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ROLE OF REFUGE ON DYNAMICS OF PREY-PREDATOR MODEL WITH INFECTED PREY

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Abstract. The paper deals with a prey predator model with infection and refuge in prey. It is assumed that the predator is predating on both susceptible and infected prey. We have considered two models, one without refuge and another with refuge. We have studied local stability, bifurcation and permanence of both models. Further, the dynamics of the system is studied numerically on the basis of interaction rate of prey - predator (β) and refuge size (m). The study reveals that refuge sizes of susceptible and infected prey are key parameters that control the dynamics of the system.

Keywords: refuge; infection; Hopf-bifurcation; stability; permanence.

2010 AMS Subject Classification: 15A18, 92D25, 92D30.

1. INTRODUCTION

Eco-epidemiology deals with dynamics of disease spread in interacting species. The dynamic relationship between interactive species is one of the dominant themes of mathematical ecology [1]. The problems are often very complicated and challenging, although they seem to be simple at first sight. Interactive species can be prey-predator, competitive species, etc. Many researchers have studied problems related to such eco-epidemiological issues [2, 3, 4, 5, 6, 7, 8,

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9, 10, 11, 12, 13]. A prey-predator model with prey infection was investigated by Anderson and May [2] who observed destabilization due to the spread of infectious diseases within animal and plant communities. A comparative study on the role of prey infection in a prey-predator system with several functional responses was studied by [13].

Modeling with differential equations for interactive species is a classical application of Mathematical biology. Theoretical observations on dynamics of prey refuges have two major influences on prey predator models. They can either stabilize or destabilize the system [13, 14, 15, 16, 17, 18, 19, 20]. A continuous time prey-predator model assuming the rate of prey moving to refuges is proportional to predator density was studied by [16]. The author showed that prey refuge has stabilizing effect on the dynamics of the system. Also, Wang and Jianzhong Wang [23], studied a diffusive prey-predator system with Michaelis-Menten functional response subject to prey refuge. Olivares and Jiliberto [24] studied the dynamic consequences of the simplest forms of refuge used by the prey by an analytical approach in which they incorporated prey refuges in a widely known continuous model satisfying the principle of biomass conversion. Further Ma. et al. [21] observed stabilizing and destabilizing effect in a prey-predator model with the functional response incorporating effect of prey refuges.

In this paper, we have studied the extension of the above models by incorporating infection in prey population. Also, we have assumed interaction between susceptible prey-infected prey can be a bilinear mass action called as Holling Type I. This term assumes that an individual interact with the whole other population and the product of both is the outcome. We also consider the fact that infected prey is more vulnerable to predation in comparison to susceptible prey. But the interactions between susceptible prey and infected prey with predator are technically by Holling type II term.

The paper is organized as follows: In section 2, we give the formulation of infected prey-predator with and without refuge. Boundedness of the above model is discussed in section 3. The existence of equilibrium points and local stability of boundary equilibrium points of both the models have been discussed in section 4 and 5 respectively. In section 6, we have obtained the local stability of interior equilibrium point and bifurcation around refuge m. Permanence of models is discussed in section 7 followed by numerical simulation and conclusion in section 8 and 9 respectively.

Biological Background: Refugia can influence population dynamics to great extent. Ecological effects of refuges can be seen in Coral reefs. Nearly 25 percent of ocean species is contained in refuge-rich coral reefs. In order to minimize their chances of being caught by predator, many prey animals systematically migrate between refuges and predator-rich feeding grounds, for example, small European perch exhibit a daily horizontal migration in some lakes in Finland. They move away from the vegetated areas, into more turbid open water areas, during the day as predation threat in the clear water is great, moving back at night because of the greater availability of zooplankton among the aquatic plants. Keeping in view the above points we develop our model in the next section.

2. MATHEMATICAL MODEL

In this section, we discuss the development of mathematical models describing infected prey-predator model with and without refuge.

Model A: Infected Prey-Predator Model Without Refuge

(1) The susceptible prey population is growing logistically at the rate r with carrying capacity K. It is assumed that the interaction between susceptible prey and infected prey follows bilinear mass action called as Holling Type I. This term assumes that an individual interacts with whole other population and the product of both is the outcome. Also, the interaction between susceptible prey and predator follows Holling type II term, which assumes that processing of food and searching for food are mutually exclusive behaviors. Thus, the equation is as follows:

$$\frac{dS}{d\tau} = rS(1-\frac{S}{K}) - \sigma SI - \frac{\beta_1 SP}{1+a_1 S} \tag{1}$$

where, β_1 is the rate of interaction of susceptible prey and predator, σ is infection rate and $1/a_1$ is the density of susceptible prey necessary to achieve one half of that rate.

(2) The evolutionary equation for infected prey is as follows:

$$\frac{dI}{d\tau} = \sigma SI - \frac{\beta_2 IP}{1+a_2 I} - \gamma_1 I \tag{2}$$

where, β_2 is the rate of interaction of infected prey and predator, γ_1 is the natural death rate of infected prey and $1/a_2$ is the density of infected prey necessary to achieve one half of that rate.

(3) The equation for predator population is given as follows:

$$\frac{dP}{d\tau} = \frac{c_1\beta_1 SP}{1+a_1 S} + \frac{c_2\beta_2 IP}{1+a_2 I} - \gamma_2 P \tag{3}$$

where $0 < c_1 < 1$ is the conversion factor denoting the number of newly born predators for each captured susceptible prey, $0 < c_2 < 1$ is the conversion factor denoting the number of newly born predators for each captured infected prey, γ_2 is the death rate of predator population. Further, $c_1 < c_2$ as infected prey are more vulnerable to predation.

Model B: Infected Prey-Predator Model With Refuge

Now, we formulate infected prey-predator model with refuge. In model B, we incorporate refuge in susceptible and infected prey, protecting mS of susceptible prey, and mI of infected prey, where $m \in [0,1)$ and is constant. Our aim is to study dynamics of the model under the effect of refuge and infection in prey. We have discussed model with different refuge rates for susceptible and infected prey, in brief in Appendix. Results obtained for the case when refuge rate in susceptible prey is not same as refuge rate in infected prey are not much different from the case when refuge rate is same in both susceptible and infected prey. Thus, to reduce the complexity, we consider same refuge rate. The model with same refuge rate is as follows:

$$\frac{dS}{d\tau} = rS(1-\frac{S}{K}) - \sigma SI - \frac{\beta_1(1-m)SP}{1+a_1(1-m)S} \tag{4}$$

$$\frac{dI}{d\tau} = \sigma SI - \frac{\beta_2(1-m)IP}{1+a_2(1-m)I} - \gamma_1 I \tag{5}$$

$$\frac{dP}{d\tau} = \frac{c_1\beta_1(1-m)SP}{1+a_1(1-m)S} + \frac{c_2\beta_2(1-m)IP}{1+a_2(1-m)I} - \gamma_2 P \tag{6}$$

In order to simplify the calculations, we now non-dimensionalise Model A by redefining the variables.

$$s = \frac{S}{K}, i = \frac{I}{K}, p = \frac{P}{K}, t = r\tau$$

Hence , equations in model A reduce to:

$$\frac{ds}{dt} = s(1-s) - asi - \frac{bsp}{s+c} \tag{7}$$

where $a = \frac{\sigma K}{r}, b = \frac{\beta_1}{ra_1}, c = \frac{1}{a_1 K}$

$$\frac{di}{dt} = asi - \frac{eip}{i+f} - gi \tag{8}$$

where $e = \frac{\beta_2}{ra_2}, f = \frac{1}{a_2 K}, g = \frac{\gamma_1}{r}$

$$\frac{dp}{dt} = \frac{c_1 bsp}{s+c} + \frac{c_2 eip}{i+f} - hp \tag{9}$$

where $h = \frac{\gamma_2}{r}$

Thus, the reduced model A is as follows:

$$\frac{ds}{dt} = s(1-s) - asi - \frac{bsp}{s+c} \tag{10}$$

$$\frac{di}{dt} = asi - \frac{eip}{i+f} - gi \tag{11}$$

$$\frac{dp}{dt} = \frac{c_1 bsp}{s+c} + \frac{c_2 eip}{i+f} - hp \tag{12}$$

Similarly, Model B is reduced to the following:

$$\frac{ds}{dt} = s(1-s) - asi - \frac{(1-m)bsp}{(1-m)s+c} \tag{13}$$

where, $a = \frac{\sigma K}{r}, b = \frac{\beta_1}{ra_1}, c = \frac{1}{a_1 K}$

$$\frac{di}{dt} = asi - \frac{eip(1-m)}{(1-m)i+f} - gi \tag{14}$$

where $e = \frac{\beta_2}{ra_2}, f = \frac{1}{a_2 K}, g = \frac{\gamma_1}{r}$

$$\frac{dp}{dt} = \frac{c_1 bsp(1-m)}{(1-m)s+c} + \frac{c_2 eip(1-m)}{(1-m)i+f} - hp \tag{15}$$

where $h = \frac{\gamma_2}{r}$

Hence, the system is reduced to:

$$\frac{ds}{dt} = s(1-s) - asi - \frac{(1-m)bsp}{(1-m)s+c} \tag{16}$$

$$\frac{di}{dt} = asi - \frac{eip(1-m)}{(1-m)i+f} - gi \tag{17}$$

$$\frac{dp}{dt} = \frac{c_1 bsp(1-m)}{(1-m)s+c} + \frac{c_2 eip(1-m)}{(1-m)i+f} - hp \tag{18}$$

3. BOUNDEDNESS

In this section, we will establish that Model A and Model B described above are bounded. Let $w = s + i + p$, then,

$$\begin{aligned}
\dot{w} = & \ \dot{s} + \dot{i} + \dot{p} \\
= & \ s(1-s) - \frac{bsp}{s+c}(1-c_1) - \frac{eip}{i+f}(1-c_2) - gi - hp \\
\leq & \ s(1-s) - gi - hp
\end{aligned}$$

Let $\mu = min\{g,h\}$. Now, we consider,

$$\begin{aligned}
\dot{w}+\mu w \leq & \ s(1-s)+\mu(s+i+p)-gi-hp \\
\leq & \ s-s^2+\mu s-(g-\mu)i-(h-\mu)p \\
\leq & \ s-s^2+\mu s \\
\leq & \ -s^2+(1+\mu)s+(\tfrac{1+\mu}{2})^2-(\tfrac{1+\mu}{2})^2 \\
\leq & \ -(s-(\tfrac{1+\mu}{2}))^2+(\tfrac{1+\mu}{2})^2 \\
\leq & \ (\tfrac{1+\mu}{2})^2 = C(Constant)
\end{aligned}$$

Hence, the system is bounded.

Boundedness of Model B will be on the same lines.

In the next section, we will discuss the existence of the boundary and interior equilibrium points of Model A and Model B.

4. EXISTENCE OF EQUILIBRIUM POINTS

Model A: We will consider four boundary equilibrium points:

(i) trivial equilibrium point $E_0(0,0,0)$

(ii) equilibrium point $E_1(1,0,0)$

(iii) predator extinction equilibrium point $E_2(\bar{s},\bar{i},0)$,where, $\bar{s} = \dfrac{g}{a}$ and $\bar{i} = \dfrac{(a-g)}{a^2}$ provided $a > g$ i.e., $R_0 > 1$ where, $R_0 = \dfrac{a}{g}$

(iv) disease-free equilibrium point $E_3(\tilde{s},0,\tilde{p})$

where, $\tilde{s} = \dfrac{hc}{c_1b-h}$ and $\tilde{p} = \dfrac{cc_1(c_1b-h-hc)}{(c_1b-h)^2}$ provided $c_1b-h > hc$ i.e $R_1 > 1$ where $R_1 = \dfrac{c_1b}{h(1+c)}$.

The interior equilibrium point $E^*(s^*,i^*,p^*)$ is given by the following equation:

$$Q_1x^3+Q_2x^2+Q_3x+Q_4 = 0$$

where,

$Q_1 = ah(c_1b+c_2e-h)$

$Q_2 = (c_2eg-gh)(bc_1-h)+(h-bc_1-hc)(c_1ae-c_2ae+ah)+(ec_1-fah)(bc_1a-bc_2a-ha)-$

$ec_1(bc_2g-hac)$

$Q_3=(bc_2g-hac)(ec_1-fah)+fgh(bc_1a-bc_2a-ah)+(h-bc_1-hc)(c_2eg-gh)+hc(c_1ae-c_2ae+ah)$

$Q_4=hc(c_2eg-gh)+fgh(bc_2g-hac)$

By Descrate's rule of sign change, interior equilibrium point exists if either of the two conditions hold:

(i) $Q_1,Q_3>0$ and $Q_2,Q_4<0$

(ii) $Q_1,Q_3<0$ and $Q_2,Q_4>0$

Model B: We will consider four boundary equilibrium points:

(i) trivial equilibrium point $E_0(0,0,0)$
(ii) equilibrium point $E_1(1,0,0)$
(iii) predator extinction equilibrium point $E_2(\bar{s},\bar{i},0)$,where, $\bar{s}=\dfrac{g}{a}$ and $\bar{i}=\dfrac{(a-g)}{a^2}$ provided $a>g$ i.e., $R_0>1$ where, $R_0=\dfrac{a}{g}$
(iv) disease-free equilibrium point $E_3(\tilde{s},0,\tilde{p})$

where, $\tilde{s}=\dfrac{hc}{(c_1b-h)(1-m)}$ and $\tilde{p}=\dfrac{((c_1b-h)(1-m)-hc))cc_1}{(c_1b-h)^2(1-m)^2}$ provided $(c_1b-h)(1-m)>hc$ i.e $\tilde{R}_1>1$ where $\tilde{R}_1=\dfrac{c_1b(1-m)}{h(1+c-m)}$.

and the interior equilibrium point $E^*(s^*,i^*,p^*)$ is given by the following equation:

$$Q_1x^3+Q_2x^2+Q_3x+Q_4=0$$

where,

$Q_1=(1-m)^2ah(c_1b+c_2e-h)$

$Q_2=(1-m)((c_2eg-gh)(bc_1-h)(1-m)+(h(1-m)-bc_1(1-m)-hc)(c_1ae-c_2ae+ah))+(1-m)(ec_1(1-m)-fah)(bc_1a-bc_2a-ha)-ec_1(1-m)(bc_2g(1-m)-hac)$

$Q_3=(bc_2g(1-m)-hac)(ec_1(1-m)-fah)+fgh(1-m)(bc_1a-bc_2a-ah)+(1-m)((h(1-m)-bc_1(1-m)-hc)(c_2eg-gh)+hc(c_1ae-c_2ae+ah))$

$Q_4 = (1-m)hc(c_2eg - gh) + fgh(bc_2g(1-m) - hac)$

By Descarte's rule of sign change, interior equilibrium point exists if either of the two conditions hold:

(i) $Q_1, Q_3 > 0$ and $Q_2, Q_4 < 0$

(ii) $Q_1, Q_3 < 0$ and $Q_2, Q_4 > 0$

5. DYNAMICAL BEHAVIOR

In this section, we discuss the local stability of boundary equilibrium points of Model A and Model B.

Theorem: Define $\bar{R}_1 = \frac{1}{h}\left(\frac{c_1bg}{g+ac} + \frac{c_2e(a-g)}{a-g+fa^2}\right)$, $R_1^* = \frac{ahcf(c_1b-h)}{ecc_1(c_1b-h-hc)+gf(c_1b-h)^2}$, and $R_2^* = \frac{(c_1b-h)c_1b}{2hcc_1b+(c_1b-h-hc)(c_1b-h)}$ then, for Model A we have the following:

(i) $E_0(0,0,0)$ is always unstable.

(ii) $E_1(1,0,0)$ is locally stable if $R_0 < 1$ and $R_1 < 1$ and unstable if $R_0 > 1$ or $R_1 > 1$.

(iii) If $R_0 > 1$ and $\bar{R}_1 < 1$, then predator extinction equilibrium point $E_2(\bar{s},\bar{i},0)$ is locally stable. If $\bar{R}_1 > 1$ then $E_2(\bar{s},\bar{i},0)$ is unstable.

(iv) If $R_1^* < 1$ and $R_2^* < 1$, then disease free equilibrium point $E_3(\tilde{s},0,\tilde{p})$ is locally asymptotically stable.

(v) If $R_1^* > 1$ or $R_2^* > 1$, then disease free equilibrium point $E_3(\tilde{s},0,\tilde{p})$ is unstable.

Proof. At any point $E_i(s,i,p)$, jacobian matrix formed for our system is given by

$$\begin{bmatrix} 1-2s-ai-\frac{bpc}{(s+c)^2} & -as & -\frac{bs}{s+c} \\ ai & as-\frac{epf}{(i+f)^2}-g & -\frac{ei}{i+f} \\ \frac{c_1bpc}{(s+c)^2} & \frac{c_2fep}{(i+f)^2} & \frac{c_1bs}{s+c}+\frac{c_2ei}{i+f}-h \end{bmatrix} \tag{19}$$

The jacobian corresponding to $E_0(0,0,0)$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -g & 0 \\ 0 & 0 & -h \end{bmatrix}$$

The corresponding eigenvalues with respect to the Jacobian are:

$\lambda_1 = 1$, $\lambda_2 = -g$ and $\lambda_3 = -h$. Since, one of the eigen values is always positive, therefore E_0 is always unstable.

The jacobian corresponding to E_1 is

$$\begin{bmatrix} -1 & -a & -\dfrac{b}{1+c} \\ 0 & a-g & 0 \\ 0 & 0 & \dfrac{c_1 b}{1+c} - h \end{bmatrix}$$

The corresponding eigenvalues with respect to the jacobian are:
$\lambda_1 = -1$, $\lambda_2 = a-g$ and $\lambda_3 = \dfrac{c_1 b}{1+c} - h$. Thus, E_1 is attractor and hence locally stable only if $a < g$ and $\dfrac{c_1 b}{1+c} < h$ i. e., $R_0 < 1$ and $R_1 < 1$, and unstable if $R_0 > 1$ or $R_1 > 1$. Thus, if $E_1(1,0,0)$ is locally stable then E_2 and E_3 does not exist.

The jacobian matrix for E_2 is given by:

$$\begin{bmatrix} -\dfrac{g}{a} & -g & -\dfrac{bg}{g+ca} \\ 1-\dfrac{g}{a} & 0 & -\dfrac{e(a-g)}{a-g+fa^2} \\ 0 & 0 & \dfrac{c_1 bg}{g+ac} + \dfrac{c_2 e(a-g)}{a-g+fa^2} - h \end{bmatrix}$$

The eigenvalues for the above matrix are:
$\dfrac{c_1 bg}{g+ac} + \dfrac{c_2 e(a-g)}{a-g+fa^2} - h$ and roots of equation $\lambda^2 + \dfrac{g}{a}\lambda + g(1-\dfrac{g}{a}) = 0$.

It can be easily seen that roots of the quadratic equation are negative as sum of roots is negative and product of roots is positive, when, $a > g$ .ie., $R_0 > 1$.

Also, $\frac{c_1bg}{g+ac}+\frac{c_2e(a-g)}{a-g+fa^2}-h$ is negative, when $\frac{c_1bg}{g+ac}+\frac{c_2e(a-g)}{a-g+fa^2}<h$, i.e., $\bar{R}_1<1$. Hence, E_2 is locally asymptotically stable, if $R_0>1$ and $\bar{R}_1<1$ and unstable if $\bar{R}_1>1$.

The jacobian corresponding to $E_3(\tilde{s},0,\tilde{p})$ is:

$$\begin{bmatrix} 1-2\tilde{s}-\frac{b\tilde{p}c}{(\tilde{s}+c)^2} & -a\tilde{s} & \frac{-h}{c_1} \\ 0 & a\tilde{s}-\frac{e\tilde{p}}{f}-g & 0 \\ \frac{c_1b\tilde{p}c}{(\tilde{s}+c)^2} & \frac{c_2e\tilde{p}}{f} & 0 \end{bmatrix}$$

One of the eigenvalue for the above matrix is,
$\frac{ecc_1(c_1b-h-hc)+gf(c_1b-h)^2}{f(c_1b-h)^2}\left(\frac{ahcf(c_1b-h)}{ecc_1(c_1b-h-hc)+gf(c_1b-h)^2}-1\right)$ and the other two are given by roots of the equation:

$$\lambda^2-A_1\lambda+A_2=0$$

where,
$A_1=1-\frac{2hcc_1b+(c_1b-h-hc)(c_1b-h)}{(c_1b-h)c_1b}$
$A_2=\frac{hb\tilde{p}c}{(\tilde{s}+c)^2}$
$E_3(\tilde{s},0,\tilde{p})$ will be stable if $\frac{ahcf(c_1b-h)}{ecc_1(c_1b-h-hc)+gf(c_1b-h)^2}-1<0$ i.e., $R_1^*<1$, and
$1-\frac{2hcc_1bc_1b+(c_1b-h-hc)(c_1b-h)}{(c_1b-h)c_1b}<0$ i.e., $R_2^*<1$ and $A_2>0$.

□

Now, we will establish the following theorem:

Theorem: Define $\bar{R}_{B_1}=\frac{1}{h}\left(\frac{c_1bg(1-m)}{g(1-m)+ac}+\frac{c_2e(a-g)(1-m)}{(a-g)(1-m)+fa^2}\right)$,
$R_{B_1}^*=\frac{ahcf(c_1b-h)}{ecc_1((c_1b-h)(1-m)-hc)+gf(c_1b-h)^2(1-m)}$ and
$R_{B_2}^*=\frac{(c_1b-h)(1-m)c_1b}{2hcc_1b+(c_1b-h)((c_1b-h)(1-m)-hc)}$, then for Model B, we have the following:

(i) $E_0(0,0,0)$ is always unstable.

(ii) $E_1(1,0,0)$ is locally stable if $R_0<1$ and $\tilde{R}_1<1$ and unstable if $R_0>1$ or $\tilde{R}_1>1$.

(iii) If $R_0 > 1$ and $\bar{R}_{B1} < 1$, then predator extinction equilibrium point $E_2(\bar{s}, \bar{i}, 0)$ is locally stable. If $\bar{R}_{B1} > 1$ then $E_2(\bar{s}, \bar{i}, 0)$ is unstable.

(iv) If $R^*_{B_1} < 1$ and $R^*_{B_2} < 1$, then disease free equilibrium point $E_3(\tilde{s}, 0, \tilde{p})$ is locally asymptotically stable.

(v) If $R^*_{B_1} > 1$ or $R^*_{B_2} > 1$, then disease free equilibrium point $E_3(\tilde{s}, 0, \tilde{p})$ is unstable.

Proof. :

At any point $E_i(s,i,p)$, the jacobian matrix formed for our system is given by:

$$\begin{bmatrix} 1-2s-ai-\dfrac{bpc(1-m)}{(s(1-m)+c)^2} & -as & -\dfrac{-bs(1-m)}{s(1-m)+c} \\ ai & as-\dfrac{epf(1-m)}{(i(1-m)+f)^2}-g & -\dfrac{ei(1-m)}{i(1-m)+f} \\ \dfrac{c_1bpc(1-m)}{(s(1-m)+c)^2} & \dfrac{c_2fep(1-m)}{(i(1-m)+f)^2} & \dfrac{c_1bs(1-m)}{s(1-m)+c}+\dfrac{c_2ei(1-m)}{i(1-m)+f}-h \end{bmatrix} \tag{20}$$

The jacobian corresponding to $E_0(0,0,0)$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -g & 0 \\ 0 & 0 & -h \end{bmatrix}$$

The corresponding eigenvalues with respect to the Jacobian are:

$\lambda_1 = 1$, $\lambda_2 = -g$ and $\lambda_3 = -h$. Since, one of the eigenvalues is always positive, therefore E_0 is locally asymptotically unstable.

The jacobian corresponding to E_1 is:

$$\begin{bmatrix} -1 & -a & -\dfrac{b(1-m)}{(1-m)+c} \\ 0 & a-g & 0 \\ 0 & 0 & \dfrac{c_1b(1-m)}{(1-m)+c}-h \end{bmatrix}$$

The corresponding eigenvalues with respect to the jacobian are:

$\lambda_1 = -1$, $\lambda_2 = a-g$ and $\lambda_3 = \dfrac{c_1b(1-m)}{(1-m)+c} - h$. Thus, E_1 is attractor and hence locally stable only if $a < g$ and $\dfrac{c_1b(1-m)}{(1-m)+c} < h$ i. e., $R_0 < 1$ and $\tilde{R}_1 < 1$, and unstable if $R_0 > 1$ or $\tilde{R}_1 < 1$. Thus, if $E_1(1,0,0)$ is locally stable then E_2 and E_3 does not exist.

The jacobian matrix for the above system with the equilibrium point $E_2(\bar{s},\bar{i},0)$ is given by:

$$\begin{bmatrix} -\dfrac{g}{a} & -g & -\dfrac{bg(1-m)}{g(1-m)+ca} \\ 1-\dfrac{g}{a} & 0 & -\dfrac{e(a-g)(1-m)}{(a-g)(1-m)+fa^2} \\ 0 & 0 & \dfrac{c_1bg(1-m)}{g(1-m)+ac}+\dfrac{c_2e(a-g)(1-m)}{(a-g)(1-m)+fa^2}-h \end{bmatrix}$$

The eigenvalues for the above matrix are: $\dfrac{c_1bg(1-m)}{g(1-m)+ac}+\dfrac{c_2e(a-g)(1-m)}{(a-g)(1-m)+fa^2}-h$ and roots of equation $\lambda^2+\dfrac{g}{a}\lambda+g(1-\dfrac{g}{a})=0$.

It can be easily seen that the roots of the quadratic equation are negative as sum of the roots is negative and product of roots is positive, when, $a>g$, .i.e. $R_0>1$. Also, $\dfrac{c_1bg(1-m)}{g(1-m)+ac}+\dfrac{c_2e(a-g)(1-m)}{(a-g)(1-m)+fa^2}-h$ is negative, when $\dfrac{c_1bg(1-m)}{g(1-m)+ac}+\dfrac{c_2e(a-g)(1-m)}{(a-g)(1-m)+fa^2}<h$, i.e. $\bar{R}_{B1}<1$.

Hence, E_2 is locally asymptotically stable when $R_0>1$ and $\bar{R}_{B1}<1$ and unstable when $\bar{R}_{B1}>1$.

The jacobian corresponding to $E_3(\tilde{s},0,\tilde{p})$:

$$\begin{bmatrix} 1-2\tilde{s}-\dfrac{b\tilde{p}c(1-m)}{(\tilde{s}(1-m)+c)^2} & -a\tilde{s} & -\dfrac{h}{c_1} \\ 0 & a\tilde{s}-\dfrac{e\tilde{p}(1-m)}{f}-g & 0 \\ \dfrac{c_1b\tilde{p}c(1-m)}{(\tilde{s}(1-m)+c)^2} & \dfrac{c_2e\tilde{p}(1-m)}{f} & 0 \end{bmatrix} \tag{21}$$

One of the eigenvalues of the above matrix is,
$\dfrac{ecc_1((c_1b-h)(1-m)-hc)+gf(c_1b-h)^2(1-m)}{f(c_1b-h)^2(1-m)}\left(R^*_{B_1}-1\right)$
where, $R^*_{B_1}=\dfrac{ahcf(c_1b-h)}{ecc_1((c_1b-h)(1-m)-hc)+gf(c_1b-h)^2(1-m)}$ and the other two eigenvalues are given by roots of equation $\lambda^2+A_1\lambda+A_2=0$, where,

$$A_1=1-\frac{2hcc_1b+(c_1b-h)((c_1b-h)(1-m)-hc)}{(c_1b-h)(1-m)c_1b}$$

$$A_2=\frac{hb\tilde{p}c(1-m)}{(\tilde{s}(1-m)+c)^2}$$

$E_3(\tilde{s},0,\tilde{p})$ will be stable if $\dfrac{ahcf(c_1b-h)}{ecc_1((c_1b-h)(1-m)-hc)+gf(c_1b-h)^2(1-m)}-1<0$ i.e $R^*_{B_1}<1$ and $1-\dfrac{2hcc_1b+(c_1b-h)((c_1b-h)(1-m)-hc)}{(c_1b-h)(1-m)c_1b}<0$ i.e., $R^*_{B_2}<1$ and $A_2>0$. □

In the next section, we will study the local stability analysis of interior equilibrium point.

6. INTERIOR EQUILIBRIUM POINT AND ITS LOCAL STABILITY

The jacobian of Model A, for the interior point $E^*(s^*,i^*,p^*)$ is:

$$\begin{bmatrix} 1-2s^*-ai^*-\frac{bp^*c}{(s^*+c)^2} & -as^* & -\frac{bs^*}{s^*+c} \\ ai^* & as^*-\frac{ep^*f}{(i^*+f)^2}-g & -\frac{ei^*}{i^*+f} \\ \frac{c_1bp^*c}{(s^*+c)^2} & \frac{c_2fep^*}{(i^*+f)^2} & \frac{c_1bs^*}{s^*+c}+\frac{c_2ei^*}{i^*+f}-h \end{bmatrix}$$

Now,the Jacobian matrix for the interior point $E^*(s^*,i^*,p^*)$ can be written as:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where,

$A_{11}=1-2s^*-ai^*-\dfrac{bp^*c}{(s^*+c)^2}=\dfrac{bsp}{(s+c)^2}-s, A_{12}=-as^*,$

$A_{13}=-\dfrac{bs^*}{s^*+c}, A_{21}=ai^*,$

$A_{22}=as^*-\dfrac{ep^*f}{(i^*+f)^2}-g=\dfrac{epi}{(i+f)^2}, A_{23}=-\dfrac{ei^*}{i^*+f}$

$A_{31}=\dfrac{c_1bp^*c}{(s^*+c)^2}, A_{32}=\dfrac{c_2fep^*}{(i^*+f)^2}, A_{33}=\frac{c_1bs^*}{s^*+c}+\frac{c_2ei^*}{i^*+f}-h=0$

we can form a characteristic equation of the jacobian in the following way:

$$\lambda^3+\sigma_1\lambda^2+\sigma_2\lambda+\sigma_3=0$$

where,

$$\begin{aligned} \sigma_1 &= -(A_{11}+A_{22}) \\ \sigma_2 &= A_{11}A_{22}-A_{23}A_{32}-A_{12}A_{21}-A_{13}A_{31} \\ \sigma_3 &= A_{11}A_{23}A_{32}+A_{13}A_{31}A_{22}-A_{12}A_{23}A_{31}-A_{13}A_{21}A_{32} \end{aligned}$$

By substituting values of A's, we can see that, $\sigma_1>0$, if $s>\dfrac{bsp}{(s+c)^2}+\dfrac{epi}{(i+f)^2}$.

Now, we will evaluate the value of $(\sigma_1\sigma_2-\sigma_3)$, which is as follows:

$\sigma_1\sigma_2-\sigma_3 = A_{11}(-A_{11}A_{22}+A_{12}A_{21}+A_{13}A_{31})+A_{22}(-A_{11}A_{22}+A_{12}A_{21}+A_{23}A_{32})+A_{12}A_{23}A_{31}+A_{13}A_{21}A_{32}$.

It is easy to see that, $\sigma_1\sigma_2-\sigma_3>0$, if $(-A_{11}A_{22}+A_{12}A_{21}+A_{13}A_{31})<0, (-A_{11}A_{22}+A_{12}A_{21}+A_{23}A_{32})>0$ and $A_{12}A_{23}A_{31}+A_{13}A_{21}A_{32}>0$.

Now, $(-A_{11}A_{22}+A_{12}A_{21}+A_{13}A_{31})<0$ if,

$$(s-\frac{bsp}{(s+c)^2})(\frac{epi}{(i+f)^2})-a^2si-\frac{bs}{s+c}\frac{cc_1bp}{(s+c)^2}<0 \tag{22}$$

Also, $(-A_{11}A_{22}+A_{12}A_{21}+A_{23}A_{32})>0$, if,

$$(s-\frac{bsp}{(s+c)^2})(\frac{epi}{(i+f)^2})-a^2si-\frac{ei}{i+f}\frac{c_2fep}{(i+f)^2}>0 \tag{23}$$

From (22), (23) and by substituting values of all A's in $A_{12}A_{23}A_{31}+A_{13}A_{21}A_{32}>0$ it is easy to compute that, $\sigma_1\sigma_2-\sigma_3>0$ if following conditions hold.

(i) $\frac{c_2fie^2p}{(i+f)^3}<\left(s-\frac{bsp}{(s+c)^2})(\frac{epi}{(i+f)^2}\right)-a^2si<\frac{cc_1b^2sp}{(s+c)^3}$

(ii) $cc_1(i+f)-c_2f(s+c)>0$

As, $A_{11}A_{23}A_{32}>0$, we can see that, $\sigma_3>0$ if,

$$A_{13}A_{31}A_{22}-A_{12}A_{23}A_{31}-A_{13}A_{21}A_{32}>0. \tag{24}$$

Again substituting values of A's in (24), we can easily compute that $\sigma_3>0$ if $(-c_1cbp-ac_1c(s+c)(i+f)+ac_2f(s+c)^2)>0$.

Since, sufficient conditions of Routh Hurwitz criteria are satisfied, thus $E^*(s^*,i^*,p^*)$ is locally asymptotically stable for Model A.

Now, we will establish the conditions for local stability of interior equilibrium point of Model B.

The jacobian of Model B, for the interior point $E^*(s^*,i^*,p^*)$ is:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where,

$A_{11} = 1-2s^*-ai^*-\frac{(1-m)bp^*c}{((1-m)s^*+c)^2} = \frac{(1-m)^2bsp}{((1-m)s+c)^2}-s, A_{12} = -as^*,$

$A_{13} = -\frac{(1-m)bs^*}{(1-m)s^*+c}, A_{21} = ai^*,$

$A_{22} = as^*-\frac{(1-m)ep^*f}{((1-m)i^*+f)^2}-g = \frac{(1-m)^2epi}{((1-m)i+f)^2}, A_{23} = -\frac{(1-m)ei^*}{(1-m)i^*+f}$

$A_{31} = \frac{(1-m)c_1bp^*c}{((1-m)s^*+c)^2}, A_{32} = \frac{(1-m)c_2fep^*}{((1-m)i^*+f)^2},$

$A_{33} = \frac{(1-m)c_1bs^*}{(1-m)s^*+c}+\frac{(1-m)c_2ei^*}{(1-m)i^*+f}-h = 0$

The local stability analysis of interior equilibrium point for Model B will be on the same lines. $E^*(s^*,i^*,p^*)$ is locally asymptotically stable for Model B if following conditions hold:

(i) $s > \frac{(1-m)^2bsp}{((1-m)s+c)^2}+\frac{(1-m)^2epi}{((1-m)i+f)^2}.$

(ii) $\frac{(1-m)^2c_2fie^2p}{((1-m)i+f)^3} < \left(s-\frac{(1-m)^2bsp}{((1-m)s+c)^2})(\frac{(1-m)^2epi}{((1-m)i+f)^2}\right)-a^2si < \frac{(1-m)^2cc_1b^2sp}{((1-m)s+c)^3}$

(iii) $cc_1((1-m)i+f)-c_2f((1-m)s+c) > 0$

(iv) $-(1-m)^2c_1cbp-ac_1c((1-m)s+c)((1-m)i+f)+ac_2f((1-m)s+c)^2 > 0.$

Now, we will study the dynamical behavior of interior equilibrium point of Model B, depending upon parameter m.

6.1. Transverslity condition for Hopf Bifurcation

The system enters into the Hopf-bifurcation around the positive equilibrium E^* for the parameter $m = m^*$ if the transversality criteria [26]

$$[\frac{dRe(\lambda(m))}{dm}]_{m=m^*} \neq 0$$

holds .

We will show that in our case, if the following conditions hold, then the transversality criteria is satisfied and the system transits into instability i.e., we find the point at which there exists a pair of conjugate symmetric eigenvalues on the imaginary axis. The conditions are as follows:

(i) $\sigma_1(m^*) > 0$.
(ii) $\sigma_1(m^*)\sigma_2(m^*) - \sigma_3(m^*) = 0$.
(iii) $(\sigma_1(m^*)\sigma_2(m^*))' < \sigma_3'(m^*)$.

Let m be the refuge rate. We assume that the interior point E^* is asymptotically stable. Our interest is on the parameter m that whether E^* loses its stability with the change in the parameter m i.e., we assume m as the bifurcation parameter, then there exists a critical value m^* such that $\sigma_1(m^*) > 0$, $\sigma_1(m^*)\sigma_2(m^*) - \sigma_3(m^*) = 0$, $[\sigma_1(m^*)\sigma_2(m^*)]' < \sigma_3'(m^*)$. For the occurrence of Hopf-bifurcation, the characteristic equation must be obtained of the kind,:

$$(\lambda^2\ (m^*) + \sigma_2(m^*))(\lambda(m^*) + \sigma_1(m^*)) = 0 \tag{25}$$

which has corresponding eigenvalues $\lambda_1(m^*) = i\sqrt{\sigma_2(m^*)}$, $\lambda_2 = -i\sqrt{\sigma_2(m^*)}$, $\lambda_3 = -\sigma_1(m^*) < 0$.

The roots for all m tends to be in the form:
$\lambda_1(m) = \mu(m) + i\nu(m)$,
$\lambda_2(m) = \mu(m) - i\nu(m)$,
$\lambda_3(m) = -\sigma_1(m)$

Now, the transversality condition $[\frac{dRe(\lambda_j(m))}{dm}]_{m=m^*} \neq 0, j = 1,2$ will be verified on substituting $\lambda_j(m) = \mu(m) \pm i\nu(m)$ in the equation (25) and then the derivative is calculated, we get

$$F(m)\mu'(m) - G(m)\nu'(m) + H(m) = 0 \tag{26}$$

$$G(m)\mu'(m) + F(m)\nu'(m) + I(m) = 0 \tag{27}$$

where,

$$F(m) = 3\mu^2(m) + 2\sigma_1(m)\mu(m) + \sigma_2(m) + \sigma_2(m) - 3\nu^2(m)$$

$$G(m) = 6\mu(m)\nu(m) + 2\sigma_1(m)\nu(m)$$

$$H(m) = \mu^2(m)\sigma'_(m) + \sigma'_2(m)\mu(m) + \sigma'_(m) - \sigma'_1(m)\nu^2(m)$$

$$I(m) = 2\mu(m)\nu(m)\sigma'_1(m) + \sigma'_2(m)\nu(m)$$

We Know that $\mu(m^*) = 0$, $\nu(m^*) = \sqrt{\sigma_2(m^*)}$, which results in

$$F(m^*) = -2\sigma_2(m^*)$$

$$G(m^*) = 2\sigma_1(m^*)\sqrt{\sigma_2(m^*)}$$

$$H(m^*) = \sigma'_3(m^*) - \sigma'_1(m^*)\sigma_2(m^*)$$

$$I(m^*) = \sigma'_2(m^*)\sqrt{\sigma_2(m^*)}$$

We now determine the value for $\mu'(m^*)$ using the equation (26),(27) we get,

$$[\frac{dRe(\lambda_j(m))}{dm}]_{m=m^*} = \mu^{'}(m)_{m=m^*} = -\frac{G(m^*)I(m^*) + F(m^*)H(m^*)}{F^2(m^*) + G^2(m^*)}$$

$$= \frac{\sigma'{}_3(m^*) - \sigma_1^{'}(m^*)\sigma_2(m^*) - \sigma_1(m^*)\sigma_2^{'}(m^*)}{\sigma_1^2(m^*) + \sigma_2(m^*)} > 0 \tag{28}$$

If $[\sigma_1(m^*)\sigma_2(m^*)]^{'} < \sigma_3^{'}(m^*)$ and $\lambda_3(m^*) = -\sigma_1(m^*) < 0$ which clearly implies that the transversality conditions hold. So, therefore,at $m = m^*$, the Hopf-bifurcation will take place.

Remark: Notice that (28) is a stronger statement than the transversality condition since it implies that at critical value, the eigenvalues cross the imaginary axis from left to right with non-zero velocity, i.e., the system transits from stability to instability via Hopf bifurcation. We note that the system transits into instability when the very first conjugate pair of eigenvalues, crosses the imaginary axis from left to right (as dictated by the positivity of the derivative in (28)).

7. PERMANENCE

In this section, we will establish the permanence of Model A and Model B.

Theorem: If $R_0 > 1$, $R_1 > 1$, $\bar{R_1} > 1$ and $R_1^* > 1$, then Model A is permanent.

Proof. We consider the average Lyapunov function $V = s^{a_1} i^{a_2} p^{a_3}$ where $a_i > 0$ $(i = 1,2,3)$ then,

$$\begin{aligned}\frac{\dot{V}}{V} &= a_1\tfrac{\dot{s}}{s} + a_2\tfrac{\dot{i}}{i} + a_3\tfrac{\dot{p}}{p}\\ &= a_1(1-s-ai-\frac{bp}{s+c}) + a_2(as-\frac{ep}{i+f}-g) + a_3(\frac{c_1bs}{s+c}+\frac{c_2ei}{i+f}-h) = \psi(s,i,p)\end{aligned}$$

Let $a_1 > a_2g + a_3h$. Then,

$$\begin{aligned}\psi(E_0) &= a_1 - a_2g - a_3h > 0\\ \psi(E_1) &= a_2(a-g) + a_3(\tfrac{c_1b}{1+c} - h) > 0 \text{ when } R_0 > 1 \text{ and } R_1 > 1\\ \psi(E_2) &= a_3(\bar{R_1} - 1) > 0 \text{ when } \bar{R_1} > 1\\ \psi(E_3) &= a_2(R_1^* - 1) > 0 \text{ when } R_1^* > 1\end{aligned}$$

□

Thus, Model A is permanent if $R_0 > 1$, $R_1 > 1$, $\bar{R_1} > 1$ and $R_1^* > 1$ by [22].

Proof for permanence of Model B will follow on similar lines.

8. NUMERICAL EXAMPLE

In this section, we have studied numerically dynamical behavior of the following systems:

(a) Model without infection and without refuge (m = 0)

(b) Model without infection and with refuge (m = 0.5, m = 0.34)

(c) Model with infection and without refuge (m = 0)

(d) Model with infection and with refuge (m = 0.5)

The above models have been studied with the following set of parameters: $r = 10, m = 0, 0.34, 0.5, \beta_1 = 0.2, 0.6, \beta_2 = 0.6, K = 100, \gamma = 0.09, c_1 = 0.01, c_2 = 0.02, a = 0.02$

1. Model without infection and without refuge is showing an oscillatory behavior, but as we increase m (refuge), oscillations have damped and by further increasing m to 0.5 the model exhibits stable behavior (Figure 1, 2, 3), which means that if $0 \leq m \leq 0.5$, then the system is unstable and becomes stable by increasing m (refuge).

2. Model with infection and without refuge exhibits oscillatory behavior (Figure 4, 5). Although, Model with infection and with refuge (m = 0.5) also exhibits oscillatory behavior (Figure 6, 7) but oscillations have damped thus showing that refuge can decrease chaos in the system but still the system is not stable due to infection (Figure 3, 6). Also, the graphs show that the survival of the predator is more when there is no refuge as more prey is available for predation (Figure 4, 6, 5, 7).

3. Interaction rate of predator with susceptible(β_1) and infected prey (β_2) also plays a very important role in shaping the dynamics of the system. If $\beta_1 < \beta_2$, then the oscillations of prey population are more damped in comparison to the case ($\beta_1 = \beta_2$) for the model with infection and with refuge. (Figure 6, 7).

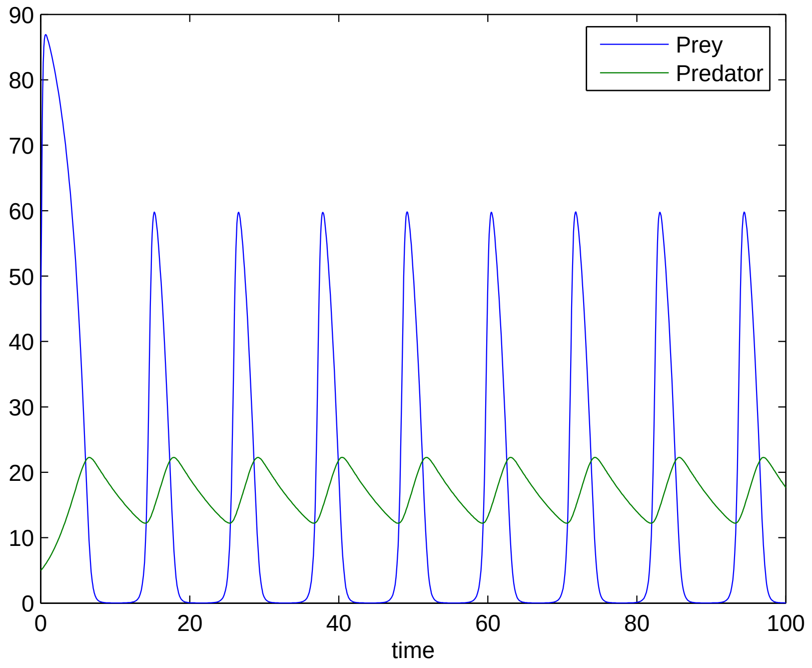

FIGURE 1. Without Infection, m =0, $\beta_1 = 0.6$, K= 100, $\gamma = 0.09$

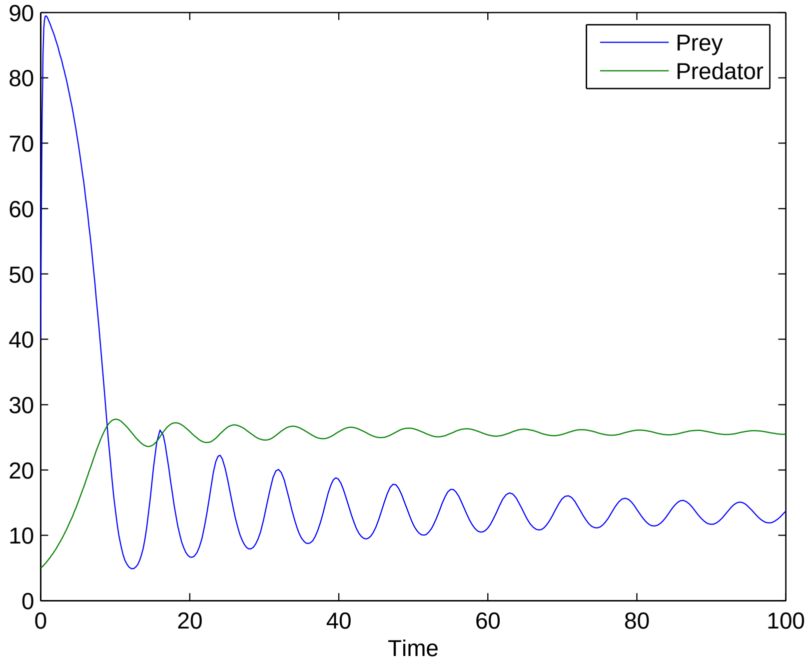

FIGURE 2. Without Infection, m =0.34, $\beta_1 = 0.6$, K= 100, $\gamma = 0.09$

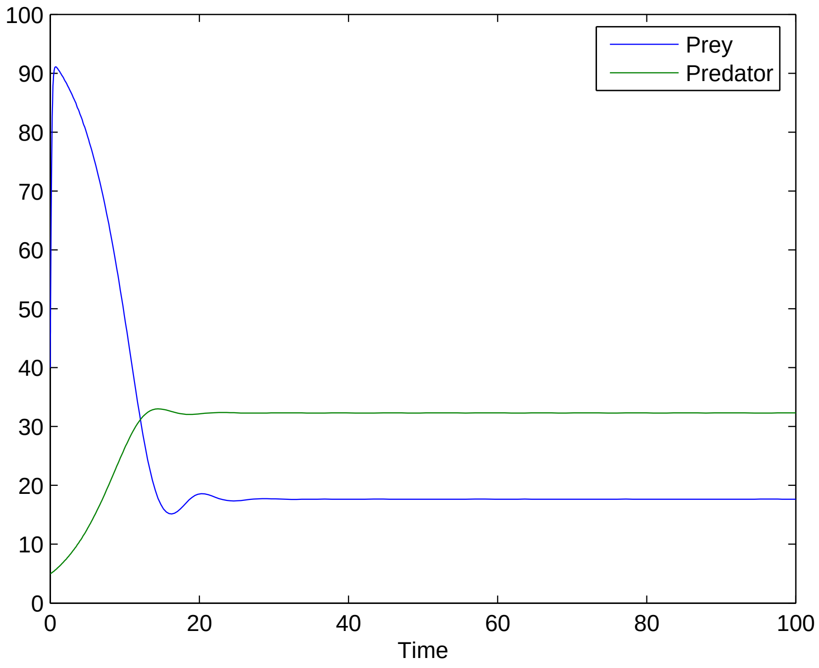

FIGURE 3. Without Infection, m =0.5, $\beta_1 = 0.6$, K= 100, $\gamma = 0.09$

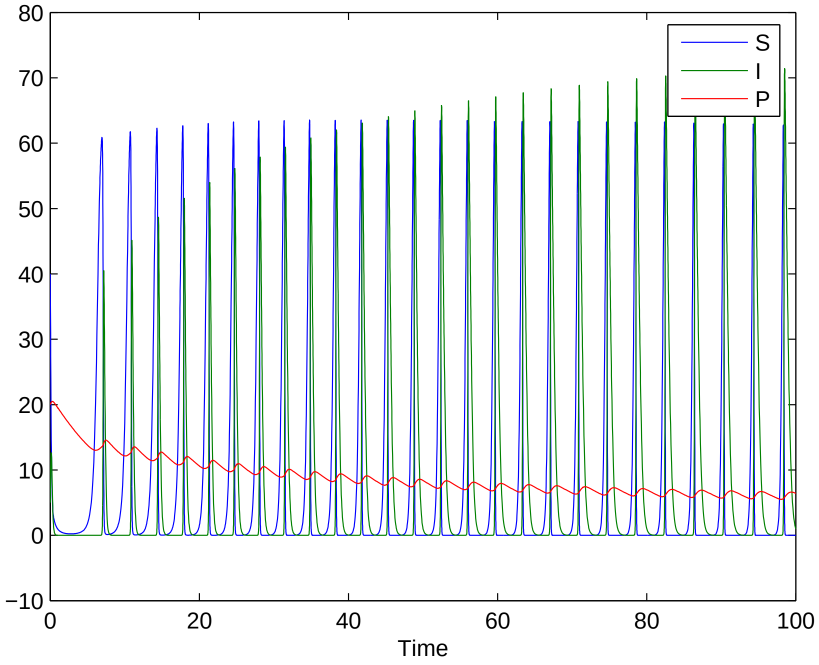

FIGURE 4. With Infection m = 0, $\beta_1 = 0.6, \beta_2 = 0.6$, K= 100, $\gamma = 0.09$

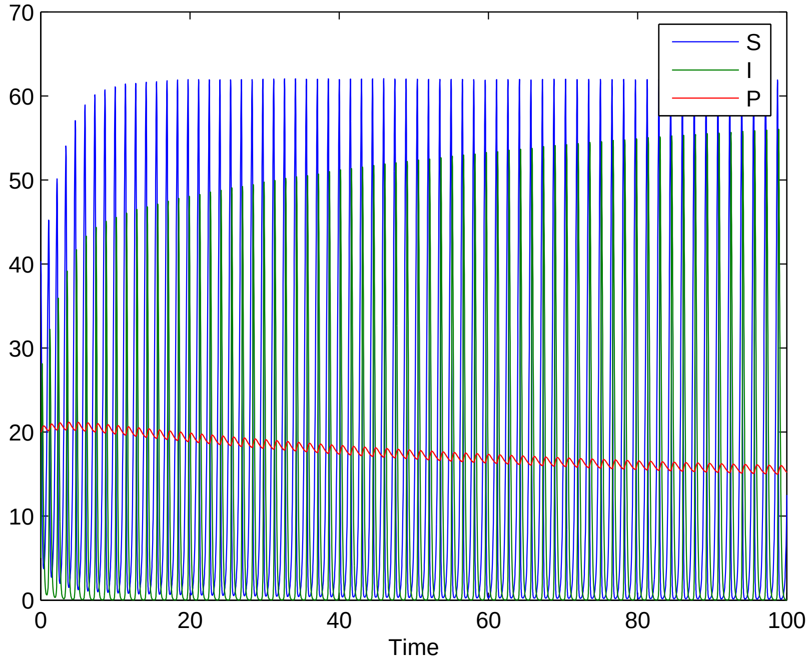

FIGURE 5. With Infection m = 0, $\beta_1 = 0.2, \beta_2 = 0.6$, K= 100, $\gamma = 0.09$

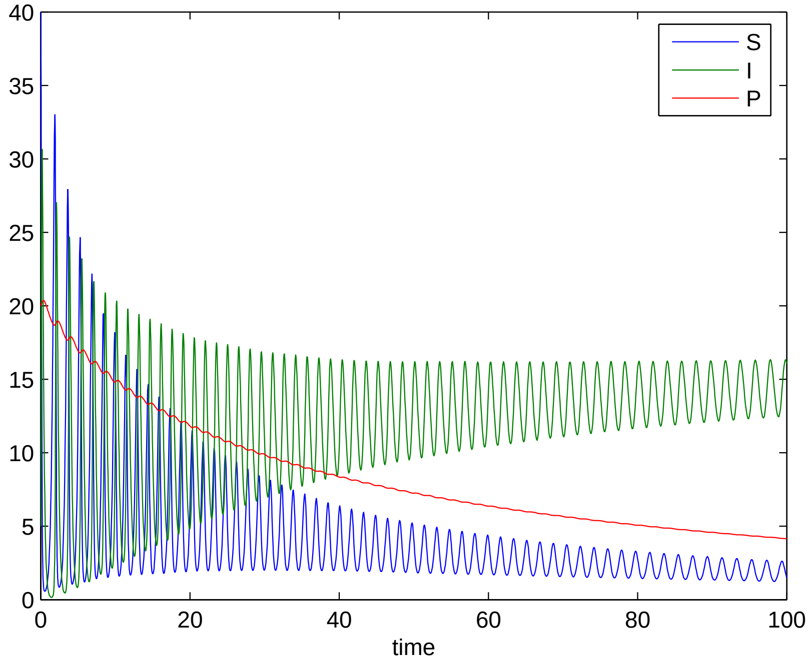

FIGURE 6. With Infection m = 0.5, $\beta_1 = 0.6, \beta_2 = 0.6$, K= 100, $\gamma = 0.09$

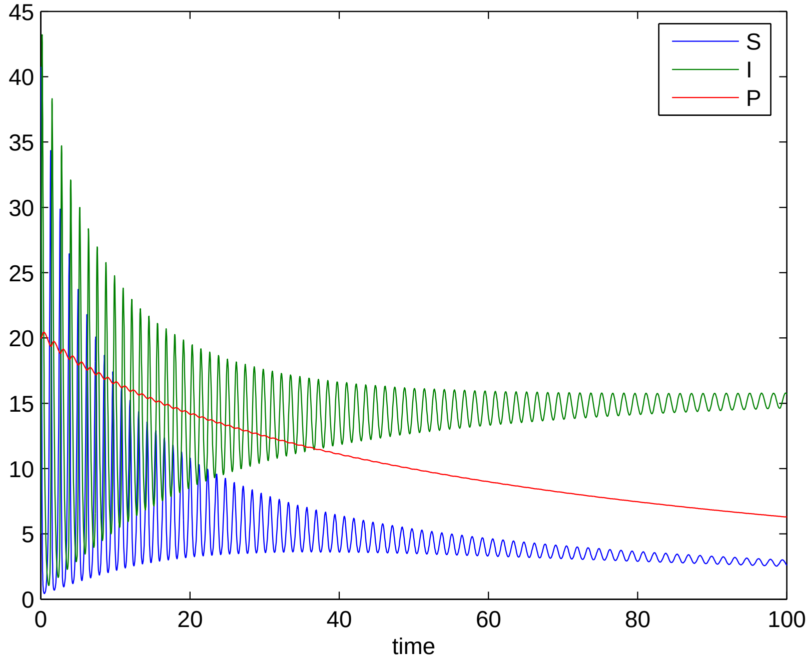

FIGURE 7. With Infection m = 0.5, $\beta_1 = 0.2, \beta_2 = 0.6$, K= 100, $\gamma = 0.09$

9. CONCLUSION

The prime focus of this paper is to study the effect of refuge and infection on a prey-predator model. It is assumed, that the prey is infected and infected prey is more vulnerable to predation as compared to susceptible prey. The existence and local stability analysis of the model with and without refuge has also been studied. The parameter m (refuge) plays a vital role in shaping the dynamics of the model. Interior equilibrium point of the system is unstable in the range $0 \leq m < 0.5$ and approaches stability when $m \geq 0.5$ in case of model without infection. In case of model with infection, the system exhibits oscillatory behavior even at $m = 0.5$ due to the effect of infection. Availability of refuge for prey population can control the oscillatory behavior of prey predator system. Further, if the interaction rate between susceptible and predator population β_1 is less than the interaction rate between infected and predator population β_2, as is the case in most real life situations where infected is less likely to defend itself from predation, then the interior equilibrium point is $S = 2.5974, I = 15.6748, P = 6.2930$ (Figure 7) whereas $\beta_1 = \beta_2$, then the interior equilibrium point is $S = 1.4991, I = 16.067, P = 4.1647$ (Figure 6) which means more susceptible, and less infected will survive in the case when $\beta_1 < \beta_2$ as expected. Lastly, the persistence of the system has also been studied.

Conflict of Interests

The authors declare that there is no conflict of interests.

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APPENDIX

In this section, we consider the case where refuge rate in susceptible prey is not same as in infected prey, i.e., $m_1 \neq m_2$. In this case, the system is as follows:

$$\frac{ds}{dt} = s(1-s) - asi - \frac{(1-m_1)bsp}{(1-m_1)s+c} \tag{29}$$

$$\frac{di}{dt} = asi - \frac{eip(1-m_2)}{(1-m_2)i+f} - gi \tag{30}$$

$$\frac{dp}{dt} = \frac{c_1bsp(1-m_1)}{(1-m_1)s+c} + \frac{c_2eip(1-m_2)}{(1-m_2)i+f} - hp \tag{31}$$

Above model will have the following four boundary equilibrium points:

(i) trivial equilibrium point $E_0(0,0,0)$

(ii) equilibrium point $E_1(1,0,0)$

(iii) predator extinction equilibrium point $E_2(\bar{s},\bar{i},0)$,where, $\bar{s} = \frac{g}{a}$ and $\bar{i} = \frac{(a-g)}{a^2}$ provided $a > g$ i.e., $R_0 > 1$ where, $R_0 = \frac{a}{g}$

(iv) disease-free equilibrium point $E_3(\tilde{s},0,\tilde{p})$

where, $\tilde{s} = \frac{hc}{(c_1b-h)(1-m_1)}$ and $\tilde{p} = \frac{((c_1b-h)(1-m_1)-hc))cc_1}{(c_1b-h)^2(1-m_1)^2}$ provided $(c_1b-h)(1-m_1) > hc$ i.e $\tilde{R}_1 > 1$ where $\tilde{R}_1 = \frac{c_1b(1-m_1)}{h(1+c-m_1)}$.

and the interior equilibrium point $E^*(s^*,i^*,p^*)$ is given by the following equation:

$$Q_1x^3 + Q_2x^2 + Q_3x + Q_4 = 0$$

where,

$Q_1 = (1-m_1)(1-m_2)ah(c_1b+c_2e-h)$

$Q_2 = (1-m_2)((c_2eg-gh)(bc_1-h)(1-m_1) + (h(1-m_1) - bc_1(1-m_1) - hc)(c_1ae - c_2ae + ah)) + (1-m_1)(ec_1(1-m_2) - fah)(bc_1a - bc_2a - ha) - ec_1(1-m_2)(bc_2g(1-m_1) - hac)$

$Q_3 = (bc_2g(1-m_1) - hac)(ec_1(1-m_2) - fah) + fgh(1-m_1)(bc_1a - bc_2a - ah) + (1-m_2)((h(1-m_1) - bc_1(1-m_1) - hc)(c_2eg - gh) + hc(c_1ae - c_2ae + ah))$

$Q_4 = (1-m_2)hc(c_2eg-gh)+fgh(bc_2g(1-m_1)-hac)$

By Descarte's rule of sign change, interior equilibrium point exists if either of the two conditions hold:

(i) $Q_1, Q_3 > 0$ and $Q_2, Q_4 < 0$

(ii) $Q_1, Q_3 < 0$ and $Q_2, Q_4 > 0$

Also, we will get the following theorem:

Theorem: Define $\bar{R}_{B_1} = \frac{1}{h}\left(\frac{c_1bg(1-m_1)}{g(1-m_1)+ac} + \frac{c_2e(a-g)(1-m_2)}{(a-g)(1-m_2)+fa^2}\right)$,
$R^*_{B_1} = \frac{ahcf(c_1b-h)(1-m_1)}{ecc_1(1-m_2)((c_1b-h)(1-m_1)-hc)+gf(c_1b-h)^2(1-m_1)^2}$ and
$R^*_{B_2} = \frac{(c_1b-h)(1-m_1)c_1b}{2hcc_1b+(c_1b-h)((c_1b-h)(1-m_1)-hc)}$, then for the above model, we have the following:

(i) $E_0(0,0,0)$ is always unstable.

(ii) $E_1(1,0,0)$ is locally stable if $R_0 < 1$ and $\tilde{R}_1 < 1$ and unstable if $R_0 > 1$ or $\tilde{R}_1 > 1$.

(iii) If $R_0 > 1$ and $\bar{R}_{B1} < 1$, then predator extinction equilibrium point $E_2(\bar{s}, \bar{i}, 0)$ is locally stable. If $\bar{R}_{B1} > 1$ then $E_2(\bar{s}, \bar{i}, 0)$ is unstable.

(iv) If $R^*_{B_1} < 1$ and $R^*_{B_2} < 1$, then disease free equilibrium point $E_3(\tilde{s}, 0, \tilde{p})$ is locally asymptotically stable.

(v) If $R^*_{B_1} > 1$ or $R^*_{B_2} > 1$, then disease free equilibrium point $E_3(\tilde{s}, 0, \tilde{p})$ is unstable.

Proof. Proof of this theorem will be similar to the proof of the theorem for the case when $m_1 = m_2 = m$. □

Now, we will establish the conditions for local stability of interior equilibrium point of the above Model.

The jacobian of the above Model, for the interior point $E^*(s^*, i^*, p^*)$ is:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where,

$A_{11} = 1 - 2s^* - ai^* - \dfrac{(1-m_1)bp^*c}{((1-m_1)s^*+c)^2} = \dfrac{(1-m_1)^2bsp}{((1-m_1)s+c)^2} - s, A_{12} = -as^*,$

$A_{13} = -\dfrac{(1-m_1)bs^*}{(1-m_1)s^*+c}, A_{21} = ai^*,$

$A_{22} = as^* - \dfrac{(1-m_2)ep^*f}{((1-m_2)i^*+f)^2} - g = \dfrac{(1-m_2)^2epi}{((1-m_2)i+f)^2}, A_{23} = -\dfrac{(1-m_2)ei^*}{(1-m_2)i^*+f}$

$A_{31} = \dfrac{(1-m_1)c_1bp^*c}{((1-m_1)s^*+c)^2}, A_{32} = \dfrac{(1-m_2)c_2fep^*}{((1-m_2)i^*+f)^2},$

$A_{33} = \dfrac{(1-m_1)c_1bs^*}{(1-m_1)s^*+c} + \dfrac{(1-m_2)c_2ei^*}{(1-m_2)i^*+f} - h = 0$

$E^*(s^*, i^*, p^*)$ is locally asymptotically stable, if following conditions hold:

(i) $s > \dfrac{(1-m_1)^2bsp}{((1-m_1)s+c)^2} + \dfrac{(1-m_2)^2epi}{((1-m_2)i+f)^2}.$

(ii) $\dfrac{(1-m_2)^2c_2fie^2p}{((1-m_2)i+f)^3} < \left(s - \dfrac{(1-m_1)^2bsp}{((1-m_1)s+c)^2})(\dfrac{(1-m_2)^2epi}{((1-m_1)i+f)^2}\right) - a^2si < \dfrac{(1-m_1)^2cc_1b^2sp}{((1-m_1)s+c)^3}$

(iii) $cc_1((1-m_2)i+f) - c_2f((1-m_1)s+c) > 0$

(iv) $-(1-m_1)(1-m_2)c_1cbp - ac_1c((1-m_1)s+c)((1-m_2)i+f) + ac_2f((1-m_1)s+c)^2 > 0.$

We can obtain conditions for Hopf-bifurcation in this case, which will be similar to the conditions for Hopf-bifurcation in the case when $m_1 = m_2 = m$. Also, proof for permanence of the model with different refuge rate will be on the same lines as for the case when $m_1 = m_2 = m$ as well.

Remark: We can see that values of $\bar{R}_{B_1}, R^*_{B_1}, R^*_{B_2}$ in the case, when $m_1 \neq m_2$ are not much different from the values of these parameters when $m_1 = m_2 = m$. Also, the conditions for stability of interior equilibrium point are similar to the case of same refuge rate. Thus, we have studied the model with same refuge rate in susceptible and infected prey, as our interest was to study the effect of refuge on dynamics of the prey- predator model with infected prey.