Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2017, 2017:12
ISSN: 2052-2541

THE STABILITY ANALYSIS OF AN EPIDEMIC MODEL WITH AGE-STRUCTURE IN THE EXPOSED AND INFECTIOUS CLASSES

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Communicated by A. Elaiw

Abstract. In this paper, we propose an epidemic model with age-structure in the exposed and infectious classes for a disease like hepatitis-B. Asymptotic smoothness of semi-flow generated by the model is studied. By calculating the basic reproduction number and analyzing the characteristic equation, we study the local stability of disease-free and endemic steady states. By using Lyapunov functionals and LaSalle's invariance principle, it is proved that if the basic reproduction number is less than unity, the disease-free steady state is globally asymptotically stable; if the basic reproduction number is greater than unity, the endemic steady state is globally asymptotically stable.

Keywords: age-structured model; asymptotic smoothness; Lyapunov functional; global stability.

2010 AMS Subject Classification: 34K20, 34K25, 92D30.

1. Introduction

Hepatitis B is a worldwide disease and it has become a serious threat to human health. To study the transmission of Hepatitis B, several epidemic models for the infection of Hepatitis B

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Received April 17, 2017

have been studied extensively. Hepatitis B virus carriers may become acute hepatitis or chronic hepatitis B patients after incubation, and two types hepatitis B patients' scaled probability of infection are different. In [1], Liu considered an HBV infection epidemic model as follows:

$$\begin{aligned}
&\dot{S}(t) = \Lambda - (\mu + p)S(t) - \beta_1 S(t)I_1(t) - \beta_3 S(t)I_2(t),\\
&\dot{V}(t) = pS(t) - (\mu + \rho)V(t) - \beta_2 V(t)I_1(t) - \beta_4 V(t)I_2(t),\\
&\dot{E}(t) = S(t)\left(\beta_1 I_1(t) + \beta_3 I_2(t)\right) + V(t)\left(\beta_2 I_1(t) + \beta_4 I_2(t)\right) - (\mu + \gamma_1 + \gamma_2)E(t),\\
&\dot{I}_1(t) = \gamma_1 E(t) - (\mu + \delta_1 + \varepsilon_1)I_1(t),\\
&\dot{I}_2(t) = \gamma_2 E(t) - (\mu + \delta_2 + \varepsilon_2)I_2(t),\\
&\dot{R}(t) = \rho V(t) + \delta_1 I_1(t) + \delta_2 I_2(t) - \mu R(t),
\end{aligned} \tag{1.1}$$

where the variables $S(t)$, $V(t)$, $E(t)$, $I_1(t)$, $I_2(t)$ and $R(t)$ represent the numbers of susceptible individuals, vaccinees, hepatitis B virus carriers, patients with acute hepatitis B, patients with chronic hepatitis B and recovered individuals at time t, respectively. Liu assume that different individuals in the same class have the same behavior and waiting time. Then system (1.1) can be regarded as ODEs(see, for example, [2-7]). In [1], Liu has proved that if the basic reproduction number is less than unity, the disease-free steady state is globally asymptotically stable; if the basic reproduction number is greater than unity, the endemic steady state is globally asymptotically stable.

However, there are differences in individuals' physical condition and social environment, Hepatitis B virus carriers' incubation stage and Hepatitis B patients' convalescence differ from man to man. Several medical studies show that the scaled probability of Hepatitis B virus infection is in connection with age of infection and the risk per unit time of activation appears to be higher in the early stages of infection than in later stages. In [8], McCluskey has shown that the risk of activation can be modeled as a function of duration age, and this form can be employed to describe more generality in the distribution of waiting time by introducing the duration age in latent class as a variable. Therefore, it's necessary to incorporate the duration age into modeling.

Motivated by the above works, in this paper, we propose an $\mathrm{SVEI_1I_2R}$ epidemic model with continuous age-dependent latency, acute hepatitis B infection and chronic hepatitis B infection as follows:

$$\begin{aligned}
\dot{S}(t) &= \Lambda-(\mu+p)S(t)-S(t)\int_0^\infty \beta_1(a)i(t,a)da-S(t)\int_0^\infty \beta_3(a)j(t,a)da,\\
\dot{V}(t) &= pS(t)-(\mu+\rho)V(t)-V(t)\int_0^\infty \beta_2(a)i(t,a)da-V(t)\int_0^\infty \beta_4(a)j(t,a)da,\\
&\frac{\partial e(t,a)}{\partial a}+\frac{\partial e(t,a)}{\partial t} = -(\mu+\gamma_1(a)+\gamma_2(a))e(t,a),\\
&\frac{\partial i(t,a)}{\partial a}+\frac{\partial i(t,a)}{\partial t} = -(\mu+\delta_1(a)+\varepsilon_1(a)+\xi(a))i(t,a),\\
&\frac{\partial j(t,a)}{\partial a}+\frac{\partial j(t,a)}{\partial t} = -(\mu+\delta_2(a)+\varepsilon_2(a))j(t,a),\\
\dot{R}(t) &= \rho V(t)+\int_0^\infty \delta_1(a)i(t,a)da+\int_0^\infty \delta_2(a)j(t,a)da-\mu R(t),
\end{aligned} \tag{1.2}$$

with boundary conditions

$$\begin{aligned}
e(t,0) &= S(t)\int_0^\infty \beta_1(a)i(t,a)da+S(t)\int_0^\infty \beta_3(a)j(t,a)da\\
&\quad +V(t)\int_0^\infty \beta_2(a)i(t,a)da+V(t)\int_0^\infty \beta_4(a)j(t,a)da,\\
i(t,0) &= \int_0^\infty \gamma_1(a)e(t,a)da,\\
j(t,0) &= \int_0^\infty \gamma_2(a)e(t,a)da+\int_0^\infty \xi(a)i(t,a)da,
\end{aligned} \tag{1.3}$$

and initial conditions

$$\begin{aligned}
&S(0)=S_0\geq 0, V(0)=V_0\geq 0, R(0)=\varphi_R\geq 0,\\
&e(0,a)=\varphi_e(a)\in L^1_+(0,\infty), i(0,a)=\varphi_i(a)\in L^1_+(0,\infty),\\
&j(0,a)=\varphi_j(a)\in L^1_+(0,\infty),
\end{aligned} \tag{1.4}$$

where, $e(t,a)$ represents the density of exposed individuals with age of latency a at time t. $i(t,a)$, $j(t,a)$ represent the density of patients with acute hepatitis B and chronic hepatitis B with age of infection a at time t, respectively. The parameters of model (1.2) are biologically explained as in Table 1.

TABLE 1. Parameters and their biological meaning in model (1.2)

Parameter	Interpretation
Λ	constant recruitment rate
μ	natural death rate
p	the rate for susceptible individuals to be vaccinated
ρ	the rate for vaccinees to obtain immunity and move into recovered population
$\beta_1(a)$	the rate for acute hepatitis B patients infecting susceptible individuals at age a
$\beta_2(a)$	the rate for acute hepatitis B patients infecting vaccinees at age a
$\beta_3(a)$	the rate for chronic hepatitis B patients infecting susceptible individuals at age a
$\beta_4(a)$	the rate for chronic hepatitis B patients infecting vaccinees at age a
$\gamma_1(a)$	the rate for exposed individuals being acute hepatitis B patients at age a
$\gamma_2(a)$	the rate for exposed individuals being chronic hepatitis B patients at age a
$\varepsilon_1(a)$	acute hepatitis B death rate at age a
$\varepsilon_2(a)$	chronic hepatitis B death rate at age a
$\delta_1(a)$	the rate for acute hepatitis B patients being recovered population at age a
$\delta_2(a)$	the rate for chronic hepatitis B patients being recovered population at age a

Note: All these constants are assumed to be positive.

In order to simplify model (1.2), denote

$$\theta_1(a) = \mu + \gamma_1(a) + \gamma_2(a), \quad \theta_2(a) = \mu + \delta_1(a) + \varepsilon_1(a) + \xi(a), \quad \theta_3(a) = \mu + \delta_2(a) + \varepsilon_2(a).$$

Since the variable $R(t)$ does not appear in the first five equations of (1.2), in this paper, we consider the following reduced system

$$\begin{aligned}
\dot{S}(t) &= \Lambda - (\mu + p)S(t) - S(t)\int_0^\infty \beta_1(a)i(t,a)da - S(t)\int_0^\infty \beta_3(a)j(t,a)da, \\
\dot{V}(t) &= pS(t) - (\mu + \rho)V(t) - V(t)\int_0^\infty \beta_2(a)i(t,a)da - V(t)\int_0^\infty \beta_4(a)j(t,a)da, \\
&\frac{\partial e(t,a)}{\partial a} + \frac{\partial e(t,a)}{\partial t} = -\theta_1(a)e(t,a), \\
&\frac{\partial i(t,a)}{\partial a} + \frac{\partial i(t,a)}{\partial t} = -\theta_2(a)i(t,a), \\
&\frac{\partial j(t,a)}{\partial a} + \frac{\partial j(t,a)}{\partial t} = -\theta_3(a)j(t,a).
\end{aligned} \tag{1.5}$$

This paper is organized as follows. In section 2, we introduce some basic results of system (1.5), including state space, assumptions and boundedness of the solutions. Asymptotic smoothness of the semi-flow is analyzed in section 3, which is generated by the system (1.5). Then we study the existence of equilibria and obtain the expression of the basic reproduction number R_0 in section 4. And the local stability of equilibria is proved in section 5. Finally, in section 6, we give proof of the global stability of equilibria.

More details concerning the global stability analysis of epidemic model approach, we refer readers to [9-19].

2. Preliminaries

To make the model be biologically significant, we list the assumption as follows:

Assumption 2. We assume that
(i) $\beta_1(a)$, $\beta_2(a)$, $\beta_3(a)$, $\beta_4(a)$, $\theta_1(a)$, $\theta_2(a)$, $\theta_3(a)$, $\xi(a)$ are non-negative and belong to $L^\infty_+(0,\infty)$ with respective essential upper bound $\bar{\beta}_1$, $\bar{\beta}_2$, $\bar{\beta}_3$, $\bar{\beta}_4$, $\bar{\theta}_1$, $\bar{\theta}_2$, $\bar{\theta}_3$, $\bar{\xi} \in (0,\infty)$;
(ii) $\beta_1(a)$, $\beta_2(a)$, $\beta_3(a)$, $\beta_4(a)$, $\xi(a)$ are Lipschitz continuous on $\mathbb{R}_+$ with coefficients M_{β_1}, M_{β_2}, M_{β_3}, M_{β_4}, M_ξ, respectively;
(iii) There exits a positive constant $\mu_0 \in (0,\mu]$ such that $\theta_1(a) \geq \mu_0$, $\theta_2(a) \geq \mu_0$, $\theta_3(a) \geq \mu_0$ for all $a > 0$.

2.1. State space

Define the space of functions $\mathscr{X}$ as

$$\mathscr{X} = \mathbb{R}_+ \times \mathbb{R}_+ \times L^1_+(0,\infty) \times L^1_+(0,\infty) \times L^1_+(0,\infty),$$

equipped with the norm

$$\|(x_1,x_2,x_3,x_4,x_5)\|_{\mathscr{X}} = |x_1| + |x_2| + \int_0^\infty |x_3(a)|da + \int_0^\infty |x_4(a)|da + \int_0^\infty |x_5(a)|da.$$

Then, the initial values (1.4) of system (1.5) are taken to be included in $\mathscr{X}$:

$$(S(0),V(0),e(0,a),i(0,a),j(0,a)) = \big(S_0,V_0,\varphi_e(a),\varphi_i(a),\varphi_j(a)\big) \in \mathscr{X}.$$

By the standard theory of functional differential equation [20], it can be verified that system (1.5) with initial conditions (1.4) has a unique nonnegative solution. Thus, a continuous semi-flow associated with system (1.5), that is

$$\Phi_t(X_0) := X(t) = (S(t), V(t), e(t,\cdot), i(t,\cdot), j(t,\cdot)) \in \mathscr{X}, t \geq 0, \tag{2.1}$$

with

$$\begin{aligned}\|\Phi_t(X_0)\|_{\mathscr{X}} &= \|S(t), V(t), e(t,\cdot), i(t,\cdot), j(t,\cdot)\|_{\mathscr{X}} \\ &= |S(t)| + |V(t)| + \int_0^\infty |e(t,a)|da + \int_0^\infty |i(t,a)|da + \int_0^\infty |j(t,a)|da.\end{aligned}$$

Finally, we define the state space for system (1.5) as

$$\Upsilon := \{(S(t), V(t), e(t,\cdot), i(t,\cdot), j(t,\cdot)) \in \mathscr{X} : 0 \leq S(t) + V(t) + \int_0^\infty e(t,a)da + \int_0^\infty i(t,a)da + \int_0^\infty j(t,a)da \leq \frac{\Lambda}{\mu_0}\},$$

which can be proved to be positive invariant by the following proposition.

2.2. Boundedness

The last three equations of system (1.5) can be reformulated as Volterra equations by use of Volterra formulation. In order to be convenient for computation, we denote

$$B_1(a) = \exp\left(-\int_0^a \theta_1(\tau)d\tau\right), \quad B_2(a) = \exp\left(-\int_0^a \theta_2(\tau)d\tau\right), \quad B_3(a) = \exp\left(-\int_0^a \theta_3(\tau)d\tau\right).$$

From the expressions of $B_1(a)$, $B_2(a)$ and $B_3(a)$, according to assumption 2, it is easy to see that for all $a \geq 0$,

$$0 \leq B_1(a), \quad B_2(a), \quad B_3(a) \leq e^{-\mu_0 a},$$

$$B_1'(a) = -\theta_1(a)B_1(a), \quad B_2'(a) = -\theta_2(a)B_2(a), \quad B_3'(a) = -\theta_3(a)B_3(a).$$

By integrating the terms $e(t,a)$, $i(t,a)$ and $j(t,a)$ along the characteristic line $t - a =$ const, respectively, we get the following expressions:

$$e(t,a) = \begin{cases} e(t-a,0)B_1(a) & for \quad 0 \leq a \leq t, \\ \varphi_e(a-t)\dfrac{B_1(a)}{B_1(a-t)} & for \quad 0 \leq t \leq a, \end{cases} \tag{2.2}$$

$$i(t,a)=\begin{cases} i(t-a,0)B_2(a) & for \quad 0\le a\le t,\\ \varphi_i(a-t)\dfrac{B_2(a)}{B_2(a-t)} & for \quad 0\le t\le a,\end{cases} \tag{2.3}$$

$$j(t,a)=\begin{cases} j(t-a,0)B_3(a) & for \quad 0\le a\le t,\\ \varphi_j(a-t)\dfrac{B_3(a)}{B_3(a-t)} & for \quad 0\le t\le a.\end{cases} \tag{2.4}$$

In order to imply the boundedness of system (1.5), we have the following proposition.

Proposition 2.1. Consider system (1.5) and equation (2.1), we have

(i) Υ is positively invariant for Φ_t, that is, $\Phi_t(X_0)\in\Upsilon$, for $\forall t\ge 0, X_0\in\Upsilon$;

(ii) Φ_t is point dissipative: there is a bounded set that attracts all points in $\mathscr{X}$.

Proof. Note that

$$\frac{d}{dt}\|\Phi_t(X_0)\|_{\mathscr{X}}=\frac{dS(t)}{dt}+\frac{dV(t)}{dt}+\frac{d}{dt}\int_0^\infty e(t,a)da+\frac{d}{dt}\int_0^\infty i(t,a)da+\frac{d}{dt}\int_0^\infty j(t,a)da. \tag{2.5}$$

By equation (2.2), we get

$$\int_0^\infty e(t,a)da=\int_0^t e(t-a,0)B_1(a)da+\int_t^\infty \varphi_e(a-t)\frac{B_1(a)}{B_1(a-t)}da.$$

Taking the substitution $\sigma=t-a$ and $\tau=a-t$ in the first and second integral, respectively, and differentiating by t, we get

$$\begin{aligned}\frac{d}{dt}\int_0^\infty e(t,a)da&=\frac{d}{dt}\int_0^t e(\sigma,0)B_1(t-\sigma)d\sigma+\frac{d}{dt}\int_0^\infty \varphi_e(\tau)\frac{B_1(t+\tau)}{B_1(\tau)}d\tau\\ &=e(t,0)B_1(0)+\int_0^\infty \varphi_e(\tau)\frac{B_1'(t+\tau)}{B_1(\tau)}d\tau+\int_0^t e(\sigma-a)B_1'(t-\sigma)d\sigma.\end{aligned}$$

Noting that $B_1(0)=1$ and $B_1'(a)=-\theta_1(a)B_1(a)$, we obtain

$$\frac{d}{dt}\int_0^\infty e(t,a)da=e(t,0)-\int_0^\infty \theta_1(a)e(t,a)da. \tag{2.6}$$

Similarly, we have

$$\frac{d}{dt}\int_0^\infty i(t,a)da=i(t,0)-\int_0^\infty \theta_2(a)i(t,a)da. \tag{2.7}$$

$$\frac{d}{dt}\int_0^\infty j(t,a)da=j(t,0)-\int_0^\infty \theta_3(a)j(t,a)da. \tag{2.8}$$

By (2.6), (2.7) and (2.8), equation (2.5) becomes

$$\frac{d}{dt}\|\Phi_t(X_0)\|_{\mathscr{X}}=\Lambda-\mu S(t)-(\mu+\rho)V(t)-S(t)\left(\int_0^\infty \beta_1(a)i(t,a)da+\int_0^\infty \beta_3(a)j(t,a)da\right)$$

$$
\begin{aligned}
&-V(t)\left(\int_0^\infty \beta_2(a)i(t,a)da+\int_0^\infty \beta_4(a)j(t,a)da\right)\\
&+S(t)\left(\int_0^\infty \beta_1(a)i(t,a)da+\int_0^\infty \beta_3(a)j(t,a)da\right)\\
&+V(t)\left(\int_0^\infty \beta_2(a)i(t,a)da+\int_0^\infty \beta_4(a)j(t,a)da\right)\\
&-\int_0^\infty \theta_1(a)e(t,a)da+\int_0^\infty \gamma_1(a)e(t,a)da-\int_0^\infty \theta_2(a)i(t,a)da\\
&+\int_0^\infty \gamma_2(a)e(t,a)da+\int_0^\infty \xi(a)i(t,a)da-\int_0^\infty \theta_3(a)j(t,a)da\\
=&\Lambda-\mu S(t)-(\mu+\rho)V(t)-\int_0^\infty (\theta_1(a)-\gamma_1(a)-\gamma_2(a))\,e(t,a)da\\
&-\int_0^\infty (\theta_2(a)-\xi(a))\,i(t,a)da-\int_0^\infty \theta_3(a)j(t,a)da.
\end{aligned}
$$

Thus, from (iii) of assumption 2, we can get

$$
\begin{aligned}
\frac{d}{dt}\|\Phi_t(X_0)\|_{\mathscr{X}} &\le \Lambda-\mu S(t)-(\mu+\rho)V(t)-\mu_0\left(\int_0^\infty e(t,a)da+\int_0^\infty i(t,a)da\int_0^\infty j(t,a)da\right)\\
&=\Lambda-\mu_0\|\Phi_t(X_0)\|_{\mathscr{X}}.
\end{aligned}
$$

Hence, it follow from the variation of constants formula that for $t\ge 0$,

$$
\|\Phi_t(X_0)\|_{\mathscr{X}} \le \frac{\Lambda}{\mu_0}-e^{-\mu_0 t}\left(\frac{\Lambda}{\mu_0}-\|\Phi_t(X_0)\|_{\mathscr{X}}\right), \tag{2.9}
$$

which implies that $\Phi_t(X_0)\in\Upsilon$ for any solution of (1.5) satisfying $X_0\in\Upsilon$ and all $t\ge 0$. Thus, the positive invariance set of Υ for semi-flow Φ can be verified.

Moreover, by (2.9) we can make inferences that $\limsup_{t\to\infty}\|\Phi_t(X_0)\|_{\mathscr{X}}\le\Lambda/\mu_0$ for any $X_0\in\mathscr{X}$. Therefore, Φ is point dissipative and Υ attracts all points in $\mathscr{X}$. This completes the proof. □

Proposition 2.2. If $X_0\in\mathscr{X}$ and $\|X_0\|_{\mathscr{X}}\le M$ for some constant $M\ge\Lambda/\mu$, then the following statements hold for $t\ge 0$,

(i) $0\le S(t)$, $V(t)$, $\int_0^\infty e(t,a)da$, $\int_0^\infty i(t,a)da$, $\int_0^\infty j(t,a)da\le M$;
(ii) $e(t,0)\le(\bar\beta_1+\bar\beta_2+\bar\beta_3+\bar\beta_4)M^2$, $i(t,0)\le\bar\gamma_1 M$, $j(t,0)\le(\bar\gamma_2+\bar\xi)M$.

Proposition 2.3. Let $C\in\mathscr{X}$ be bounded, then

(i) $\Phi_t(C)$ is bounded;

(ii) Φ_t is eventually bounded on C.

3. Asymptotic smoothness

In order to obtain global properties of the semi-flow $\Phi(t)_{t\geq 0}$, it is necessary to prove that the semi-flow is asymptotically smooth. Before giving the results, we first introduce some lemmas for later use.

Lemma 3.1 ([8]). *Let $D \subseteq \mathbb{R}$. For j=1,2, suppose $f_j : D \to \mathbb{R}$ is a bounded Lipschitz continuous function with bound K_j and Lipschitz coefficient M_j. Then the product function $f_1 f_2$ is Lipschitz with coefficient $K_1M_2 + K_2M_1$.*

The definition of asymptotic smoothness is as follows:

Definition 3.1 ([21]). A semi-flow $\Phi(t,X_0) := \mathbb{R}^+ \times \mathscr{X} \to \mathscr{X}$ is said to be asymptotically smooth, if, for any nonempty, closed bounded set $B \subset \mathscr{X}$ for which $\Phi(t,B) \subset B$, there is a compact set $B_0 \subset B$ such that B_0 attracts B.

In order to prove the asymptotic smoothness of the semi-flow, we will apply the following results, which is based on Lemma 3.2.3 in [21].

Lemma 3.2 ([21,22]). *If the following two conditions hold then the semi-flow $\Phi(t,X_0) = \phi(t,X_0) + \varphi(t,X_0) : \mathbb{R}^+ \times \mathscr{X} \to \mathscr{X}$ is asymptotically smooth in $\mathscr{X}$.*

(i) There exists a continuous function $w : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $w(t,h) \to 0$ as $t \to \infty$ and $\|\varphi(t,X_0)\|_{\mathscr{X}} \leq w(t,h)$ if $\|X_0\|_{\mathscr{X}} \leq h$;

(ii) For $t \geq 0$, $\phi(t,X_0)$ is completely continuous.

To verify that the two conditions are fulfilled for system (1.5), we decompose $\Phi : \mathbb{R}^+ \times \mathscr{X} \to \mathscr{X}$ into the following two operators $\phi(t,X_0)$, $\varphi(t,X_0) : \mathbb{R}^+ \times \mathscr{X} \to \mathscr{X}$. Let $\phi(t,X_0) := (S(t),V(t),\tilde{e}(t,\cdot),\tilde{i}(t,\cdot),\tilde{j}(t,\cdot))$ and $\varphi(t,X_0) := (0,0,\tilde{\varphi}_e(t,\cdot),\tilde{\varphi}_i(t,\cdot),\tilde{\varphi}_j(t,\cdot))$ where

$$\tilde{\varphi}_e(t,a) := \begin{cases} 0 & for \quad 0 \leq a \leq t, \\ e(t,a) & for \quad 0 \leq t \leq a. \end{cases} \quad and \quad \tilde{e}(t,a) := \begin{cases} e(t,a) & for \quad 0 \leq a \leq t, \\ 0 & for \quad 0 \leq t \leq a. \end{cases} \tag{3.1}$$

$$\tilde{\varphi}_i(t,a) := \begin{cases} 0 & for \quad 0 \le a \le t, \\ i(t,a) & for \quad 0 \le t \le a. \end{cases} \quad and \quad \tilde{i}(t,a) := \begin{cases} i(t,a) & for \quad 0 \le a \le t, \\ 0 & for \quad 0 \le t \le a. \end{cases} \tag{3.2}$$

$$\tilde{\varphi}_j(t,a) := \begin{cases} 0 & for \quad 0 \le a \le t, \\ j(t,a) & for \quad 0 \le t \le a. \end{cases} \quad and \quad \tilde{j}(t,a) := \begin{cases} j(t,a) & for \quad 0 \le a \le t, \\ 0 & for \quad 0 \le t \le a. \end{cases} \tag{3.3}$$

Then we have $\Phi(t,X_0) = \varphi(t,X_0) + \phi(t,X_0)$ for all $t \ge 0$. In order to verify condition (i) of lemma 3.2 holds true, we turn to prove the following proposition.

Proposition 3.1 ([22]). For $h > 0$, let $w(t,h) = he^{-\mu_0 t}$. Then $\lim_{t\to\infty} w(t,h) = 0$ and $\|\varphi(t,X_0)\|_{\mathscr{X}} \le w(t,h)$ if $\|X_0\|_{\mathscr{X}} \le h$.

Proof. Obviously, $\lim_{t\to\infty} w(t,h) = 0$. For $X_0 \in \Upsilon$ and $\|X_0\|_{\mathscr{X}} \le h$, we have

$$\begin{aligned}
\|\varphi(t,X_0)\|_{\mathscr{X}} &= |0| + |0| + \int_0^\infty |\tilde{\varphi}_e(t,a)| da + \int_0^\infty |\tilde{\varphi}_i(t,a)| da + \int_0^\infty |\tilde{\varphi}_j(t,a)| da \\
&= \int_t^\infty \left| \varphi_e(a-t) \frac{B_1(a)}{B_1(a-t)} \right| da + \int_t^\infty \left| \varphi_i(a-t) \frac{B_2(a)}{B_2(a-t)} \right| da \\
&\quad + \int_t^\infty \left| \varphi_j(a-t) \frac{B_3(a)}{B_3(a-t)} \right| da \\
&= \int_0^\infty \left| \varphi_e(\tau) \frac{B_1(t+\tau)}{B_1(\tau)} \right| d\tau + \int_0^\infty \left| \varphi_i(\tau) \frac{B_2(t+\tau)}{B_2(\tau)} \right| d\tau \\
&\quad + \int_0^\infty \left| \varphi_j(\tau) \frac{B_3(t+\tau)}{B_3(\tau)} \right| d\tau \\
&= \int_0^\infty \left| \varphi_e(\tau) \exp\left(-\int_\tau^{t+\tau} \theta_1(\upsilon) d\upsilon \right) \right| d\tau + \int_0^\infty \left| \varphi_i(\tau) \exp\left(-\int_\tau^{t+\tau} \theta_2(\upsilon) d\upsilon \right) \right| d\tau \\
&\quad + \int_0^\infty \left| \varphi_j(\tau) \exp\left(-\int_\tau^{t+\tau} \theta_3(\upsilon) d\upsilon \right) \right| d\tau.
\end{aligned}$$

By (iii) of assumption 2, $\theta_1(a)$, $\theta_2(a)$, $\theta_3(a) \ge \mu_0$ for $a \ge 0$, we have

$$\begin{aligned}
\|\varphi(t,X_0)\|_{\mathscr{X}} &\le e^{-\mu_0 t} \left(|0| + |0| + \textstyle\int_0^\infty |\varphi_e(\tau)| d\tau + \int_0^\infty |\varphi_i(\tau)| d\tau + \int_0^\infty |\varphi_j(\tau)| d\tau \right) \\
&= e^{-\mu_0 t} \|X_0\|_{\mathscr{X}} \le he^{-\mu_0 t} \triangleq w(t,h).
\end{aligned}$$

This completes the proof. □

To verify (ii) of lemma 3.2, we need to prove the following lemma.

Lemma 3.3 ([23,22]). *Let $K \subset L^p(0,\infty)$ be closed and bounded where $p \ge 1$. Then K is compact if the following conditions hold true.*

(i) $\lim_{h\to 0}\int_0^\infty |f(z+h)-f(z)|^p dz = 0$ *uniformly for* $f \in K$.

(ii) $\lim_{h\to\infty}\int_h^\infty |f(z)|^p dz = 0$ *uniformly for* $f \in K$.

Proposition 3.2 ([22]). For $t \geq 0$, $\phi(t, X_0)$ is completely continuous.

Proof. From lemma 3.3, for any closed and bounded set $B \subset \mathscr{X}$, we have $\phi(t,B)$ is compact. According to proposition 2.2, $S(t)$ and $V(t)$ remain in the compact set $[0,\Lambda/\mu_0] \subset [0,M]$, where $M \geq \Lambda/\mu_0$ is bound for B. Thus, it is only to show that $\tilde{e}(t,a)$, $\tilde{i}(t,a)$ and $\tilde{j}(t,a)$ remain in a precompact subset of $L^1_+(0,\infty)$, which is independent of $X_0 \in \Upsilon$. It suffices to verify that (i) and (ii) in lemma 3.3 hold.

Now, from (2.2) and (3.1) we have

$$0 \leq \tilde{e}(t,a) = \begin{cases} e(t-a,0)B_1(a), & for \quad 0 \leq a \leq < t; \\ 0, & for \quad 0 \leq t \leq a. \end{cases}$$

Then, combing (i) of proposition 2.2, we have

$$\tilde{e}(t,a) \leq (\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3 + \bar{\beta}_4)M^2 e^{-\mu_0 a},$$

which implies that (ii) in lemma 3.3 is satisfied. To check condition (i), for sufficiently small $h \in (0,t)$, we have

$$\begin{aligned}\int_0^\infty |\tilde{e}(t,a+h)-\tilde{e}(t,a)|da &= \int_0^t |e(t,a+h)-e(t,a)|da \\ &= \int_0^{t-h} |e(t-a-h,0)B_1(a) - e(t-a,0)B_1(a)|da \\ &\quad + \int_{t-h}^t |0 - e(t-a,0)B_1(a)|da \\ &\leq \int_0^{t-h} e(t-a-h,0)|B_1(a+h)-B_1(a)|da \\ &\quad + \int_0^{t-h} B_1(a)|e(t-a-h,0)-e(t-a,0)|da \\ &\quad + \int_{t-h}^t |e(t-a,0)B_1(a)|da\end{aligned}$$

Recall that $0 \le B_1(a) \le e^{-\mu_0 a} \le 1$ and $B_1(a)$ is non-increasing function with respect to a, we have

$$\begin{aligned}\int_0^{t-h} |B_1(a+h)-B_1(a)|da &= \int_0^{t-h} B_1(a)da - \int_0^{t-h} B_1(a+h)da \\ &= \int_0^{t-h} B_1(a)da - \int_h^{t} B_1(a)da \\ &= \int_0^{t-h} B_1(a)da - \int_h^{t-h} B_1(a)da - \int_{t-h}^{t} B_1(a)da \\ &= \int_0^{h} B_1(a)da - \int_{t-h}^{t} B_1(a)da \le h.\end{aligned}$$

Hence, from (ii) of proposition 3.2, we have

$$\int_0^{\infty} |\tilde{e}(t,a+h)-\tilde{e}(t,a)|da \le 2(\bar{\beta}_1+\bar{\beta}_2+\bar{\beta}_3+\bar{\beta}_4)M^2h+\Delta. \tag{3.4}$$

where,

$$\Delta = \int_0^{t-h} B_1(a)|e(t-a-h,0)-e(t-a,0)|da.$$

From (i) of proposition 3.2, we find that $|dS(t)/dt|$ is bounded by $M_S = \Lambda + (\mu+p)M + \bar{\beta}_1 M^2 + \bar{\beta}_3 M^2$ and $|dV(t)/dt|$ is bounded by $M_V = (\mu+p)M + \bar{\beta}_2 M^2 + \bar{\beta}_4 M^2$. Therefore, $S(\cdot)$ and $V(\cdot)$ are Lipschitz on $[0,\infty)$ with coefficients M_S and M_V. By Lemma 3.1 of [24], there exist some Lipschitz coefficients M_{I_1}, M_{I_2}, M_{J_1}, M_{J_2} for $\int_0^\infty \beta_1(a)i(\cdot,a)da$, $\int_0^\infty \beta_2(a)i(\cdot,a)da$, $\int_0^\infty \beta_3(a)j(\cdot,a)da$, $\int_0^\infty \beta_4(a)j(\cdot,a)da$, respectively. Thus, $\int_0^\infty \beta_1(a)i(\cdot,a)S(\cdot)da$, $\int_0^\infty \beta_2(a)i(\cdot,a)$ $V(\cdot)da$, $\int_0^\infty \beta_3(a)j(\cdot,a)S(\cdot)da$, $\int_0^\infty \beta_4(a)j(\cdot,a)V(\cdot)da$ are Lipschitz continuous on $[0,\infty)$ with coefficients $M_{SI} = KM_{I_1} + KM_S$, $M_{SJ} = KM_{J_1} + KM_S$, $M_{VI} = KM_{I_2} + KM_V$, $M_{VJ} = KM_{J_2} + KM_V$, respectively. Denote $M = M_{SI} + M_{SJ} + M_{VI} + M_{VJ}$. Then,

$$\Delta \le Mh \int_0^{t-h} e^{-\mu_0 a}da \le \frac{Mh}{\mu_0}. \tag{3.5}$$

Finally, by (3.4) and (3.5), we have

$$\int_0^{\infty} |\tilde{e}(t,a+h)-\tilde{e}(t,a)|da \le \left(2\bar{\beta}M^2 + \frac{M}{\mu_0}\right)h, \tag{3.6}$$

where, $\bar{\beta} = \bar{\beta}_1+\bar{\beta}_2+\bar{\beta}_3+\bar{\beta}_4$. The right hand of (3.6) converges uniformly to 0 as $h \to 0$ and condition (i) is proved for $\tilde{e}(t,a)$. Noting that (3.4) holds for any $X_0 \in B$, thus, $\tilde{e}(t,a)$ remains in a precompact subset $B_{\tilde{e}}$ of $L^1_+(0,\infty)$. Similarly, $\tilde{i}(t,a)$ and $\tilde{j}(t,a)$ remain in a precompact subset $B_{\tilde{i}}$, $B_{\tilde{j}}$ of $L^1_+(0,\infty)$ respectively. Thus, the proof is completed. □

From proposition 3.1 and 3.2, we apply lemma 3.2 and conclude that the following theorem holds.

Theorem 3. *The semi-flow $\Phi(t)_{t\geq 0}$ generated by system (1.5) is asymptotically smooth.*

Then we have the following proposition.

Proposition 3.3. The semi-flow $\Phi(t)_{t\geq 0}$ has a global attractor $\mathscr{A}$ contained in $\mathscr{X}$, which attracts the bounded sets of $\mathscr{X}$.

4. The existence of equilibria

System (1.5) always has the equilibrium $E_0 = (S_0, V_0, 0, 0, 0)$, where

$$S_0 = \frac{\Lambda}{\mu + p}, \quad V_0 = \frac{p\Lambda}{(\mu + p)(\mu + \rho)}.$$

By calculating the basic reproduction number, we get

$$\begin{aligned} R_0 := S_0 \bigg(\int_0^\infty & \beta_1(a)B_2(a) \int_0^\infty \gamma_1(a)B_1(a)da \\ & + \beta_3(a)B_3(a) \int_0^\infty (\gamma_2(a)B_1(a) + \xi(a)B_2(a) \int_0^\infty \gamma_1(a)B_1(a)da)da \bigg) da \\ & + V_0 \bigg(\int_0^\infty \beta_2(a)B_2(a) \int_0^\infty \gamma_1(a)B_1(a)da \\ & + \beta_4(a)B_3(a) \int_0^\infty (\gamma_2(a)B_1(a) + \xi(a)B_2(a) \int_0^\infty \gamma_1(a)B_1(a)da)da \bigg) da. \end{aligned}$$

Now we consider the positive equilibrium of system (1.5). The steady state $(S^*, V^*, e^*(\cdot), i^*(\cdot), j^*(\cdot))$ of system (1.5) satisfies the following equalities

$$\begin{aligned} & \Lambda - (\mu + p)S^* - S^* \int_0^\infty \beta_1(a)i^*(a)da - S^* \int_0^\infty \beta_3(a)j^*(a)da = 0, \\ & pS^* - (\mu + \rho)V^* - V^* \int_0^\infty \beta_2(a)i^*(a)da - V^* \int_0^\infty \beta_4(a)j^*(a)da = 0, \\ & \frac{de^*(a)}{da} = -\theta_1(a)e^*(a), \\ & \frac{di^*(a)}{da} = -\theta_2(a)i^*(a), \\ & \frac{dj^*(a)}{da} = -\theta_3(a)j^*(a), \end{aligned} \tag{4.1}$$

$$e^*(0) = \int_0^\infty (\beta_1(a)S^* + \beta_2(a)V^*)\, i^*(a)da + \int_0^\infty (\beta_3(a)S^* + \beta_4(a)V^*)\, j^*(a)da,$$
$$i^*(0) = \int_0^\infty \gamma_1(a)e^*(a)da,$$
$$j^*(0) = \int_0^\infty \gamma_2(a)e^*(a)da + \int_0^\infty \xi(a)i^*(a)da.$$

Solving the third, fourth and fifth equations of (4.1) yields that

$$e^*(a) = e^*(0)B_1(a), \quad i^*(a) = i^*(0)B_2(a), \quad j^*(a) = j^*(0)B_3(a).$$

Put it into the last three equations, we get

$$\begin{aligned} e^*(0) &= \int_0^\infty (\beta_1(a)S^* + \beta_2(a)V^*)\, i^*(0)B_2(a)da \\ &\quad + \int_0^\infty (\beta_3(a)S^* + \beta_4(a)V^*)\, j^*(0)B_3(a)da, \\ i^*(0) &= \int_0^\infty \gamma_1(a)e^*(0)B_1(a), \\ j^*(0) &= \int_0^\infty \gamma_2(a)e^*(0)B_1(a) + \int_0^\infty \xi(a)i^*(0)B_2(a). \end{aligned} \tag{4.2}$$

Denote $K_1(a) = \gamma_1(a)B_1(a)$, $K_2(a) = \gamma_2(a)B_1(a)$, $K_3(a) = \beta_1(a)B_2(a)$, $K_4(a) = \beta_3(a)B_3(a)$, $K_5(a) = \beta_2(a)B_2(a)$, $K_6(a) = \beta_4(a)B_3(a)$, $K_7(a) = \xi(a)B_2(a)$, $K_m = \int_0^\infty K_m(a)da$, $m = 1, 2, \cdots 7$. Furthermore, we obtain

$$\begin{aligned} e^*(0) &= (K_3S^* + K_5V^*)\, i^*(0) + (K_4S^* + K_6V^*)\, j^*(0), \\ i^*(0) &= K_1e^*(0), \\ j^*(0) &= K_2e^*(0) + K_7i^*(0). \end{aligned} \tag{4.3}$$

In (4.3), put the last two equations into the first equation, we get

$$e^*(0) = [(K_1K_3 + K_2K_4 + K_1K_4K_7)S^* + (K_1K_5 + K_2K_6 + K_1K_6K_7)V^*]\, e^*(0). \tag{4.4}$$

Because $i^*(0) \neq 0$, then we have $e^*(0) \neq 0$. From equation (4.4) we get

$$(K_1K_3 + K_2K_4 + K_1K_4K_7)S^* + (K_1K_5 + K_2K_6 + K_1K_6K_7)V^* = 1. \tag{4.5}$$

Denote $A_1 = K_1K_3 + K_2K_4 + K_1K_4K_7$, $A_2 = K_1K_3 + K_2K_4 + K_1K_4K_7$. Then $R_0 = A_1S_0 + A_2V_0$. It follows from the first and second equations of (4.1) that

$$S^* = \frac{\Lambda}{\mu + p + A_1e^*(0)}, \quad V^* = \frac{p\Lambda}{(\mu + p + A_1e^*(0))(\mu + \rho + A_2e^*(0))}.$$

Plugging it into equation (4.5) yields

$$a_0(e^*(0))^2 + a_1 e^*(0) + a_2 = 0, \tag{4.6}$$

where, $a_0 = A_1A_2$, $a_1 = A_1(\mu+\rho) + A_2(\mu+p) - A_1A_2\Lambda$, $a_2 = (\mu+p)(\mu+\rho)(1-R_0)$. Obviously, $a_0 > 0$. When $R_0 \le 1$, then $a_2 \ge 0$ and equation (4.6) has not positive root. When $R_0 > 1$, (4.6) has a unique positive real root and

$$e^*(0) = \frac{-a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_0}.$$

Therefore, if $R_0 > 1$, there exists a unique positive equilibrium E^* of system (1.5). where, $E^* = (S^*, V^*, e^*(a), i^*(a), j^*(a))$, $e^*(a) = e^*(0)B_1(a)$, $i^*(a) = K_1e^*(0)B_2(a)$, $j^*(a) = (K_2 + K_1K_7)e^*(0)B_3(a)$.

From the above discussions, we have the following theorem.

Theorem 4. *System (1.5) always has a steady state $E_0(S_0, V_0, 0, 0, 0)$, where $S_0 = \Lambda/(\mu+p)$, $V_0 = p\Lambda/(\mu+p)(\mu+\rho)$; if and only if $R_0 > 1$, system (1.5) has a unique positive equilibrium $E^*(S^*, V^*, e^*(\cdot), i^*(\cdot), j^*(\cdot))$.*

5. Local stability

This section is mainly used to prove the local stability of equilibrium, and to verify that the basic reproduction number is related to the stability of the equilibrium.

Theorem 5.1. *The equilibrium E_0 is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$.*

Proof. First, we introduce the change of variables as follows

$$s_1(t) = S(t) - S_0, \quad v_1(t) = V(t) - V_0, \quad e_1(t,a) = e(t,a), \quad i_1(t,a) = i(t,a),$$
$$j_1(t,a) = j(t,a).$$

Linearizing system (1.5) at the equilibrium E_0 yields the following system

$$\begin{aligned}
&\dot{s}_1(t) = -(\mu+p)s_1(t) - S_0\int_0^\infty \beta_1(a)i_1(t,a)da - S_0\int_0^\infty \beta_3(a)j_1(t,a)da,\\
&\dot{v}_1(t) = ps_1(t) - (\mu+\rho)v_1(t) - V_0\int_0^\infty \beta_2(a)i_1(t,a)da - V_0\int_0^\infty \beta_4(a)j_1(t,a)da,\\
&\frac{\partial e_1(t,a)}{\partial a} + \frac{\partial e_1(t,a)}{\partial t} = -\theta_1(a)e_1(t,a),\\
&\frac{\partial i_1(t,a)}{\partial a} + \frac{\partial i_1(t,a)}{\partial t} = -\theta_2(a)i_1(t,a),\\
&\frac{\partial j_1(t,a)}{\partial a} + \frac{\partial j_1(t,a)}{\partial t} = -\theta_3(a)j_1(t,a),\\
&e_1(t,0) = \int_0^\infty (\beta_1(a)S_0+\beta_2(a)V_0)i_1(t,a)da + \int_0^\infty (\beta_3(a)S_0+\beta_4(a)V_0)j_1(t,a)da,\\
&i_1(t,0) = \int_0^\infty \gamma_1(a)e_1(t,a)da,\\
&j_1(t,0) = \int_0^\infty \gamma_2(a)e_1(t,a)da + \int_0^\infty \xi(a)i_1(t,a)da,
\end{aligned}$$

Set

$$s_1(t) = s_1^0e^{\lambda t}, v_1(t) = v_1^0e^{\lambda t}, e_1(t,a) = e_1^0(a)e^{\lambda t}, i_1(t,a) = i_1^0(a)e^{\lambda t}, j_1(t,a) = j_1^0(a)e^{\lambda t},$$

where s_1^0, v_1^0, $e_1^0(a)$, $i_1^0(a)$, $j_1^0(a)$ will be determined later. We can get

$$\begin{aligned}
&\lambda s_1^0 = -(\mu+p)s_1^0 - S_0\int_0^\infty \beta_1(a)i_1^0(a)e^{-\lambda a}da - S_0\int_0^\infty \beta_3(a)j_1^0(a)e^{-\lambda a}da,\\
&\lambda v_1^0 = ps_1^0 - (\mu+\rho)v_1^0 - V_0\int_0^\infty \beta_2(a)i_1^0(a)e^{-\lambda a}da - V_0\int_0^\infty \beta_4(a)j_1^0(a)e^{-\lambda a}da,\\
&\frac{de_1^0(a)}{da} = -(\lambda+\theta_1(a))e_1^0(a),\\
&\frac{di_1^0(a)}{da} = -(\lambda+\theta_2(a))i_1^0(a),\\
&\frac{dj_1^0(a)}{da} = -(\lambda+\theta_3(a))j_1^0(a),\\
&e_1^0(0) = \int_0^\infty (\beta_1(a)S_0+\beta_2(a)V_0)i_1^0(a)da + \int_0^\infty (\beta_3(a)S_0+\beta_4(a)V_0)j_1^0(a)da,\\
&i_1^0(0) = \int_0^\infty \gamma_1(a)e_1^0(a)da,\\
&j_1^0(0) = \int_0^\infty \gamma_2(a)e_1^0(a)da + \int_0^\infty \xi(a)i_1^0(a)da,
\end{aligned} \tag{5.1}$$

Integrating the third, the forth and the fifth equation of (5.1) from 0 to a yields

$$\begin{aligned} e_1^0(a) &= e_1^0(0)\exp\left(-\int_0^a(\lambda+\theta_1(\tau))d\tau\right),\\ i_1^0(a) &= i_1^0(0)\exp\left(-\int_0^a(\lambda+\theta_2(\tau))d\tau\right),\\ j_1^0(a) &= j_1^0(0)\exp\left(-\int_0^a(\lambda+\theta_3(\tau))d\tau\right), \end{aligned} \tag{5.2}$$

Plugging (5.2) into (5.1) and solving (5.1) gives

$$\begin{aligned} e_1^0(0) = e_1^0(0)&\int_0^\infty\left(S_0K_3(a)e^{-\lambda a}+V_0K_5(a)e^{-\lambda a}\right)da\int_0^\infty K_1(a)e^{-\lambda a}da\\ &+e_1^0(0)\int_0^\infty\left(S_0K_4(a)e^{-\lambda a}+V_0K_6(a)e^{-\lambda a}\right)da\left[\int_0^\infty K_2(a)e^{-\lambda a}da\right.\\ &\left.+\int_0^\infty K_1(a)e^{-\lambda a}da\int_0^\infty K_7(a)e^{-\lambda a}da\right]. \end{aligned}$$

Denote $U_m(\lambda)=\int_0^\infty K_m(a)e^{-\lambda a}da, m=1,2,\cdots,7$. Then we can get

$$\begin{aligned} e_1^0(0) = e_1^0(0)&[S_0U_3(\lambda)+V_0U_5(\lambda)]U_1(\lambda)+e_1^0(0)[S_0U_4(\lambda)+V_0U_6(\lambda)]U_2(\lambda)\\ &+e_1^0(0)[S_0U_4(\lambda)+V_0U_6(\lambda)]U_1(\lambda)U_7(\lambda). \end{aligned}$$

If $e_1^0(0)=0$, then $i_1^0(0)=0$, $j_1^0(0)=0$. Plugging it into (5.1), we have

$$(\lambda+\mu+p)s_1^0=0, \quad (\lambda+\mu+\rho)v_1^0-ps_1^0=0.$$

For $s_1^0\neq 0$ and $v_1^0\neq 0$, it is easy to get

$$\lambda=-(\mu+p).$$

If $e_1^0(0)\neq 0$, then

$$1=\left(S_0U_3(\lambda)+V_0U_5(\lambda)\right)U_1(\lambda)+\left(S_0U_4(\lambda)+V_0U_6(\lambda)\right)\left(U_2(\lambda)+U_1(\lambda)U_7(\lambda)\right).$$

Then we denote

$$\mathscr{A}_1(\lambda)=U_1(\lambda)U_3(\lambda)+U_2(\lambda)U_4(\lambda)+U_1(\lambda)U_4(\lambda)U_7(\lambda),$$

$$\mathscr{A}_2(\lambda)=U_1(\lambda)U_5(\lambda)+U_2(\lambda)U_6(\lambda)+U_1(\lambda)U_6(\lambda)U_7(\lambda).$$

The characteristic equation is

$$(\lambda+\mu)s_1^0+(\lambda+\mu+\rho)v_1^0+e_1^0=0,$$

where,

$$s_1^0=-\frac{\mathscr{A}_1(\lambda)S_0e_1^0}{\lambda+\mu+p},\quad v_1^0=-\frac{\mathscr{A}_2(\lambda)V_0e_1^0}{\lambda+\mu+\rho}.$$

That is

$$\frac{\lambda+\mu}{\lambda+\mu+p}\mathscr{A}_1(\lambda)S_0+\mathscr{A}_2(\lambda)V_0=1. \tag{5.3}$$

Assume that Re$\lambda\geq 0$, then $|\mathscr{A}_1(\lambda)|\leq A_1$ and $|\mathscr{A}_2(\lambda)|\leq A_2$ hold. Hence, the modulus of the left-hand side of equation (5.3) satisfies

$$\begin{aligned}&\left|\frac{\lambda+\mu}{\lambda+\mu+p}\mathscr{A}_1(\lambda)S_0+\mathscr{A}_2(\lambda)V_0\right|\\ \leq&\left|\frac{\lambda+\mu}{\lambda+\mu+p}\mathscr{A}_1(\lambda)S_0\right|+|\mathscr{A}_2(\lambda)V_0|<A_1S_0+A_2V_0=R_0.\end{aligned}$$

It follows from (5.3) that there is a contradiction. Thus, all the roots of equation (5.3) have negative real part if and only if $R_0<1$ and have at least one eigenvalue with positive real part if $R_0>1$. Therefore, the equilibrium E_0 is locally asymptotically stable if $R_0<1$ and unstable if $R_0>1$. This completes the proof. □

Theorem 5.2. *The equilibrium E^* is locally asymptotically stable if $R_0>1$.*

Proof. Linearizing system (1.5) at the equilibrium E^* under introducing the perturbation variables

$$s_2(t)=S(t)-S^*,\quad v_2(t)=V(t)-V^*,\quad e_2(t,a)=e(t,a)-e^*(a),$$

$$i_2(t,a)=i(t,a)-i^*(a),\quad j_2(t,a)=j(t,a)-j^*(a),$$

we obtain the following system

$$
\begin{aligned}
\dot{s}_2(t) &= -(\mu+p)s_2(t) - s_2(t)\int_0^\infty \beta_1(a)i^*(a)da - S^*\int_0^\infty \beta_1(a)i_2(t,a)da \\
&\quad - s_2(t)\int_0^\infty \beta_3(a)j^*(a)da - S^*\int_0^\infty \beta_3(a)j_2(t,a)da, \\
\dot{v}_2(t) &= ps_2(t) - (\mu+\rho)v_2(t) - v_2(t)\int_0^\infty \beta_2(a)i^*(a)da - V^*\int_0^\infty \beta_2(a)i_2(t,a)da \\
&\quad - v_2(t)\int_0^\infty \beta_4(a)j^*(a)da - V^*\int_0^\infty \beta_4(a)j_2(t,a)da, \\
&\frac{\partial e_2(t,a)}{\partial a} + \frac{\partial e_2(t,a)}{\partial t} = -\theta_1(a)e_2(t,a), \\
&\frac{\partial i_2(t,a)}{\partial a} + \frac{\partial i_2(t,a)}{\partial t} = -\theta_2(a)i_2(t,a), \\
&\frac{\partial j_2(t,a)}{\partial a} + \frac{\partial j_1(t,a)}{\partial t} = -\theta_3(a)j_2(t,a), \\
e_2(t,0) &= \int_0^\infty (\beta_1(a)S^* + \beta_2(a)V^*)i_2(t,a)da + \int_0^\infty (\beta_3(a)S^* + \beta_4(a)V^*)j_2(t,a)da, \\
i_2(t,0) &= \int_0^\infty \gamma_1(a)e_2(t,a)da, \\
j_2(t,0) &= \int_0^\infty \gamma_2(a)e_2(t,a)da + \int_0^\infty \xi(a)i_2(t,a)da,
\end{aligned}
$$

Set

$$
s_2(t) = s_2^0 e^{\lambda t}, \quad v_2(t) = v_2^0 e^{\lambda t}, \quad e_2(t,a) = e_2^0(a)e^{\lambda t}, \quad i_2(t,a) = i_2^0(a)e^{\lambda t},
$$
$$
j_2(t,a) = j_2^0(a)e^{\lambda t},
$$

where s_2^0, v_2^0, $e_2^0(a)$, $i_2^0(a)$, $j_2^0(a)$ will be determined later. We get

$$
\begin{aligned}
\lambda s_2^0 &= -(\mu+p)s_2^0 - S^*\int_0^\infty \beta_1(a)i_2^0(a)da - s_2^0\int_0^\infty \beta_1(a)i^*(a)da \\
&\quad - S^*\int_0^\infty \beta_3(a)j_2^0(a)da - s_2^0\int_0^\infty \beta_3(a)j^*(a)da, \\
\lambda v_2^0 &= ps_2^0 - (\mu+\rho)v_2^0 - V^*\int_0^\infty \beta_2(a)i_2^0(a)da - v_2^0\int_0^\infty \beta_2(a)i^*(a)da \\
&\quad - V^*\int_0^\infty \beta_4(a)j_2^0(a)da - v_2^0\int_0^\infty \beta_4(a)j^*(a)da,
\end{aligned}
$$

$$
\begin{aligned}
\frac{de_2^0(a)}{da} &= -(\lambda + \theta_1(a))e_2^0(a), \\
\frac{di_2^0(a)}{da} &= -(\lambda + \theta_2(a))i_2^0(a), \\
\frac{dj_2^0(a)}{da} &= -(\lambda + \theta_3(a))j_2^0(a),
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
e_2^0(0) &= \int_0^\infty (\beta_1(a)S^* + \beta_2(a)V^*)i_2^0(a)da + \int_0^\infty (\beta_3(a)S^* + \beta_4(a)V^*)j_2^0(a)da, \\
&\quad + \int_0^\infty \left(\beta_1(a)s_2^0 + \beta_2(a)v_2^0\right) i^*(a)da + \int_0^\infty \left(\beta_3(a)s_2^0 + \beta_4(a)v_2^0\right) j^*(a)da \\
i_2^0(0) &= \int_0^\infty \gamma_1(a)e_2^0(a)da, \\
j_2^0(0) &= \int_0^\infty \gamma_2(a)e_2^0(a)da + \int_0^\infty \xi(a)i_2^0(a)da,
\end{aligned}
$$

Integrating the third, the forth and the fifth equation of (5.4) from 0 to a yields

$$
\begin{aligned}
e_2^0(a) &= e_2^0(0)\exp\left(-\int_0^a (\lambda + \theta_1(\tau))d\tau\right), \\
i_2^0(a) &= i_2^0(0)\exp\left(-\int_0^a (\lambda + \theta_2(\tau))d\tau\right), \\
j_2^0(a) &= j_2^0(0)\exp\left(-\int_0^a (\lambda + \theta_3(\tau))d\tau\right),
\end{aligned} \tag{5.5}
$$

Plugging (5.5) into (5.4) and solving (5.4) gives

$$
\begin{aligned}
e_2^0(0) = e_2^0(0)&\int_0^\infty \left(S_0K_3(a)e^{-\lambda a} + V_0K_5(a)e^{-\lambda a}\right) da \int_0^\infty K_1(a)e^{-\lambda a}da \\
+e_2^0(0)&\int_0^\infty \left(S_0K_4(a)e^{-\lambda a} + V_0K_6(a)e^{-\lambda a}\right) da \int_0^\infty K_2(a)e^{-\lambda a}da \\
+e_2^0(0)&\int_0^\infty \left(S_0K_4(a)e^{-\lambda a} + V_0K_6(a)e^{-\lambda a}\right) da \int_0^\infty K_1(a)e^{-\lambda a}da \int_0^\infty K_7(a)e^{-\lambda a}da.
\end{aligned}
$$

The characteristic equation is

$$
(\lambda + \mu)s_2^0 + (\lambda + \mu + \rho)v_2^0 + e_2^0 = 0,
$$

where,

$$
s_2^0 = -\frac{\mathscr{A}_1(\lambda)S^*e_2^0}{\lambda + \mu + p + A_1e^*(0)}, \quad v_2^0 = -\frac{\mathscr{A}_2(\lambda)V^*e_2^0}{\lambda + \mu + \rho + A_2e^*(0)}.
$$

That is

$$
\frac{\lambda + \mu}{\lambda + \mu + p + A_1e^*(0)}\mathscr{A}_1(\lambda)S^* + \frac{\lambda + \mu + \rho}{\lambda + \mu + \rho + A_2e^*(0)}\mathscr{A}_2(\lambda)V^* = 1. \tag{5.6}
$$

Assume that $\mathrm{Re}\lambda \geq 0$, then $|\mathscr{A}_1(\lambda)| \leq A_1$ and $|\mathscr{A}_2(\lambda)| \leq A_2$ hold. Hence, the modulus of the left-hand side of equation (5.6) satisfies

$$\begin{aligned}&\left|\frac{\lambda+\mu}{\lambda+\mu+p+A_1e^*(0)}\mathscr{A}_1(\lambda)S^*+\frac{\lambda+\mu+\rho}{\lambda+\mu+\rho+A_2e^*(0)}\mathscr{A}_2(\lambda)V^*\right|\\ &\leq\left|\frac{\lambda+\mu}{\lambda+\mu+p+A_1e^*(0)}\mathscr{A}_1(\lambda)S^*\right|+\left|\frac{\lambda+\mu+\rho}{\lambda+\mu+\rho+A_2e^*(0)}\mathscr{A}_2(\lambda)V^*\right|<A_1S^*+A_2V^*=1.\end{aligned}$$

It follows from (5.6) that there is a contradiction. Therefore, $\mathrm{Re}\lambda < 0$. This means that all the roots of (5.6) have negative real parts. Consequently, if $R_0 > 1$, the steady state E^* is locally asymptotically stable. This completes the proof. □

6. Global stability

This section is devoted to the global stability of equilibria. Before going into details, we make some preparations.

First, we introduce an important function which is obtained from the linear combination of Volterra-type functions of the form

$$g(x) = x - 1 - \ln x.$$

Obviously, $g(x) \geq 0$ for $x > 0$ and $g'(x) = 1 - 1/x$. Then, $g(x)$ has a global minimum at $x = 1$ and $g(1) = 0$.

Theorem 6.1. *If $R_0 < 1$, the disease-free equilibrium E_0 is globally asymptotically stable.*

Proof. Consider the following Lyapunov functional defined as

$$V_1 = V_{11} + V_{12} + V_{13} + V_{14},$$

where,

$$V_{11} = S_0 g(S(t)/S_0) + V_0 g(V(t)/V(0)), \quad V_{12} = \int_0^\infty \omega_1(a)e(t,a)da,$$

$$V_{13} = \int_0^\infty \omega_2(a)i(t,a)da, \quad V_{14} = \int_0^\infty \omega_3(a)j(t,a)da,$$

where,

$$\omega_1(a)=\int_a^{\infty}(\omega_3(0)\gamma_2(x)+\omega_2(0)\gamma_1(x))\exp\left(-\int_a^x\theta_1(\tau)d\tau\right)dx,$$
$$\omega_2(a)=\int_a^{\infty}(S_0\beta_1(x)+V_0\beta_2(x)+\omega_3(0)\xi(x))\exp\left(-\int_a^x\theta_2(\tau)d\tau\right)dx,$$
$$\omega_3(a)=\int_a^{\infty}(S_0\beta_3(x)+V_0\beta_4(x))\exp\left(-\int_a^x\theta_3(\tau)d\tau\right)dx,$$

then

$$\omega_2(0)\gamma_1(a)+\omega_3(0)\gamma_2(a)+\omega_1'(a)-\theta_1(a)\omega_1(a)=0.$$

$$S_0\beta_1(a)+V_0\beta_2+\omega_3(0)\xi(a)+\omega_2'(a)-\theta_2(a)\omega_2(a)=0,$$

$$S_0\beta_3(a)+V_0\beta_4+\omega_3'(a)-\theta_3(a)\omega_3(a)=0.$$

The derivative of V_{11} along with the solution of system (1.5) can be calculated as

$$\begin{aligned}\frac{dV_{11}}{dt}&=\left(1-\frac{S_0}{S(t)}\right)\left(\Lambda-(\mu+p)S(t)-S(t)\int_0^{\infty}(\beta_1(a)i(t,a)+\beta_3(a)j(t,a))\,da\right)\\&\quad+\left(1-\frac{V_0}{V(t)}\right)(pS(t)-(\mu+\rho)V(t)\ -V(t)\int_0^{\infty}\beta_2(a)i(t,a)da\\&\quad-V(t)\int_0^{\infty}\beta_4(a)j(t,a)da\Big)\\&=\mu S_0\left(2-\frac{S(t)}{S_0}-\frac{S_0}{S(t)}\right)+pS_0\left(3-\frac{V(t)}{V_0}-\frac{S_0}{S(t)}-\frac{S(t)V_0}{S_0V(t)}\right)\\&\quad-(S(t)-S_0)\int_0^{\infty}(\beta_1(a)i(t,a)+\beta_3(a)j(t,a))\,da\\&\quad-(V(t)-V_0)\int_0^{\infty}(\beta_2(a)i(t,a)\ +\beta_4(a)j(t,a))\,da.\end{aligned}\tag{6.1}$$

The derivative of V_{12} along with the solution of system (1.5) can be calculated as

$$\begin{aligned}\frac{dV_{12}}{dt}&=\frac{d}{dt}\int_0^t\omega_1(a)e(t-a,0)\exp\left(-\int_0^a\theta_1(\tau)d\tau\right)da\\&\quad+\frac{d}{dt}\int_t^{\infty}\omega_1(a)\varphi_e(a-t)\exp\left(-\int_{a-t}^a\theta_1(\tau)d\tau\right)da.\end{aligned}$$

Let $r = t - a$, then

$$\begin{aligned}\frac{dV_{12}}{dt} &= \frac{d}{dt}\int_0^t \omega_1(t-r)e(r,0)\exp\left(-\int_0^{t-r}\theta_1(\tau)d\tau\right)dr\\ &\quad+\frac{d}{dt}\int_t^\infty \omega_1(t+r)\varphi_e(r)\exp\left(-\int_r^{t+r}\theta_1(\tau)d\tau\right)dr\\ &= \omega_1(0)e(t,0)+\int_0^\infty \left(\omega'(a)-\theta_1(a)\omega_1(a)\right)e(t,a)da.\end{aligned} \tag{6.2}$$

Similarly, we can get

$$\begin{aligned}\frac{dV_{13}}{dt} &= \omega_2(0)\int_0^\infty \gamma_1(a)e(t,a)da+\int_0^\infty \left(\omega_2'(a)-\theta_2(a)\omega_2(a)\right)i(t,a)da,\\ \frac{dV_{14}}{dt} &= \omega_3(0)\int_0^\infty (\gamma_2(a)e(t,a)+\xi(a)i(t,a))da+\int_0^\infty \left(\omega_3'(a)-\theta_3(a)\omega_3(a)\right)j(t,a)da.\end{aligned} \tag{6.3}$$

Combining the (6.1)-(6.3), it is easy to get

$$\begin{aligned}\frac{dV_1}{dt} &= \mu S_0\left(2-\frac{S_0}{S(t)}-\frac{S(t)}{S_0}\right)+pS_0\left(3-\frac{V(t)}{V_0}-\frac{S_0}{S(t)}-\frac{S(t)V_0}{S_0V(t)}\right)\\ &\quad -e(t,0)+\omega_1(0)e(t,0)\\ &\quad+\int_0^\infty(S_0\beta_1(a)+V_0\beta_2(a)+\omega_3(0)\xi(a)+\omega_2'(a)-\theta_2(a)\omega_2(a))i(t,a)da\\ &\quad+\int_0^\infty(S_0\beta_3(a)+V_0\beta_4(a)+\omega_3'(a)-\theta_3(a)\omega_3(a))j(t,a)da\\ &\quad+\int_0^\infty(\omega_2(0)\gamma_1(a)+\omega_3(0)\gamma_2(a)+\omega'(a)-\theta_1(a)\omega_1(a))e(t,a)da\\ &= \mu S_0\left(2-\frac{S_0}{S(t)}-\frac{S(t)}{S_0}\right)+pS_0\left(3-\frac{V(t)}{V_0}-\frac{S_0}{S(t)}-\frac{S(t)V_0}{S_0V(t)}\right)+(R_0-1)e(t,0).\end{aligned}$$

Therefore, $R_0 \leq 1$ ensures that $dL_{\mathrm{DFE}}/dt \leq 0$ holds. Furthermore, the strict equality holds only if $S = S_0$, $V = V_0$, $e(t,a) = 0$, $i(t,a) = 0$ and $j(t,a) = 0$, simultaneously. Thus, $M_0 = E_0 \subset \Upsilon$ is the largest invariant subset of $dL_{\mathrm{DFE}}/dt = 0$, and by the Lyapunov-LaSalle invariance principle, the equilibrium E_0 is globally asymptotically stable when $R_0 \leq 1$.

Theorem 6.2. *If $R_0 > 1$, the equilibrium E^* is globally asymptotically stable.*

Proof. Constructing the Lyapunov functional as follows

$$V_2 = V_{21} + V_{22} + V_{23} + V_{24},$$

where,

$$\begin{aligned}
V_{21} &= g(S(t)/S^*) + g(V(t)/V^*),\\
V_{22} &= (K_3S^* + K_5V^*)\int_0^\infty q_1(a)g(e(t,a)/e^*(a))da\\
&\quad +(K_4S^* + K_6V^*)\int_0^\infty [q_2(a)g(e(t,a)/e^*(a)) + q_3(a)g(i(t,a)/i^*(a))]da,\\
V_{23} &= \int_0^\infty q_i(a)g(i(t,a)/i^*(a))da,\\
V_{24} &= \int_0^\infty q_j(a)g(j(t,a)/j^*(a)da,
\end{aligned}$$

where,

$$\begin{aligned}
q_1(a) &= \int_a^\infty \gamma_1(\sigma)e^*(\sigma)d\sigma, \quad q_2(a) = \int_a^\infty \gamma_2(\sigma)e^*(\sigma)d\sigma, \quad q_3(a) = \int_a^\infty \xi(\sigma)i^*(\sigma)d\sigma,\\
q_i(a) &= \int_a^\infty (S^*\beta_1(\sigma) + V^*\beta_2(\sigma))i^*(\sigma)d\sigma, \quad q_j(a) = \int_a^\infty (S^*\beta_3(\sigma) + V^*\beta_4(\sigma))j^*(\sigma)d\sigma.
\end{aligned}$$

Calculating the time derivative of V_{21} along equations (1.5), we have

$$\begin{aligned}
\frac{dV_{21}}{dt} &= \left(1 - \frac{S^*}{S(t)}\right)\left(\Lambda - (\mu + p)S(t) - S(t)\int_0^\infty [\beta_1(a)i(t,a) + \beta_3(a)j(t,a)]da\right)\\
&\quad + \left(1 - \frac{V^*}{V(t)}\right)\left(pS(t) - (\mu + \rho)V(t) - V(t)\int_0^\infty \beta_2(a)i(t,a)da\right.\\
&\quad \left. - V(t)\int_0^\infty \beta_4(a)j(t,a)da\right)\\
&= \mu S^*\left(2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)}\right) + pS^*\left(3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)}\right)\\
&\quad + S^*\int_0^\infty (\beta_1(a)i^*(a) + \beta_3(a)j^*(a))da - S(t)\int_0^\infty (\beta_1(a)i(t,a)\\
&\quad + \beta_3(a)j(t,a))da - \frac{S^*}{S(t)}S^*\int_0^\infty (\beta_1(a)i^*(a) + \beta_3(a)j^*(a))da\\
&\quad + S^*\int_0^\infty (\beta_1(a)i(t,a) + \beta_3(a)j(t,a))da - V^*\int_0^\infty (\beta_2 i^*(a)\\
&\quad + \beta_4(a)j^*(a))da + V^*\int_0^\infty (\beta_2(a)i(t,a) + \beta_4(a)j(t,a))da\\
&\quad - V(t)\int_0^\infty (\beta_2(a)i(t,a) + \beta_4(a)j(t,a))da - (\mu + \rho)V(t) + pS^*\frac{V(t)}{V^*}.
\end{aligned}$$

Applying the following equation

$$pS^* = (\mu+\rho)V^* + V^*\int_0^\infty (\beta_2(a)i^*(a)+\beta_4(a)j^*(a))da, \tag{6.4}$$

we have

$$\begin{aligned}
\frac{dV_{21}}{dt} &= \mu S^*\left(2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)}\right)+pS^*\left(3-\frac{V(t)}{V^*}-\frac{S^*}{S(t)}-\frac{S(t)V^*}{S^*V(t)}\right)\\
&+\int_0^\infty (S^*\beta_1(a)i^*(a)+S^*\beta_3(a)j^*(a))da-\frac{S(t)}{S^*}\int_0^\infty (S^*\beta_1(a)i(t,a)+S^*\beta_3(a)j(t,a))da\\
&-\frac{S^*}{S(t)}\int_0^\infty (S^*\beta_1(a)i^*(a)+S^*\beta_3(a)j^*(a))da+\int_0^\infty (S^*\beta_1(a)i(t,a)+S^*\beta_3(a)j(t,a))da\\
&-\int_0^\infty (V^*\beta_2(a)i^*(a)+V^*\beta_4(a)j^*(a))da+\int_0^\infty (V^*\beta_2(a)i(t,a)+V^*\beta_4(a)j(t,a))da\\
&-\frac{V(t)}{V^*}\int_0^\infty (V^*\beta_2(a)i(t,a)+V^*\beta_4(a)j(t,a))da+\frac{V(t)}{V^*}\int_0^\infty (V^*\beta_2(a)i^*(a)\\
&+V^*\beta_4(a)j^*(a))\,da
\end{aligned} \tag{6.5}$$

The derivative of V_{22}, V_{23} and V_{24} can be calculated as follows:

$$\begin{aligned}
\frac{dV_{22}}{dt} &= (K_3S^*+K_5V^*)\int_0^\infty \gamma_1(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)}-\frac{e(t,a)}{e^*(a)}+\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}da\\
&+(K_4S^*+K_6V^*)\int_0^\infty \gamma_2(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)}-\frac{e(t,a)}{e^*(a)}+\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}da\\
&+(K_4S^*+K_6V^*)\int_0^\infty \xi(a)i^*(a)\left\{\frac{i(t,0)}{i^*(0)}-\frac{i(t,a)}{i^*(a)}+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}da,
\end{aligned} \tag{6.6}$$

$$\frac{dV_{23}}{dt} = \int_0^\infty (S^*\beta_1(a)+V^*\beta_2(a))i^*(a)\left\{\frac{i(t,0)}{i^*(0)}-\frac{i(t,a)}{i^*(a)}+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}da, \tag{6.7}$$

$$\frac{dV_{24}}{dt} = \int_0^\infty (S^*\beta_3(a)+V^*\beta_4(a))j^*(a)\left\{\frac{j(t,0)}{j^*(0)}-\frac{j(t,a)}{j^*(a)}+\ln\frac{j(t,a)}{j^*(a)}-\ln\frac{j(t,0)}{j^*(0)}\right\}da. \tag{6.8}$$

Combining (6.5)-(6.8), we get

$$\begin{aligned}
\frac{dV_2}{dt} &= \mu S^*\left(2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)}\right)+pS^*\left(3-\frac{V(t)}{V^*}-\frac{S^*}{S(t)}-\frac{S(t)V^*}{S^*V(t)}\right)\\
&+\int_0^\infty S^*\beta_1(a)i^*(a)\left\{1-\frac{S(t)i(t,a)}{S^*i^*(a)}-\frac{S^*}{S(t)}+\frac{i(t,0)}{i^*(0)}+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}da\\
&+\int_0^\infty V^*\beta_2(a)i^*(a)\left\{-1-\frac{V(t)i(t,a)}{V^*i^*(a)}+\frac{V^*}{V(t)}+\frac{i(t,0)}{i^*(0)}+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}da
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty S^*\beta_3(a)j^*(a)\left\{1-\frac{S(t)j(t,a)}{S^*j^*(a)}-\frac{S^*}{S(t)}+\frac{j(t,0)}{j^*(0)}+\ln\frac{j(t,a)}{j^*(a)}-\ln\frac{j(t,0)}{j^*(0)}\right\}da\\
&+\int_0^\infty V^*\beta_4(a)j^*(a)\left\{-1-\frac{V(t)j(t,a)}{V^*j^*(a)}+\frac{V^*}{V(t)}+\frac{j(t,0)}{j^*(0)}+\ln\frac{j(t,a)}{j^*(a)}-\ln\frac{j(t,0)}{j^*(0)}\right\}da\\
&+(K_3S^*+K_5V^*)\int_0^\infty \gamma_1(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)}-\frac{e(t,a)}{e^*(a)}+\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}da\\
&+(K_4S^*+K_6V^*)\int_0^\infty \gamma_2(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)}-\frac{e(t,a)}{e^*(a)}+\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}da\\
&+(K_4S^*+K_6V^*)\int_0^\infty \xi(a)i^*(a)\left\{\frac{i(t,0)}{i^*(0)}-\frac{i(t,a)}{i^*(a)}+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}da\\
=&\,\mu S^*\left(2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)}\right)+pS^*\left(3-\frac{V(t)}{V^*}-\frac{S^*}{S(t)}-\frac{S(t)V^*}{S^*V(t)}\right)\\
&+\int_0^\infty S^*\beta_1(a)i^*(a)\left\{1-\frac{S^*}{S(t)}+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}da\\
&+\int_0^\infty V^*\beta_2(a)i^*(a)\left\{-1+\frac{V^*}{V(t)}+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}da\\
&+\int_0^\infty S^*\beta_3(a)j^*(a)\left\{1-\frac{S^*}{S(t)}+\ln\frac{j(t,a)}{j^*(a)}-\ln\frac{j(t,0)}{j^*(0)}\right\}da\\
&+\int_0^\infty V^*\beta_4(a)j^*(a)\left\{-1+\frac{V^*}{V(t)}+\ln\frac{j(t,a)}{j^*(a)}-\ln\frac{j(t,0)}{j^*(0)}\right\}da\\
&+(K_3S^*+K_5V^*)\int_0^\infty \gamma_1(a)e^*(a)\left\{\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}da\\
&+(K_4S^*+K_6V^*)\int_0^\infty \gamma_2(a)e^*(a)\left\{\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}da\\
&+(K_4S^*+K_6V^*)\int_0^\infty \xi(a)i^*(a)\left\{\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}da\\
&+(K_3S^*+K_5V^*)\int_0^\infty \gamma_1(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)}\right\}da\\
&+(K_4S^*+K_6V^*)\int_0^\infty \gamma_2(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)}\right\}da\\
&+(K_4S^*+K_6V^*)\int_0^\infty \xi(a)i^*(a)\left\{\frac{i(t,0)}{i^*(0)}\right\}da-\int_0^\infty S^*\beta_1(a)i^*(a)\frac{S(t)i(t,a)}{S^*i^*(a)}da\\
&-\int_0^\infty V^*\beta_2(a)i^*(a)\frac{V(t)i(t,a)}{V^*i^*(a)}da-\int_0^\infty S^*\beta_3(a)j^*(a)\frac{S(t)j(t,a)}{S^*j^*(a)}da\\
&-\int_0^\infty V^*\beta_4(a)j^*(a)\frac{V(t)j(t,a)}{V^*j^*(a)}da-(K_3S^*+K_5V^*)\int_0^\infty \gamma_1(a)e^*(a)\frac{e(t,a)}{e^*(a)}da\\
&-(K_4S^*+K_6V^*)\int_0^\infty \gamma_2(a)e^*(a)\frac{e(t,a)}{e^*(a)}da-(K_4S^*+K_6V^*)\int_0^\infty \xi(a)i^*(a)\frac{i(t,a)}{i^*(a)}da\\
&+\int_0^\infty (S^*\beta_1(a)+V^*\beta_2(a))i^*(a)\frac{i(t,0)}{i^*(0)}da+\int_0^\infty (S^*\beta_3(a)+V^*\beta_4(a))j^*(a)\frac{j(t,0)}{j^*(0)}da.
\end{aligned}
$$

it is easy to see that the last twelve terms of the above equation equal to 0. Thus , using equation (6.4), we have

$$\begin{aligned}
\frac{dV_2}{dt} &= \mu S^* \left(2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)}\right) + (\mu+\rho)V^* \left(3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)}\right) \\
&+ \int_0^\infty (V^*\beta_2(a)i^*(a) + V^*\beta_4(a)j^*(a)) \left\{3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)}\right\} da \\
&+ \int_0^\infty S^*\beta_1(a)i^*(a) \left\{1 - \frac{S^*}{S(t)} + \ln\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)}\right\} da \\
&+ \int_0^\infty V^*\beta_2(a)(a)i^*(a) \left\{-1 + \frac{V^*}{V(t)} + \ln\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)}\right\} da \\
&+ \int_0^\infty S^*\beta_3(a)j^*(a) \left\{1 - \frac{S^*}{S(t)} + \ln\frac{j(t,a)}{j^*(a)} - \ln\frac{j(t,0)}{j^*(0)}\right\} da \\
&+ \int_0^\infty V^*\beta_4(a)j^*(a) \left\{-1 + \frac{V^*}{V(t)} + \ln\frac{j(t,a)}{j^*(a)} - \ln\frac{j(t,0)}{j^*(0)}\right\} da \\
&+(K_3S^* + K_5V^*) \int_0^\infty \gamma_1(a)e^*(a) \left\{\ln\frac{e(t,a)}{e^*(a)} - \ln\frac{e(t,0)}{e^*(0)}\right\} da \\
&+(K_4S^* + K_6V^*) \int_0^\infty \gamma_2(a)e^*(a) \left\{\ln\frac{e(t,a)}{e^*(a)} - \ln\frac{e(t,0)}{e^*(0)}\right\} da \\
&+(K_4S^* + K_6V^*) \int_0^\infty \xi(a)i^*(a) \left\{\ln\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)}\right\} da.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\frac{dV_2}{dt} &= \mu S^* \left(2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)}\right) + (\mu+\rho)V^* \left(3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)}\right) \\
&- \int_0^\infty [(S^*\beta_1(a) + V^*\beta_2(a))i^*(a) + (S^*\beta_3(a) + V^*\beta_4(a))j^*(a)] g\left(\frac{S^*}{S(t)}\right) da \\
&-(K_3S^* + K_5V^*) \int_0^\infty \gamma_1(a)e^*(a) g\left(\frac{e(t,a)}{e^*(a)i(t,0)}\right) da \\
&-(K_4S^* + K_6V^*) \int_0^\infty \gamma_2(a)e^*(a) g\left(\frac{e(t,a)j^*(0)}{e^*(a)j(t,0)}\right) da \\
&-(K_4S^* + K_6V^*) \int_0^\infty \xi(a)i^*(a) g\left(\frac{i(t,a)j^*(0)}{i^*(a)j(t,0)}\right) da \\
&- \int_0^\infty S^*\beta_1(a)i^*(a) g\left(\frac{S(t)i(t,a)e^*(0)}{S^*i^*(a)e(t,0)}\right) da \\
&- \int_0^\infty V^*\beta_2(a)i^*(a) g\left(\frac{V(t)i(t,a)e^*(0)}{V^*i^*(a)e(t,0)}\right) da \\
&- \int_0^\infty S^*\beta_3(a)j^*(a) g\left(\frac{S(t)j(t,a)e^*(0)}{S^*j^*(a)e(t,0)}\right) da \\
&- \int_0^\infty V^*\beta_4(a)j^*(a) g\left(\frac{V(t)j(t,a)e^*(0)}{V^*j^*(a)e(t,0)}\right) da
\end{aligned}$$

$$
-\int_0^\infty V^*\beta_2(a)i^*(a)g\left(\frac{S(t)V^*}{S^*V(t)}\right)da
$$
$$
-\int_0^\infty V^*\beta_4(a)j^*(a)g\left(\frac{S(t)V^*}{S^*V(t)}\right)da
$$

Hence, $dV_2/dt \leq 0$ holds. Furthermore, the strict equality holds only if $S = S^*$, $V = V^*$, $e(t,a) = e^*(a)$, $i(t,a) = i^*(a)$, $j(t,a) = j^*(a)$. Thus, $M^* = \{E^*\} \subset \Omega$ is the largest invariant subset of $dV_2/dt = 0$, and by the Lyapunov-LaSalle invariance principle, when $R_0 > 1$, the equilibrium E^* is globally asymptotically stable. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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