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PHYTOPLANKTON DIFFUSIVE MODEL WITH PULSE AND VIRAL INFECTION

YU ZANG, HAIYAN XU, ZHIGUI LIN*

School of Mathematical Science, Yangzhou University, Yangzhou 225002, China

Abstract. This paper deals with a reaction-diffusion-advection problem, which describes phytoplankton model with pulse and virus infection. Firstly, the corresponding periodic problem is considered, and the principal eigenvalues of periodic eigenvalue problems are calculated, and the conditions related to the eigenvalues for the existence of disease-free periodic solution are given. Finally, the asymptotic behavior of disease-free periodic solution is studied, and the sufficient conditions for extinction of virus and persistence of phytoplankton are obtained.

Keywords: phytoplankton; diffusion; harvest pulse; virus extinction.

2020 AMS Subject Classification: 35K57, 35R12, 92D25.

1. Introduction

As we know, phytoplankton is a general term for tiny algal organisms floating in lakes, oceans and other waters, which rely on light energy and carbon dioxide for photosynthesis [1, 2]. Nutrients and lights are necessity for their survival. Many studies about states of phytoplankton multi-population communities have been set up [3, 4, 5, 6, 7], including dynamic models with time-delay and diffusive models. In addition, the habitat of plankton is mostly changing continuously, so the model in an evolving area [8, 9] has also attracted a lot of attention.

On the other hand, the influence of viruses on the aquatic system cannot be ignored, which can directly affect the population density of the host and may lead to its perish. A phytoplankton

*Corresponding author

E-mail address: zglin68@hotmail.com

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SIS (susceptible-infected-susceptible) model was established in [10], the local stability of equilibrium point was analyzed and the condition that the periodic solution does not exist was obtained. Recently, the virus-infected phytoplankton-zooplankton system was studied in [11], where planktonic bloom oscillations were observed for increasing viral infection in both species, and in [12], convergence analysis was set up in a fractional model with phytoplankton-toxic.

It is well-known that the flooding of phytoplankton can cause great impacts on the ecosystem, so we should take strategies to control the density of phytoplankton in a short period of time by regular harvesting and killing. Mathematically, pulse differential equations can be used to describe this discrete-continuous process. In 2012, Lewis and Li [13] proposed an impulsive reaction-diffusion problem that describes population dynamics with seasonal pulses, and their results showed how pulses affect the extinction and persistence of species. Recently, [14] studied a pulsed logistic problem in an evolving domain and explored the role of pulsed harvesting in the dynamics of invasive species. In [15], a reaction-diffusion problem with population growth and multi-pulse disturbance was studied.

In this paper, we consider a phytoplankton model with virus and periodic harvesting, where the number of phytoplankton species changes rapidly when harvesting occurs, otherwise, species spread freely and grow under the influence of light. Inspired by [10] and [15], a phytoplankton population model with harvest pulse and virus infection is considered as follows

$$
(1)\quad \begin{cases}
S_t = DS_{xx} - \alpha S_x + [g(w(S,I,x,t)) - d(x) - mS]S & \\
\qquad -\beta(x)\frac{SI}{S+I} + \gamma(x)I, & x \in (0,L_0),\ t \in ((nT)^+,(n+1)T], \\
I_t = DI_{xx} - \alpha I_x + [g(w(S,I,x,t)) - d(x) & \\
\qquad -mI - \delta(x)]I + \beta(x)\frac{SI}{S+I} - \gamma(x)I, & x \in (0,L_0),\ t \in ((nT)^+,(n+1)T], \\
DS_x - \alpha S = DI_x - \alpha I = 0, & x = 0, L_0,\ t \in ((nT)^+,(n+1)T], \\
S(x,0) = S_0(x) \geq 0, \not\equiv 0, & x \in [0,L_0], \\
I(x,0) = I_0(x) \geq 0, \not\equiv 0, & x \in [0,L_0], \\
S(x,(nT)^+) = P(S(x,nT)), & x \in (0,L_0), \\
I(x,(nT)^+) = P(I(x,nT)), n = 0,1,2,\cdots, & x \in (0,L_0),
\end{cases}
$$

where $S(x,t)$ and $I(x,t)$ represent the densities of susceptible and infected phytoplankton at space x and time t, respectively. The positive constant D represents the diffusive coefficient of phytoplankton, and $\alpha \in R$ is the sinking ($\alpha < 0$) or rising ($\alpha > 0$) speed of phytoplankton. x represents the depth of the water body from 0 (top) to L_0 (bottom). The phytoplankton mortality related to space is denoted by $d(x) > 0$, which is a smooth and monotonically increasing function with respect to x and $\delta(x)$ means the additional mortality of infected phytoplankton. $\beta(x)$ is a positive bounded continuous function describing the infection rate and $\gamma(x)$ is the recovery rate. The positive constant m is the competitive coefficient, and g represents the specific growth rate of phytoplankton, which is a function of light intensity $w(S,I,x,t)$ with

$$w(S,I,x,t) = w_0 \exp(-k_0 x - \int_0^x k[S(y,t)+I(y,t)]dy),$$

where w_0, k_0 and k are positive constants. The initial values $S_0(x), I_0(x)$ are nonnegative smooth functions for $x \in [0,L_0]$. As in [4], we assume that g is a smooth function that satisfies $g(0) = 0$ and $g'(w) > 0$ for all positive w. Impulse function P is a smooth positive function and satisfies

$$P(0) = 0 \text{ and } P(v) \leq P'(0)v \text{ for } v > 0.$$

2. The Principal Eigenvalue Problem

We first claim here that problem (1) admits a global classical solution $(S(x,t), I(x,t))$, where $S, I \in C^{2,1}((0,L_0) \times (nT,(n+1)T]) \cap C([0,L_0] \times (nT,(n+1)T])$ for $n = 0,1,\cdots$. In fact, if the initial value $S_0(x), I_0(x) \in C^2([0,L_0])$ and $P \in C^2[0,\infty)$, we have that $S(x,0^+), I(x,0^+) \in C^2([0,L_0])$. By virtue of standard theory for parabolic equations, we can deduce that $S(x,t), I(x,t) \in C^{2,1}([0,L_0] \times (0,T])$. Then, $S(x,T^+) = P(S(x,T)), I(x,T^+) = P(I(x,T))$ are also twice continuously differentiable in x. Hence, let $S(x,T^+), I(x,T^+)$ be new initial values for $t \in (T^+, 2T]$, then $S, I \in C^{2,1}([0,L_0] \times (T,2T])$. Eventually, we can obtain the solution $(S(x,t), I(x,t))$ of problem (1) for $t \geq 0$ and $x \in [0,l_0]$ by the same procedures.

We now consider the disease-free periodic solution $(S^*(x,t), 0)$ to problem (1), where $S^*(x,t)$ satisfies the following problem

$$(2)\qquad \begin{cases} S_t = DS_{xx} - \alpha S_x + [G(S,x,t) - d(x) - mS]S, & x \in (0,L_0),\ t \in (0^+,T], \\ DS_x - \alpha S = 0, & x = 0, L_0,\ t \in (0^+,T], \\ S(x,0) = S(x,T), & x \in (0,L_0), \\ S(x,0^+) = P(S(x,0)), & x \in (0,L_0) \end{cases}$$

with $G(S,x,t) = g(w(S,0,x,t))$.

After linearization at $S=0$, the corresponding periodic eigenvalue problem to (2) becomes

$$(3)\qquad \begin{cases} \phi_t = D\phi_{xx} - \alpha\phi_x + [G(x) - d(x)]\phi + \lambda_1\phi, & x \in (0,L_0),\ t \in (0^+,T], \\ D\phi_x - \alpha\phi = 0, & x = 0, L_0,\ t \in (0^+,T], \\ \phi(x,0) = \phi(x,T), & x \in (0,L_0), \\ \phi(x,0^+) = P'(0)\phi(x,0), & x \in (0,L_0), \end{cases}$$

where $G(x) = g(w_0 e^{-k_0 x})$. In the following, we first present the expression of λ_1.

Theorem 2.1. *The principal eigenvalue of problem* (3) *is*

$$\lambda_1 = \lambda_1^* - \frac{\ln P'(0)}{T},$$

where λ_1^* is principal eigenvalue of problem

$$\begin{cases} -D\psi_{xx} = -\alpha\psi_x + [G(x) - d(x)]\psi + \lambda_1^*\psi, & x \in (0,L_0), \\ D\psi_x - \alpha\psi = 0, & x = 0, L_0. \end{cases}$$

Proof: Let $\phi(x,t) = f(t)\psi(x)$, problem (3) can be transformed into

$$\begin{cases} f'(t)\psi(x) = Df(t)\psi''(x) - \alpha f(t)\psi'(x) \\ \qquad\qquad + [G(x) - d(x)]f(t)\psi(x) + \lambda_1 f(t)\psi(x), & x \in (0,L_0), t \in (0^+,T], \\ D\psi'(x) - \alpha\psi(x) = 0, & x = 0, L_0, \\ f(0) = f(T), \\ f(0^+) = P'(0)f(0) \end{cases}$$

by separation of variable, we then have

$$\frac{f'(t)}{f(t)} - \lambda_1 = \frac{D\psi''(x) - \alpha\psi'(x) + [G(x) - d(x)]\psi(x)}{\psi(x)} = -\lambda_1^*.$$

It is easy to see that

$$f'(t) + (\lambda_1^* - \lambda_1)f(t) = 0,$$

integrating both side of equation over 0^+ to t, we have

$$f(t) = Ce^{(-\lambda_1^* + \lambda_1)t},\ t \in (0^+, T],$$

where $C = f(0^+) = P'(0)f(0)$.

Let $t = T$, we obtain

$$f(T) = P'(0)f(0)e^{(-\lambda_1^* + \lambda_1)T},$$

which, combines with $f(0) = f(T)$, yields

$$P'(0)e^{(-\lambda_1^* + \lambda_1)T} = 1.$$

Therefore

$$\lambda_1 = \lambda_1^* - \frac{\ln P'(0)}{T}.$$

To give the expression of λ_1^*, we convert the Robin boundary condition into null Neumann boundary condition, and define

$$\psi^*(x) = e^{-\frac{\alpha}{D}x}\psi(x),$$

we get

$$\begin{cases} D\psi^*_{xx} + \alpha\psi^*_x + [G(x) - d(x) + \lambda_1^*]\psi^* = 0, & x \in (0, L_0), \\ \psi^*_x(x) = 0, & x = 0, L_0. \end{cases} \tag{4}$$

Furthermore, in order to use the variational formula, we rewrite problem (4) as

$$\begin{cases} [De^{\frac{\alpha}{D}x}\psi^*_x]_x + [G(x) - d(x) + \lambda_1^*]e^{\frac{\alpha}{D}x}\psi^* = 0, & x \in (0, L_0), \\ De^{\frac{\alpha}{D}x}\psi^*_x(x) = 0, & x = 0, L_0, \end{cases}$$

which yield

$$\lambda_1^* = \min_{\psi \in W^{1,2}(0,L), \psi \neq 0} \frac{\int_0^L De^{\frac{\alpha}{D}x}\psi_x^2 dx - \int_0^L [G(x) - d(x)]e^{\frac{\alpha}{D}x}\psi^2 dx}{\int_0^L e^{\frac{\alpha}{D}x}\psi^2 dx}.$$

$\square$

In order to facilitate the construction of upper and lower solutions, we rewrite the periodic eigenvalue problem (3) as following.

$$
(5) \qquad \begin{cases} \phi_t = D\phi_{xx} - \alpha\phi_x + [G(x) - d(x)]\phi + \mu_1\phi, & x \in (0, L_0),\ t \in (0^+, T], \\ D\phi_x - \alpha\phi = 0, & x = 0, L_0,\ t \in (0^+, T], \\ \phi(x, 0) = \phi(x, T), & x \in (0, L_0), \\ \phi(x, 0^+) = P'(0)\phi(x, 0) + \mu_1\phi(x, 0). & x \in (0, L_0). \end{cases}
$$

It is easy to draw the relationship between λ_1 and μ_1.

Lemma 2.2. *The following statements hold:*
(1) $\mu_1 = 0 \Leftrightarrow \lambda_1 = 0$;
(2) $\mu_1 > 0 \Leftrightarrow \lambda_1 > 0$.

Proof: Direct calculations show that

$$
(6) \qquad e^{T\lambda_1} = \frac{P'(0) + \mu_1}{P'(0)} e^{T\mu_1}.
$$

If $\mu_1 = 0$, we see $\lambda_1 = 0$ directly by (6). On the other hand, if $\lambda_1 = 0$, we have $\mu_1 = P'(0)(e^{-T\mu_1} - 1)$, and $\mu_1 = 0$ is the only root since that $F(x) := P'(0)(e^{-Tx} - 1) - x$ is decreasing in x, $F(-\infty) > 0$ and $F(+\infty) < 0$.

If $\mu_1 > 0$, the right side of (6) is positive, and then $\lambda_1 > 0$. On the other hand, if $\lambda_1 > 0$. Assume $\mu_1 < 0$ by contradiction, it is easy to see that the right side of (6) is less that 1, so $\lambda_1 > 0$, which is in contradiction with our assumption. Therefore, $\mu_1 > 0$. $\square$

3. The Positive Periodic Solution

Using the principal eigenvalue of problem (3), we first give the existence and uniqueness of positive solution of periodic problem (2).

Theorem 3.1. *The following asserts hold:*
(i) *If* $\lambda_1 < 0$, *problem* (2) *admits a unique positive* $T-$ *periodic solution;*
(ii) *If* $\lambda_1 \geq 0$, *problem* (2) *has no positive solution.*

Proof: To prove (i), we first consider the existence of positive solutions to problem (2) by constructing upper and lower solutions. When $\lambda_1 < 0$, it follows from Lemma 2.2 that $\mu_1 < 0$, where (μ_1, ϕ^*) satisfies the eigenvalue problem

$$\begin{cases} \phi_t = D\phi_{xx} - \alpha\phi_x + [G(x) - d(x)]\phi + \mu_1\phi, & x \in (0, L_0),\ t \in (0^+, T], \\ D\phi_x - \alpha\phi = 0, & x = 0, L_0,\ t \in (0^+, T], \\ \phi(x, 0) = \phi(x, T), & x \in (0, L_0), \\ \phi(x, 0^+) = P'(0)\phi(x, 0) + \mu_1\phi(x, 0), & x \in (0, L_0). \end{cases}$$

Let $\underline{S} = \varepsilon\phi^*$, we claim that, for ε sufficiently small, $\underline{S}$ is a lower solution to problem (2), that is, $\underline{S}$ satisfies

$$\begin{cases} \underline{S}_t \le D\underline{S}_{xx} - \alpha\underline{S}_x + [g(W(\underline{S}, 0, x, t)) - d(x) - m\underline{S}]\underline{S} & x \in (0, L_0),\ t \in (0^+, T], \\ D\underline{S}_x - \alpha\underline{S} \le 0, & x = 0, L_0,\ t \in (0^+, T], \\ \underline{S}(x, 0) \le \underline{S}(x, T), & x \in (0, L_0), \\ \underline{S}(x, 0^+) \le P(\underline{S}(x, 0)), & x \in (0, L_0). \end{cases} \tag{7}$$

In fact,

$$\begin{aligned} &\underline{S}_t - [D\underline{S}_{xx} - \alpha\underline{S}_x + [g(w(\underline{S}, 0, x, t))\underline{S} - d(x)\underline{S} - m\underline{S}^2] \\ &= \varepsilon(D\phi^*_{xx} - \alpha\phi^*_x + [g(w(0, 0, x, t)) - d(x)]\phi^* + \mu_1\phi^*) \\ &-\varepsilon[D\phi^*_{xx} - \alpha\phi^*_x + [g(w(\varepsilon\phi^*, 0, x, t))\phi^* - d(x)\phi^* - m\varepsilon\phi^*] \\ &= \varepsilon\phi[\mu_1 + g(w(0, 0, x, t)) - g(w(\varepsilon\phi^*, 0, x, t)) + \varepsilon m]. \end{aligned}$$

Noticing that $\varepsilon \to 0$, $g(w(0, 0, x, t)) - g(w(\varepsilon\phi^*, 0, x, t)) + \varepsilon m \to 0$, and recalling that $\mu_1 < 0$, we can choose ε small enough so that the first inequality in (7) holds.

Considering the impulse condition, and direct calculation yields

$$\begin{aligned} &\underline{S}(x, 0^+) - P(\underline{S}(x, 0)) \\ &= \varepsilon\phi(x, 0^+) - P(\varepsilon\phi(x, 0)) \\ &= \varepsilon[P'(0)\phi(x, 0) + \mu_1\phi(x, 0)] - P(\varepsilon\phi(x, 0)) \\ &= P'(0)\varepsilon\phi(x, 0) - P(\varepsilon\phi(x, 0)) + \varepsilon\mu_1\phi(x, 0). \end{aligned}$$

Note that $P(0)=0$ and $P(y)$ is second order continuously differentiable in $y \in [0,+\infty)$, so using Taylor expansion, we have

$$P(\varepsilon\phi(x,0)) = P'(0)\varepsilon\phi(x,0) + 1/2P''(\xi)\varepsilon^2\phi^2(x,0), \xi \in [0,\varepsilon],$$

therefore,

$$S(x,0^+) - P(S(x,0)) = \varepsilon\mu_1\phi(x,0) - 1/2P''(\psi)\varepsilon^2\phi^2(x,0) < 0$$

as long as ε is small enough. So $\underline{S} = \varepsilon\phi^*$ is the lower solution to problem (2).

Next, we look for an upper solution to problem (2), which satisfies

$$\begin{cases} \overline{S}_t \geq D\overline{S}_{xx} - \alpha\overline{S}_x + [g(W(\overline{S},0,x,t)) - d(x) - m\overline{S}]\overline{S} & x \in (0,L_0),\ t \in (0^+,T], \\ D\overline{S}_x - \alpha\overline{S} \geq 0, & x = 0, L_0,\ t \in (0^+,T], \\ \overline{S}(x,0) \geq \overline{S}(x,T), & x \in (0,L_0), \\ \overline{S}(x,0^+) \geq P(\overline{S}(x,0)), & x \in (0,L_0). \end{cases}$$

Choose $\bar{S} = M\phi$, with (λ_1,ϕ) meets (3), and then

$$\begin{aligned} &\bar{S}_t - [D\bar{S}_{xx} - \alpha\bar{S}_x + g(w(\bar{S},0,x,t))\bar{S} - d(x)\bar{S} - m\bar{S}^2] \\ &= M(D\phi_{xx} - \alpha\phi_x + [g(w(0,0,x,t)) - d(x) - m\phi]\phi + \lambda_1\phi) \\ &- M[D\phi_{xx} - \alpha\phi_x + g(w(M\phi,0,x,t))\phi - d(x)\phi - mM\phi^2] \\ &= M\phi[\lambda_1 + mM\phi + g(w(0,0,x,t)) - g(w(M\phi,0,x,t))] \\ &\geq M\phi[\lambda_1 + mM\phi] \geq 0 \end{aligned}$$

as long as M is big enough.

On the other hand, $P(v) \leq P'(0)v$ can be used to derive that

$$\begin{aligned} &\overline{S}(0^+,x) - P(\overline{S}(0,x)) \\ &= M\phi(0^+,x) - P(M\phi(0,x)) \\ &= MP'(0)\phi(x,0) - P(M\phi(0,x)) \\ &\geq 0. \end{aligned}$$

Therefore, $\bar{S} = M\phi$ is the upper solution. According to the method of upper and lower solutions, Problem (2) has at least one positive solution satisfying $\underline{S} \leq S^* \leq \bar{S}$.

Now we prove the uniqueness of the solution to problem (2). If not, we assume that problem (2) has two positive solutions S_1 and S_2. Let

$$A = \{a \in [0,1], S_2 \geq aS_1, t \in [0,T], x \in [0,L_0]\},$$

we declare that $1 \in A$. Assume on the contrary $a_0 = \sup A < 1$, denote $u = S_2 - a_0 S_1$, then we obtain $u \geq 0$, $Du_x - \alpha u = 0$ for any $x = 0, L_0$, $0 \leq t \leq T$ and $u(x,0) = u(x,T)$ for $0 < x < L_0$. By direct calculation, we also get

$$\begin{aligned}
&u_t - Du_{xx} + \alpha u_x + d(x)u + u \\
&= \mathrm{g}(w(S_2,0,x,t))S_2 - a_0\mathrm{g}(w(S_1,0,x,t))S_1 - mS_2{}^2 + a_0 mS_1{}^2 \\
&\geq g(w(S_2,0,x,t))S_2 - a_0 g(w(S_1,0,x,t))S_1 - mS_2{}^2 + a_0 mS_1{}^2 \\
&\geq g(w(S_2,0,x,t)) - g(w(S_1,0,x,t))]a_0 S_1 - m(S_2 + a_0 S_1)u \\
&\geq g(w(S_2,0,x,t)) - g(w(S_1,0,x,t))]a_0 S_1 - m(S_2 + a_0 S_1)u \\
&= \tfrac{g(w(S_2,0,x,t)) - g(w(a_0 S_1,0,x,t))}{u} a_0 S_1 u - m(S_2 + a_0 S_1)u \\
&= F(S_1, S_2, x, t)u.
\end{aligned}$$

Since g is a smooth function, $\frac{g(w(S_2,0,x,t)) - g(w(a_0 S_1,0,x,t))}{u}$ is bounded, so $F(S_1,S_2,x,t)$ is also bounded, denoted by M_1. Due to the strong maximum principle, $S_2 - a_0 S_1 > 0$ or $S_2 - a_0 S_1 \equiv 0$ is obtained for $0 < x < L_0$, $0 \leq t \leq T$. Now we derive contradictions in the following two situations:

(1) If $S_2 - a_0 S_1 > 0$ for $0 < x < L_0$, $0 \leq t \leq T$. A positive constant ε can be given such that $S_2 - a_0 S_1 > \varepsilon S_1$, so $a_0 + \varepsilon \in S$, which is in contradiction with the definition of a_0.

(2) If $S_2 - a_0 S_1 \equiv 0$. Since

$$(S_1)_t = D(S_1)_{xx} - \alpha (S_1)_x + g(w(S_1,0,x,t))S_1 - d(x)S_1 - mS_1^2$$

and

$$(S_2)_t = D(S_2)_{xx} - \alpha (S_2)_x + g(w(S_2,0,x,t))S_2 - d(x)S_2 - mS_2^2,$$

substituting $S_2 \equiv a_0 S_1$ into the equation of S_2 and multiplying both sides of equation S_1 by a_0, we obtain

$$g(w(a_0S_1,0,x,t))a_0S_1 - g(w(S_1,0,x,t))a_0S_1 - ma_0(a_0-1)S_1^2 = 0 \tag{8}$$

by subtracting these two equations. However, $g(w(S,0,x,t))$ is decreasing with respect to S and equation (8) cannot hold.

As a conclusion, $1 \in A$ and $S_1 = S_2$, the proof of existence and uniqueness of periodic solution to problem (2) is now complete.

(ii) For simplicity of calculation, we only prove the case of $\alpha = 0$. In contrast, assume that there is a positive solution $v(x,t)$ to problem (2), and satisfies

$$\begin{cases} v_t - Dv_{xx} = [g(w(v,0,x,t)) - d(x) - mv]v, & x \in (0,L_0),\ t \in (0^+,T], \\ v_x(x,t) = 0, & x = 0, L_0, t \in (0^+,T], \\ v(x,0) = v(x,T), & x \in (0,L_0), \\ v(x,0^+) = P(v(x,0)), & x \in (0,L_0). \end{cases} \tag{9}$$

Firstly, let $\psi(x,t) = \phi(x,T-t)$ in problem (5), and then $\psi(x,t)$ satisfies

$$\begin{cases} -\psi_t - D\psi_{xx} = [g(w(0,0,x,T-t)) - d(x)]\psi + \mu_1\psi, & x \in (0,L_0),\ t \in [0,T^-), \\ \psi_x(x,t) = 0, & x = 0, L_0, t \in (0,T^-), \\ \psi(x,0) = \psi(x,T), & x \in (0,L_0), \\ \psi(x,T^-) = P'(0)\psi(x,T), & x \in (0,L_0). \end{cases} \tag{10}$$

Multiplying the first equation in (9) by ψ to get

$$v_t\psi - Dv_{xx}\psi = [g(w(v,0,x,t)) - d(x) - mv]v\psi$$

and multiplying the equation in (10) by v to get

$$-v\psi_t - Dv\psi_{xx} = [g(w(0,0,x,T-t)) - d(x)]v\psi + \mu_1 v\psi(x,t).$$

Subtracting the above two equations, yields

$$(v\psi)_t = -Dv_{xx}\psi + Dv\psi_{xx} + [g(w(v,0,x,t)) - g(w(0,0,x,T-t)) - mv]\psi v - \mu_1\psi v.$$

Next, we integrate with respect to t and x over $[0^+, T^-] \times [0, l]$ to obtain

$$\int_0^l (v\psi)_t|_{0^+}^{T^-} dx = \int_{0^+}^{T^-} \int_0^l [g(w(v,0,x,t)) - g(w(0,0,x,T-t)) - mv - \mu_1]\psi v dxdt.$$

Owning to

$$\psi(x, T^-) = P'(0)\psi(x,T),\ v(x, 0^+) = P(v(x,0)),$$

we derive that

$$(v\psi)_t|_{0^+}^{T^-} = (v\psi)(x,T^-) - (v\psi)(x,0^+) = v(x,T)P'(0)\psi(x,T) - \psi(x,0)P(v(x,0)),$$

which means

$$\begin{aligned}
&\textstyle\int_{0^+}^{T^-} \int_0^l \mu_1 \psi \nu dxdt \\
&= -\textstyle\int_0^l [v(x,T)P'(0)\psi(T,x) - \psi(0,x)P(v(0,x))]dx \\
&+ \textstyle\int_{0^+}^{T^-} \int_0^l [g(w(v,0,x,t)) - g(w(0,0,x,T-t)) - mv]\psi v dxdt.
\end{aligned}$$

Recalling that

$$w(v,0,x,t) = w_0 \exp(-k_0 x - \int_0^x kv(y,t)dy)$$

and

$$w(0,0,x,t) = w_0 \exp(-k_0 x),$$

we have

$$\int_{0^+}^{T^-} \int_0^l [g(w(v,0,x,t)) - g(w(0,0,x,t-T))]dxdt < 0.$$

Recalling to the fact $p(v(0,x)) \leq p'(0)v(x,T)$ yields

$$-\int_0^l [v(x,T)P'(0)\psi(T,x) - \psi(0,x)P(v(0,x)]dx = -\int_0^l \psi(0,x)[v(x,T)P'(0) - P(v(0,x)]dx \leq 0.$$

Moreover, noticing that $\iint -m\psi v^2 < 0$, we then have $\int_{0^+}^{T^-} \int_0^l \mu_1 \psi \nu dxdt < 0$, which is in contradiction with the condition $\mu_1 \geq 0$. □

4. THE DISEASE-FREE POSITIVE PERIODIC SOLUTION

Returning to problem (1), its periodic solution satisfies

$$
(11)\quad \begin{cases}
S_t^\Delta = DS_{xx}^\Delta - \alpha S_x^\Delta + [g(w(S^\Delta, I^\Delta, x, t)) - d(x) - mS^\Delta]S^\Delta \\
\qquad -\beta \frac{S^\Delta I^\Delta}{S^\Delta + I^\Delta} + \gamma I^\Delta, & x \in (0, L_0),\ t \in (0^+, T], \\
I_t^\Delta = DI_{xx}^\Delta - \alpha I_x^\Delta + [g(w(S^\Delta, I^\Delta, x, t)) - d(x) - mI^\Delta]I^\Delta \\
\qquad -[\gamma(x) + \delta(x)]I^\Delta + \beta(x) \frac{S^\Delta I^\Delta}{S^\Delta + I^\Delta}, & x \in (0, L_0),\ t \in (0^+, T], \\
DS_x^\Delta - \alpha S^\Delta = 0, DI_x^\Delta - \alpha I^\Delta = 0, & x = 0, L_0,\ t > 0, \\
S^\Delta(x, 0) = S^\Delta(x, T), I^\Delta(x, 0) = I^\Delta(x, T), & x \in (0, L_0), \\
S^\Delta(x, 0^+) = P(S^\Delta(x, 0)), & x \in (0, L_0), \\
I^\Delta(x, 0^+) = P(I^\Delta(x, 0)), & x \in (0, L_0).
\end{cases}
$$

In order to determine its disease-free positive and periodical solution, we first consider the equation of I. Notice that $I \le \bar{I}$, where $\bar{I}$ satisfies

$$
(12)\quad \begin{cases}
\bar{I}_t = D\bar{I}_{xx} - \alpha \bar{I}_x + [g(w(0, \bar{I}, x, t)) - d(x) - m\bar{I} - \delta(x)]\bar{I} \\
\qquad +[\beta(x) - \gamma(x)]\bar{I}, & x \in (0, L_0),\ t \in (0^+, T], \\
D\bar{I}_x - \alpha \bar{I} = 0, & x = 0, L_0,\ t > 0, \\
\bar{I}(x, 0) = \bar{I}_0(x), & x \in (0, L_0), \\
\bar{I}(x, 0^+) = P(\bar{I}(x, 0)), & x \in (0, L_0).
\end{cases}
$$

Similarly, we get its corresponding periodic problem

$$
(13)\quad \begin{cases}
I_t^* = DI_{xx}^* - \alpha I_x^* + [g(w(0, I^*, x, t)) - d(x) - mI^* - \delta(x)]I^* \\
\qquad +[\beta(x) - \gamma(x)]I^*, & x \in (0, L_0),\ t \in (0^+, T], \\
DI_x^* - \alpha I^* = 0, & x = 0, L_0,\ t > 0, \\
I^*(x, 0) = I^*(x, T), & x \in (0, L_0), \\
I^*(x, 0^+) = P(I^*(x, 0)), & x \in (0, L_0).
\end{cases}
$$

After linearization at $I^* = 0$, the corresponding periodic eigenvalue problem is

$$\begin{cases} \phi_t = D\phi_{xx} - \alpha\phi_x + [G(x) - d(x) - \delta(x)]\phi & \\ \qquad\qquad +[\beta(x) - \gamma(x)]\phi + \kappa_1\phi, & x \in (0, L_0),\ t \in (0^+, T], \\ D\phi_x - \alpha\phi = 0, & x = 0, L_0,\ t > 0, \\ \phi(x,0) = \phi(x,T), & x \in (0, L_0), \\ \phi(x,0^+) = P'(0)\phi(x,0), & x \in (0, L_0), \end{cases} \tag{14}$$

where $G(x) = g(w_0 e^{-k_0 x})$.

The same as Theorems 2.1 and 3.1, we can prove the following two theorems and we omit the details here.

Theorem 4.1. *The principal eigenvalue of periodic eigenvalue problem* (14) *is*

$$\kappa_1 = \kappa_1^* - \frac{\ln P'(0)}{T},$$

where

$$\kappa_1^* = \min_{\psi \in W^{1,2}(0,L), \psi \neq 0} \frac{\int_0^L D e^{\frac{\alpha}{D}x} \psi_x^2 dx - \int_0^L [G(x) - d(x) - \delta(x) + \beta(x) - \gamma(x)] e^{\frac{\alpha}{D}x} \psi^2 dx}{\int_0^L e^{\frac{\alpha}{D}x} \psi^2 dx}$$

satisfies

$$\begin{cases} -D\psi_{xx} = -\alpha\psi_x + [G(x) - d(x) - \delta(x) + \beta(x) - \gamma(x)]\psi + \kappa_1^*\psi, & x \in (0, L_0), \\ D\psi_x - \alpha\psi = 0, & x = 0, L_0. \end{cases}$$

Theorem 4.2. *The following statements hold:*

(i) Assume that $\kappa_1 < 0$*, problem* (12) *has a unique positive and periodical solution.*

(ii) Assume that $\kappa_1 \geq 0$*, problem* (12) *admits no positive solution.*

Combining Theorem 3.1 with Theorem 4.2, we get the condition for the existence of disease-free equilibrium periodically solution $(S^*(x,t), 0)$ of problem (1).

Theorem 4.3. *If* $\lambda_1 < 0, \kappa_1 \geq 0$*, problem* (1) *has a unique disease-free periodic solution* $(S^*(x,t), 0)$.

Proof: When $\kappa_1 \geq 0$, we know from Theorem 4.2 that problem (12) has no positive solution, So problem (11) becomes (2). Further, we know from $\lambda_1 < 0$ and Theorem 3.1(i) that problem (2) has a unique positive periodic solution $S^*(x,t)$, thus problem (11) has a unique disease-free periodic solution $(S^*(x,t),0)$ and the endemic periodical solution does not exist. □

5. The Stability of Disease-Free Periodic Solution

In this section, we will discuss the asymptotic behaviors of solution to problem (1). Firstly, it is known that I has an upper solution $\overline{I}$, which satisfies (12), so we draw a conclusion as follows.

Theorem 5.1. *Suppose that* $\kappa_1 > 0$. *For any solution* $\overline{I}(x,t)$ *to problem* (12)*, we have*

$$\lim_{t\to\infty} \overline{I}(x,t) = 0$$

for $x \in [0, L_0]$.

Proof: Let $\tilde{I}(x,t) = Me^{-\lambda t}\phi(x,t)$, where $\psi(y,t)$ is the eigenfunction of periodic eigenvalue problem (14) corresponding to the principal eigenvalue κ_1, and λ is chosen to satisfy

$$0 < \lambda < \kappa_1.$$

A direct calculation yields

$$\begin{aligned}
&\tilde{I}_{\mathrm{t}} - D\tilde{I}_{xx} + \alpha\tilde{I}_x - [g(w(0,\tilde{I},x,t)) - d(x) - m\tilde{I} - \delta(x) + \beta(x)]\tilde{I} \\
&\geq \tilde{I}_t - D\tilde{I}_{xx} + \alpha\tilde{I}_x - [g(w(0,0,x,t)) - d(x) - m\tilde{I} - \delta(x) + \beta(x)]\tilde{I} \\
&= Me^{-\lambda t}\phi_t - \lambda Me^{-\lambda t}\phi - DMe^{-\lambda t}\phi_{xx} + \alpha Me^{-\lambda t}\phi_x \\
&-[g(w(0,0,x,t)) - d(x) - mMe^{-\lambda t}\phi(x,t) - \delta(x) + \beta(x)]Me^{-\lambda t}\phi(x,t) \\
&= Me^{-\lambda t}\phi[-\lambda + mMe^{-\lambda t}\phi(x,t) + \kappa_1] \\
&= \tilde{I}[-\lambda + m\tilde{I} + \kappa_1] \\
&\geq 0
\end{aligned}$$

for $(x,t) \in [0, L_0] \times [0, T]$, and

$$D\tilde{I}_x - \alpha\tilde{I} = 0,\ x = 0, L_0, t > 0,$$

$$\tilde{I}(x,0) = \tilde{I}_0(x) \geq 0,\ 0 < x < L_0,$$

$$\tilde{I}(x,(nT)^+) = P(\tilde{I}(x,nT),\ 0 < x < L_0.$$

Therefore, when M is sufficiently big, $\tilde{I}(x,t)$ is an upper solution to problem (12). Since $\lim_{t\to\infty} \tilde{I}(x,t) = 0$, we have $\lim_{t\to\infty} \bar{I}(x,t) = 0$ holds uniformly for $x \in [0,L_0]$. □

In the following, we exhibit the long-time behaviors of solution to problem (1).

Theorem 5.2. *When $\lambda_1 < 0$ and $\kappa_1 > 0$, the disease-free equilibrium periodically solution $(S^*(x,t),0)$ is globally asymptotically stable, that is, any non-negative and non-trivial solution to problem* (1) *satisfies*

$$\lim_{t\to\infty} I(x,t) = 0$$

uniformly for $x \in [0,L_0]$, and

$$\lim_{m\to\infty} S(x,t+mT) = S^*(x,t)$$

uniformly for $(x,t) \in [0,L_0]\times[0,+\infty)$.

Proof: Obviously I has an upper solution $\bar{I}$, which satisfies (12). When $\kappa_1 > 0$, the solution of (12) converges to 0 uniformly according to Theorem 5.1, so I converges to 0 when $t \to 0$ uniformly for $x \in [0,L_0]$. Therefore, for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that $0 \le I \le \varepsilon$ for all $(x,t) \in [0,L_0]\times[T_\varepsilon,+\infty)$. We now take an integer M_ε such that $M_\varepsilon T > T_\varepsilon > 0$. Now we regard the pulse point $M_\varepsilon T$ as a new starting point.

Next we will consider the asymptotic behavior of S, for which we construct the upper and lower solutions. It is easy to see

$$\begin{aligned} S_t &= DS_{xx} - \alpha S_x + [\mathrm{g}(w(S,I,x,t)) - d(x) - mS]S - \beta(x)\tfrac{SI}{S+I} + \gamma(x)I \\ &\le DS_{xx} - \alpha S_x + [\mathrm{g}(w(S,0,x,t)) - d(x) - mS]S + \gamma^*\varepsilon \end{aligned}$$

and

$$\begin{aligned} S_t - DS_{xx} + \alpha S_x &= [\mathrm{g}(w(S,I,x,t)) - d(x) - mS]S - \beta(x)\tfrac{SI}{S+I} \\ &\ge [g(w(S,0,x,t)) + \tfrac{g(w(S,\varepsilon,x,t)) - g(w(S,0,x,t))}{\varepsilon}\varepsilon - d(x)]S - M_2\varepsilon S \\ &\ge [g(w(S,0,x,t)) - M_2\varepsilon - d(x)]S - M_3\varepsilon \\ &= [g(w(S,0,x,t)) - d(x)]S - M\varepsilon S, \end{aligned}$$

where $\gamma(x)$, $\beta(x)$ and $\frac{g(w(S,\varepsilon,x,t))-g(w(S,0,x,t))}{\varepsilon}$ are both bounded functions, so there exist γ^*, M_2, M_3 and $M = M_2 + M_3$ to satisfy the above formula, and then

$$S_t \geq DS_{xx} - \alpha S_x + [g(w(S,0,x,t)) - d(x)]S - M\varepsilon S.$$

We now assume that $\bar{S}_\varepsilon$ and $\underline{S}_\varepsilon$ are solutions to

$$\begin{cases} (\bar{S}_\varepsilon)_t = D(\bar{S}_\varepsilon)_{xx} - \alpha(\bar{S}_\varepsilon)_x + [g(w(\bar{S}_\varepsilon,0,x,t)) & \\ \qquad -d(x) - m\bar{S}_\varepsilon]\bar{S}_\varepsilon + \gamma^*\varepsilon, & x \in (0,L_0), t \in ((nT)^+,(n+1)T], \\ D(\bar{S}_\varepsilon)_x - \alpha\bar{S}_\varepsilon = 0, & x = 0, L_0, t \in ((nT)^+,(n+1)T], \\ \bar{S}_\varepsilon(x, M_\varepsilon T) = S(x, M_\varepsilon T), & x \in (0,L_0), \\ \bar{S}_\varepsilon(x,(nT)^+) = P(\bar{S}_\varepsilon(x,nT)),\ n \geq M_\varepsilon, & x \in (0,L_0) \end{cases}$$

and

$$\begin{cases} (\underline{S}_\varepsilon)_t = D(\underline{S}_\varepsilon)_{xx} - \alpha(\underline{S}_\varepsilon)_x - M\varepsilon\underline{S}_\varepsilon & \\ \qquad +[g(w(\underline{S}_\varepsilon,0,x,t)) - d(x)]\underline{S}_\varepsilon - M\varepsilon\underline{S}_\varepsilon, & x \in (0,L_0), t \in ((nT)^+,(n+1)T], \\ D(\underline{S}_\varepsilon)_x - \alpha\underline{S}_\varepsilon = 0, & x = 0, L_0, t \in ((nT)^+,(n+1)T], \\ \underline{S}_\varepsilon(x, M_\varepsilon T) = S(x, M_\varepsilon T), & x \in (0,L_0), \\ \underline{S}_\varepsilon(x,(nT)^+) = P(\underline{S}_\varepsilon(x,nT)),\ n \geq M_\varepsilon, & x \in (0,L_0), \end{cases}$$

respectively.

It is easy to know $\underline{S}_\varepsilon \leq S(x,t)) \leq \bar{S}_\varepsilon$. Taking the sequences $\bar{S}_\varepsilon^{(m)}$ and ${\underline{S}_\varepsilon}^{(m)}$ to be the largest and smallest sequences that satisfy the following problem

$$(15)\quad \begin{cases} (\bar{S}_\varepsilon^{(m)})_t - D(\bar{S}_\varepsilon^{(m)})_{xx} + \alpha(\bar{S}_\varepsilon^{(m)})_x + K_1\bar{S}_\varepsilon^{(m)} = g_1(\bar{S}_\varepsilon^{(m-1)}), & x \in (0,L_0), t \in ((nT)^+,(n+1)T], \\ (\underline{S}_\varepsilon^{(m)})_t - D(\underline{S}_\varepsilon^{(m)})_{xx} + \alpha(\underline{S}_\varepsilon^{(m)})_x + K_2\underline{S}_\varepsilon^{(m)} = g_2(\underline{S}_\varepsilon^{(m-1)}), & x \in (0,L_0), t \in ((nT)^+,(n+1)T], \\ D(\bar{S}_\varepsilon^{(m)}(x,t))_y - \alpha\bar{S}_\varepsilon^{(m)}(x,t) = 0, & x = 0, L_0, t \in ((nT)^+,(n+1)T], \\ D(\underline{S}_\varepsilon^{(m)}(x,t))_y - \alpha\underline{S}_\varepsilon^{(m)}(x,t) = 0, & x = 0, L_0, t \in ((nT)^+,(n+1)T], \\ \bar{S}_\varepsilon^{(m)}(x, M_\varepsilon T) = \bar{S}_\varepsilon^{(m-1)}(x,(M_\varepsilon+1)T), & x \in (0,L_0), \\ \underline{S}_\varepsilon^{(m)}(x, M_\varepsilon T) = \underline{S}_\varepsilon^{(m-1)}(x,(M_\varepsilon+1)T), & x \in (0,L_0), \\ \bar{S}_\varepsilon^{(m)}(x,(nT)^+) = P(\bar{S}_\varepsilon^{(m)}(x,nT)),\ n \geq M_\varepsilon, & x \in (0,L_0), \\ {\underline{S}_\varepsilon}^{(m)}(x,(nT)^+) = P({\underline{S}_\varepsilon}^{(m)}(x,nT)),\ n \geq M_\varepsilon, & x \in (0,L_0), \end{cases}$$

where $m = 1,2,\cdots$. Initial iterations $\bar{S}_\varepsilon^{(0)} = \bar{S}_\varepsilon$ and $\underline{S}_\varepsilon^{(0)} = \underline{S}_\varepsilon$ will be chosen and

$$g_1(S) = [g(w(S,0,x,t)) - d(x) - mS]S + \gamma^* \varepsilon + K_1 S,$$

$$g_2(S) = [g(w(S,0,x,t)) - d(x)]S - M\varepsilon S + K_2 S,$$

$$K_1 = \sup_{x\in[0,L_0]} \{d(x) + \beta(x)\} + m \sup_{x\in[0,L_0]} ||S_0 + I_0||,\ K_2 = K_1 + M.$$

According to [16] (Lemma 3.1), sequences $\bar{S}^{(m)}$ and $\underline{S}_\varepsilon^{(m)}$ have monotonicity, that is $\underline{S}_\varepsilon \leq \underline{S}_\varepsilon^{m-1} \leq \underline{S}_\varepsilon^m \leq \bar{S}^{(m)} \leq \bar{S}^{(m-1)} \leq \bar{S}$. Furthermore, let

$$\lim_{t\to\infty} \underline{S}_\varepsilon^m = \underline{S}_\varepsilon^*,\ \lim_{t\to\infty} \bar{S}^{(m)} = \bar{S}^*,$$

we obtain

$$\underline{S}_\varepsilon \leq \underline{S}_\varepsilon^{m-1} \leq \underline{S}_\varepsilon^m \leq \underline{S}_\varepsilon^* \leq \bar{S}_\varepsilon^* \leq \bar{S}_\varepsilon^{(m)} \leq \bar{S}_\varepsilon^{(m-1)} \leq \bar{S}_\varepsilon.$$

Denote $S_m(x,t) = S(x,t+mT)$, we have $\underline{S}_\varepsilon(x,t+T) < S_1(x,t) < \bar{S}_\varepsilon(x,t+T)$ for $0 < x < L_0, t > M_\varepsilon T$.

According to initial condition in problem (15) with $m = 1$, we have

$$\bar{S}^{(1)}(x,M_\varepsilon T) = \bar{S}^{(0)}(x,(M_\varepsilon+1)T) = \bar{S}(x,(M_\varepsilon+1)T)$$

and

$$\underline{S}_\varepsilon^{(1)}(x,M_\varepsilon T) = \underline{S}_\varepsilon^{(0)}(x,(M_\varepsilon+1)T) = \underline{S}_\varepsilon(x,(M_\varepsilon+1)T).$$

Moreover,

$$\underline{S}_\varepsilon^{(1)}(x,M_\varepsilon T) \leq S_1(x,M_\varepsilon T) \leq \bar{S}^{(1)}(x,M_\varepsilon T).$$

holds for any $0 < x < L_0$.

Therefore, by the comparison principal, we obtain

$$\underline{S}_\varepsilon^{(1)}(x,t) \leq S_1(x,t) \leq \bar{S}^{(1)}(x,t),\ 0 < x < L_0, t > M_\varepsilon T$$

and induction shows

$$\underline{S}_\varepsilon^{(m)}(x,t) \leq S_m(x,t) \leq \bar{S}^{(m)}(x,t),\ 0 < x < L_0, t > M_\varepsilon T.$$

Finally, we have

$$\liminf_{m\to\infty} \underline{S}_\varepsilon^{(m)}(x,t) \leq \liminf_{m\to\infty} S_m(x,t) \leq \limsup_{m\to\infty} S_m(x,t) \leq \limsup_{m\to\infty} \bar{S}^{(m)}(x,t) \tag{16}$$

for $(x,t)\in[0,L_0]\times[M_\varepsilon T,+\infty)$.

On the other hand, for any $(x,t)\in[0,L_0]\times[M_\varepsilon T,+\infty)$, there holds

$$\lim_{m\to\infty}\underline{S}_\varepsilon^{(m)}(x,t)=\underline{S}_\varepsilon^*(x,t),\ \lim_{m\to\infty}\bar{S}^{(m)}(x,t)=\bar{S}^*(x,t),$$

where $\underline{S}_\varepsilon^*(x,t)$ meets

$$\begin{cases} S_t=DS_{xx}-\alpha S_x+[g(w(S,0,x,t))-d(x)-mS]S+\gamma^*\varepsilon, & x\in(0,L_0),\ t>M_\varepsilon T,\\ DS_x-\alpha S=0, & x=0,L_0,\ t>M_\varepsilon T,\\ S(x,M_\varepsilon T)=S(x,(M_\varepsilon+1)T), & x\in(0,L_0),\\ S(x,(M_\varepsilon T)^+)=P(S(x,M_\varepsilon T)), & x\in(0,L_0) \end{cases}$$

and $\bar{S}^*(x,t)$ satisfies

$$\begin{cases} S_t=DS_{xx}-\alpha S_x+[g(w(S,0,x,t))-d(x)]S-M\varepsilon S, & x\in(0,L_0),\ t>M_\varepsilon T,\\ DS_x-\alpha S=0, & x=0,L_0,\ t>M_\varepsilon T,\\ S(x,M_\varepsilon T)=S(x,(M_\varepsilon+1)T), & x\in(0,L_0),\\ S(x,(M_\varepsilon T)^+)=P(S(x,M_\varepsilon T)), & x\in(0,L_0). \end{cases}$$

According to continuous dependence of the solution to coefficients, we obtain $\lim_{\varepsilon\to0^+}\underline{S}_\varepsilon^*(x,t)=\lim_{\varepsilon\to0^+}\bar{S}_\varepsilon^*(x,t)=S^*(x,t)$, which combines with (16) yields

$$\lim_{m\to\infty}S(x,t+mT)=S^*(x,t)$$

for $(x,t)\in[0,L_0]\times[0,+\infty)$. □

It is easy to check that the conditions of Theorem 5.2 are satisfied when the natural mortality rate d of phytoplankton populations is small enough, or the additional mortality rate δ of the infected part is large enough, or the recovery rate γ is large enough. The above result tells us that under these conditions, the infected phytoplankton will gradually disappear, while the uninfected phytoplankton will persist and exhibit periodic oscillations.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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