1.8059	0.1183	0.3147	0.1225	0.1138	0.6806
GaMD	0.1734	0.5848	0.5988	0.6470	0.1024	0.5775	0.0665	0.9393
WD	0.1850	0.5007	1.0835	0.3155	0.1834	0.3036	0.0865	0.8675
RD	0.2566	0.1437	1.7806	0.1222	0.3222	0.1166	0.1150	0.6372
IRD	0.2567	0.1434	2.0662	0.0851	0.3978	0.0723	0.1306	0.5347

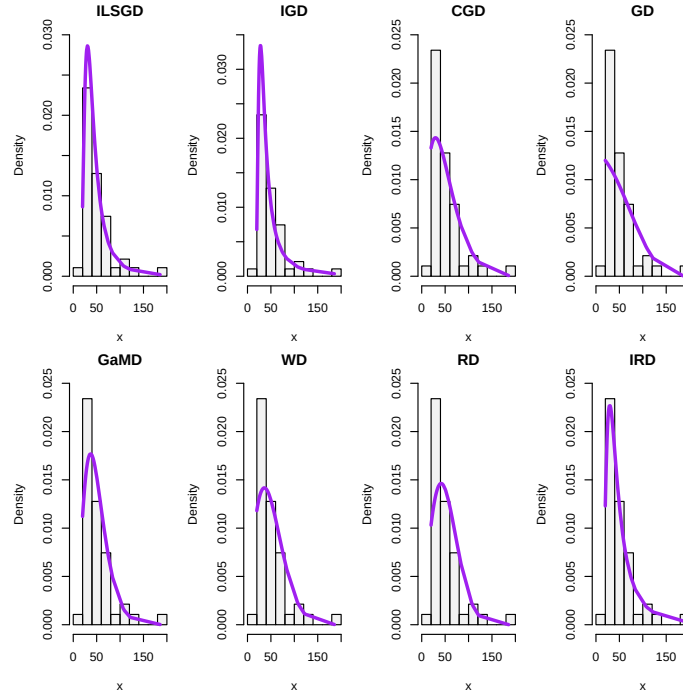


FIGURE 11. Histograms of Dataset I with fitted probability density functions for all compared models.

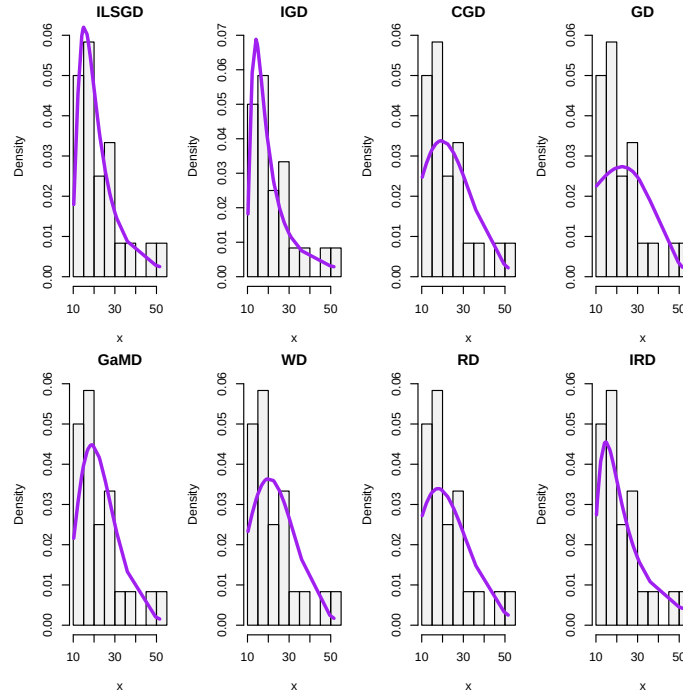


FIGURE 12. Histograms of Dataset II with fitted probability density functions for all compared models.

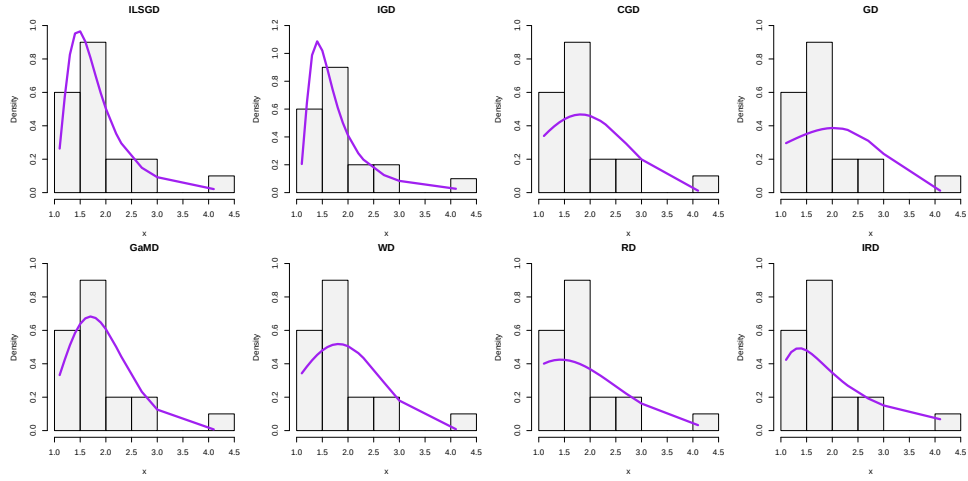


FIGURE 13. Histograms of Dataset III with fitted probability density functions for all compared models.

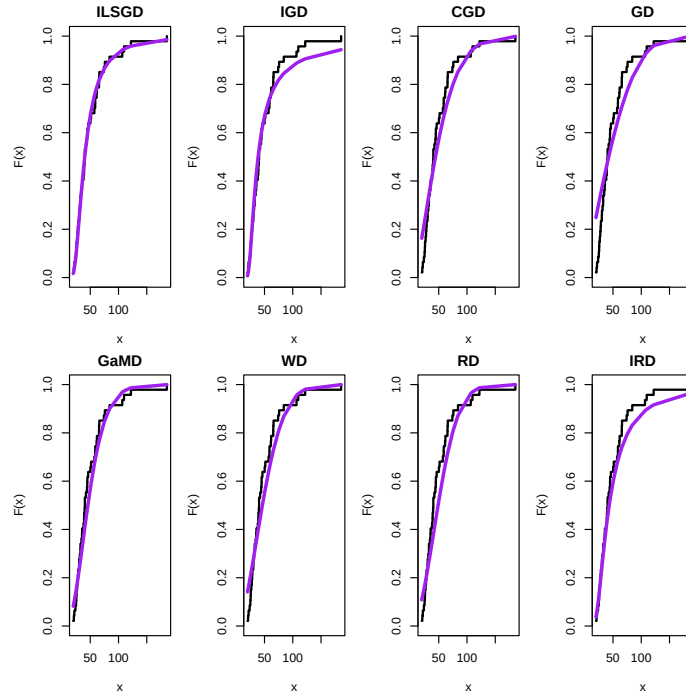


FIGURE 14. Empirical cumulative distribution functions of Dataset I with fitted CDFs for each of the eight compared models.

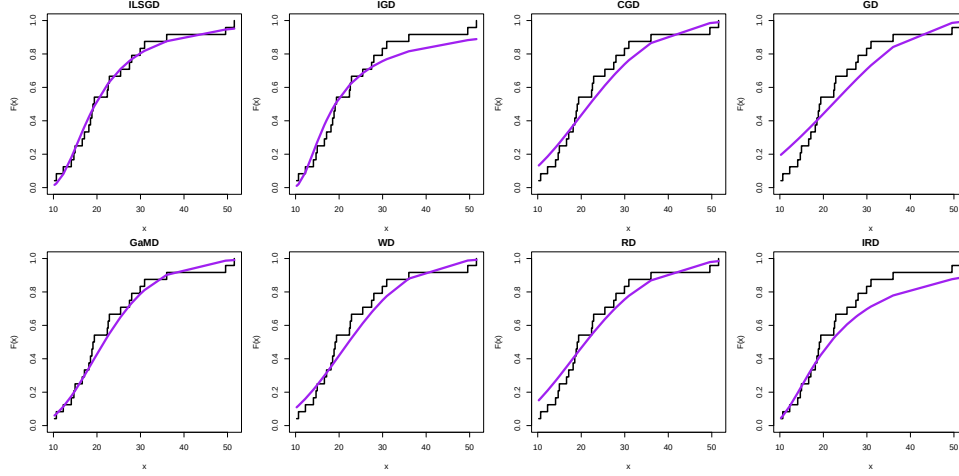


FIGURE 15. Empirical cumulative distribution functions of Dataset II with fitted CDFs for each of the eight compared models.

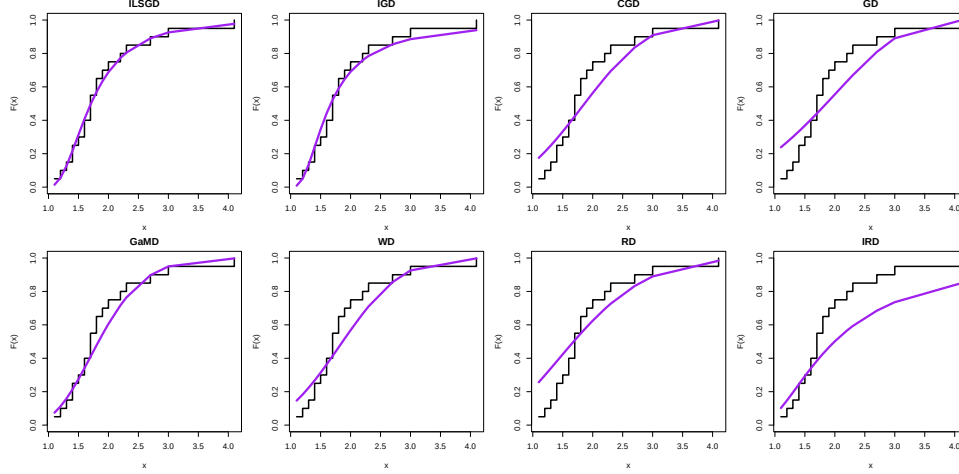


FIGURE 16. Empirical cumulative distribution functions of Dataset III with fitted CDFs for each of the eight compared models.

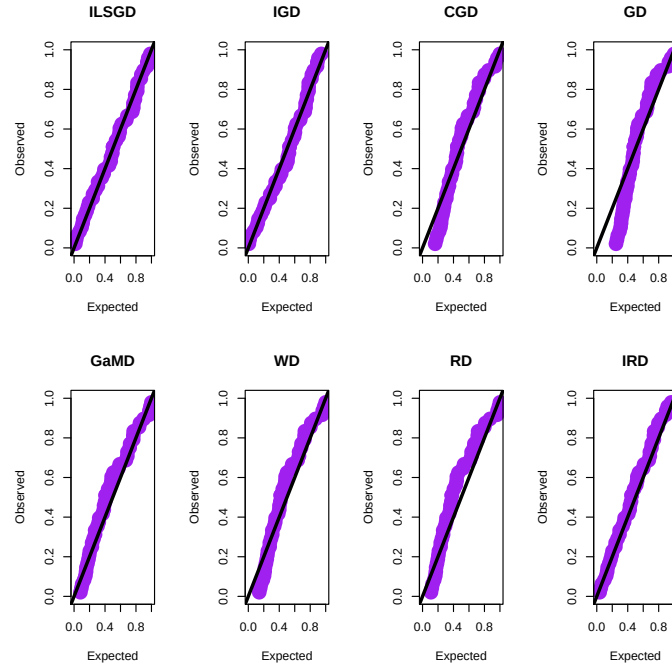


FIGURE 17. Probability-probability (P-P) plots of the fitted models for Dataset I.

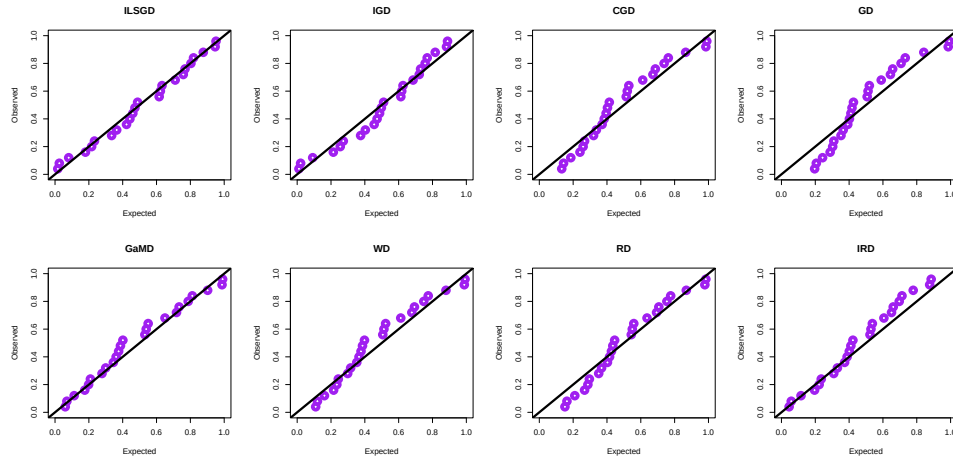


FIGURE 18. Probability-probability (P-P) plots of fitted models for Dataset II.

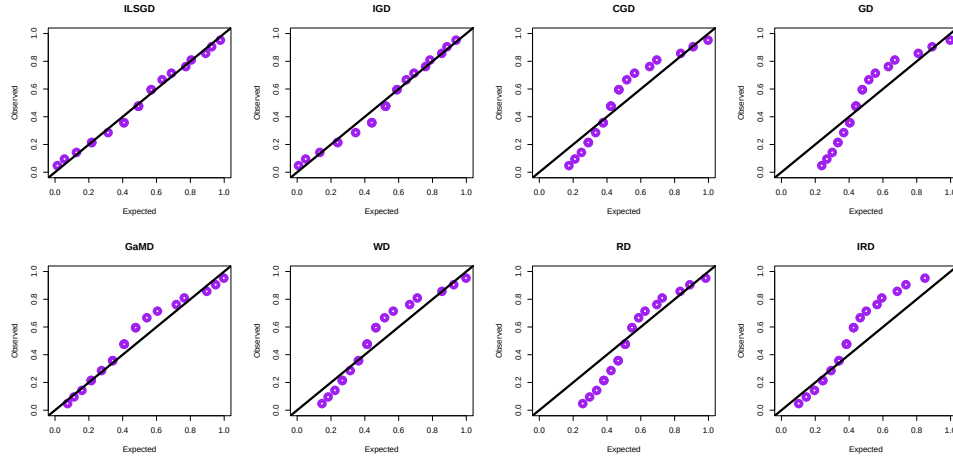


FIGURE 19. Probability-probability (P-P) plots of fitted models for Dataset III.

7. CONCLUSION

This paper proposed the inverse log-shifted Gompertz distribution for modeling data from hydrology, reliability engineering, and medical studies. The model is obtained through the inverse scheme, which modifies the tail behavior of the baseline distribution and increases its ability to represent different types of data. Statistical properties of the model are derived. Parameter estimation is studied using several classical methods. A simulation study is carried out to examine the behavior of the estimators. The results show that the estimators yielded reliable estimates. The model is applied to real datasets and compared with existing distributions. The proposed distribution provides a better fit than the competing models based on common goodness-of-fit measures. These findings indicate that the inverse log-shifted Gompertz distribution is suitable for data with complex behavior. For future studies, the model can be applied to datasets from other fields. Further research may also explore regression modeling frameworks and applications to censored lifetime data.

AUTHORS CONTRIBUTIONS

All authors have worked equally to write and review the manuscript.

DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available within the article.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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