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NUMERICAL ANALYSIS OF STABILITY FOR A NONLINEAR SIZE-STRUCTURED POPULATION MODEL WITH ELASTIC GROWTH

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Abstract. This article is concerned with an upwind difference scheme for a nonlinear size-dependent population model, into which an elastic growth is incorporated; that is, a decrease of size is possible. We make a through analysis of convergence for the numerical method, and use it as an auxiliary tool in the investigation of stability of population steady states. Two examples are presented.

Keywords: size-structure; elastic growth; upwind difference; convergence; stability.

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1. Introduction

Structured population models combine individuals variations such as age, size and gender with the dynamics of a population or community, which play a significant role in the management of populations, see e.g. Ref. [1, 2, 3]. Compared with the unstructured models described usually by ordinary differential equations, models with some structuring variables are ecologically more realistic, but, in the meantime, they are leading to partial differential equations with

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global feedback boundary conditions. A rigid analysis for the models is often difficult and analytical solutions are not expected.

In order to withdraw some insights from the structured models, an important method (even the unique one in some situations) is to solve them numerically. For age-structured models, lots of efforts have been made by many researchers. In [4], López-Marcos proposed and analyzed an upwind scheme for a kind of age-structured model, equipped with a general birth process. Using Taylor series expansion, Sulsky presented a numerical method for autonomous Gurtin-MacCamy model [5] and compared the computing accuracy with the dynamical behaviors of Blowfly and Squirrel populations. For linear Lotka-McKendrick model with finite expectancy, Iannelli and Milner in [6] delicately dealt with the breakdown of difference approximation when the death rate is unbounded. Ref. [7] contains an excellent review by Abia, Angulo and López-Marcos for numerical solutions to age-structured models, therein the accuracy and efficiency are discussed for methods of characteristic curves and finite difference. Kim and Selenge formulated a Galerkin method for the autonomous Lotka-McKendrick model based on finite elements [8], and the stability was established. A second-order numerical method based on characteristic curves was proposed by Angulo et al. in [9], and the convergence of the scheme also treated.

As a generalization, numerical approximations to size-structured models need to be done more carefully. After a rigid proof of the existence of solutions, Ito, Kappel and Peichl presented an algorithm of characteristics type for a linear size-structured model, with the reproduction process given by a general functional of the density [10]. For a kind of non-autonomous mass-structured models, Sulsky compared the performance of Eulerian method, implicit difference and the method of characteristic curves [11]. She claimed that the preferred is the method of characteristic curves combined with an adaptive grid. In [12], Ackleh and Ito considered a fully nonlinear system with an external inflow of new-borns, and proved that an implicit algorithm converges to a bounded variation solution. An interesting result, optimal rate of convergence, was derived by Angulo and López-Marcos [13] for a numerical approach based on characteristic curves to a simple model. Furthermore, the same authors compared the three numerical methods solving size-structured models [14]: Lax-Wendroff scheme, Box method, and characteristic curves. They concluded that the Box scheme is constrained less but needs more cpu-times. The authors in [7] also presented a review for the numerical methods of size-structured population dynamics [15], their efforts should be appreciated. Angulo, Durán and López-Marcos investigated the evolution of Gambussia affinis population from a numerical point of view, obtained such long-run behaviors as periodic Oscillation [16]. Ackleh, Deng and Thibodeaux established a monotone approximation based on upper and lower solution series to a hierarchical system [17], which was used to show the existence of unique solution. For a resources-population system with variable maximum individual size [18], Angulo, J.C. López-Marcos and M.A. López-Marcos obtained the approximation of singular asymptotic states.

It should be noted that all of works mentioned above are completed under an assumption that the growth rate is non-negative, with the exception of [17] and [18]. The assumption is unrealitic in most of situations. For example, animal's weights will decrease if their foods are in shortage. So we consider, in the present paper, a size-structured model with elastic growth. The remainder of this paper is structured as follows. Next section contains the basic model, and the section 3 is devoted to the upwind scheme and its convergence is established in section 4. Two examples are examined in section 5, and some remarks are included in the final section.

2. The Basic Model

The dynamics of the population is governed by the following infinite-dimensional system:

$$(1)\qquad \begin{cases} p_t(x,t)+[g(x)p(x,t)]_x=-\mu(x,Q_1(t))p(x,t), & 0<x<l,\, 0<t<\infty,\\ g(0)p(0,t)=\int_0^l \beta(x,Q_2(t))p(x,t)dx, & 0<t<\infty,\\ p(x,0)=p_0(x),\, Q_i(t)=\int_0^l q_i(x)p(x,t)dx, & 0<x<l,\, 0<t<\infty,\, i=1,2. \end{cases}$$

Here $p(x,t)$ is the population density, t denotes time and x the individual's size. Functions g, μ and β are the rates of growth, death and birth, respectively. $Q_i(t)$ gives the weighted population size and $q_i(x)$ works as a weighting function. l denotes the maximum body size, and $p_0(x)$ the initial size distribution. It is supposed that all new-born individuals are of size zero, and functional parameters are non-negative.

Remark 2.1 The well-posedness and theoretic results of equilibria stability of (1) have been established in [19]. In the following sections, we present a numerical approximation to the solutions of (1) and use it to demonstrate the stability through several examples.

3. The Upwind Scheme

Let the integer J be large enough, size step $h = l/J$, and $x_j = jh$, $0 \le j \le J$. For $\mathbf{V} = (V_1, V_2, ..., V_J) \in \mathbb{R}^J$, define the symbols as

$$Q_h^i(\mathbf{V}) = \sum_{j=1}^{J} h q_i(x_j) V_j,\ i = 1, 2; \tag{2}$$

$$\mu(\mathbf{V}) = (\mu(x_1, Q_h^1(\mathbf{V})), \mu(x_2, Q_h^1(\mathbf{V})), ..., \mu(x_J, Q_h^1(\mathbf{V}))); \tag{3}$$

$$\beta(\mathbf{V}) = (\beta(x_1, Q_h^2(\mathbf{V})), \beta(x_2, Q_h^2(\mathbf{V})), ..., \beta(x_J, Q_h^2(\mathbf{V}))). \tag{4}$$

For given $T > 0$, choose integer N sufficiently large, and then time step $k = T/N$. Denote $t_n = nk$, $0 \le n \le N$. Let

$$\mathbf{P}^n = (P_1^n, P_2^n, \cdots, P_J^n),$$

where P_j^n is the numerical approximation of the grid value $p(x_j, t_n)$ of solution $p(x,t)$, $0 \le j \le J, 0 \le n \le N$, to model (1). Then the upwind difference equations read as

$$\begin{cases} \frac{P_j^n - P_j^{n-1}}{k} + \frac{g_j P_j^{n-1} - g_{j-1} P_{j-1}^{n-1}}{h} + \mu(x_j, Q_h^1(\mathbf{P}^{n-1})) P_j^{n-1} = 0, & 1 \le j \le J, 1 \le n \le N, \\ g_0 P_0^n = h\beta(\mathbf{P}^n)\mathbf{P}^n, & 0 \le n \le N, \\ \mathbf{P}^0 = (P_1^0, P_2^0, \cdots, P_J^0), \end{cases} \tag{5}$$

where $g_j = g(x_j)$.

Denote the set $H = \{h > 0 : h = l/J, J \in \mathbb{N}\}$. For each $h \in H$, define the operator $\Phi_h : X_h \to Y_h$ on space $X_h = Y_h = \mathbb{R}^{N+1} \times (\mathbb{R}^J)^{N+1}$ as follows:

$$\Phi_h(\mathbf{V}_0, \mathbf{V}^0, \cdots, \mathbf{V}^N) = (\mathbf{U}_0, \mathbf{U}^0, \cdots, \mathbf{U}^N), \tag{6}$$

where

$$\begin{cases} \mathbf{U}_0 = (U_0^0, U_0^1, \cdots, U_0^N), \\ U_0^n = g_0 V_0^n - h\beta(\mathbf{V}^n \mathbf{V}^n), & 0 \le n \le N, \\ U_j^0 = V_j^0 - P_j^0, & 1 \le j \le J, \\ U_j^n = \dfrac{V_j^n - V_j^{n-1}}{k} + \dfrac{g_j V_j^{n-1} - g_{j-1} V_{j-1}^{n-1}}{h} + \mu(x_j, Q_h^1(\mathbf{V}^{n-1})) V_j^{n-1}, & 1 \le n \le N, \\ & 1 \le j \le J. \end{cases} \tag{7}$$

For $\mathbf{P}_h = (\mathbf{P}_0, \mathbf{P}^0, \mathbf{P}^1, \cdots, \mathbf{P}^N) \in X_h$, where

$$\mathbf{P}_0 = (P_0^0, P_0^1, \cdots, P_0^N) \in \mathbb{R}^{N+1},$$
$$\mathbf{P}^n = (P_1^n, P_2^n, \cdots, P_J^n) \in \mathbb{R}^J, 0 \le n \le N,$$

we conclude that $\mathbf{P}_h$ is a solution of (5), if and only if $\mathbf{P}_h$ is a solution of the following problem:

$$\Phi_h(\mathbf{P}_h) = 0 \in Y_h. \tag{8}$$

In order to show that how close $\mathbf{P}_h$ is to the true solution p, for grid point value vector $\mathbf{p}_h \in X_h$,

$$\mathbf{p}_h = (\mathbf{p}_0, \mathbf{p}^0, \mathbf{p}^1, \cdots, \mathbf{p}^N), \tag{9}$$

where

$$\begin{aligned} &\mathbf{p}_0 = (p_0^0, p_0^1, p_0^2, \cdots, p_0^N) \in \mathbb{R}^{N+1}, p_0^n = p(0, t_n), \quad 0 \le n \le N; \\ &\mathbf{p}^n = (p_1^n, p_2^n, \cdots, p_J^n) \in \mathbb{R}^J, \, p_j^n = p(x_j, t_n), \qquad 1 \le j \le J, 0 \le n \le N, \end{aligned} \tag{10}$$

we define the global error

$$\mathbf{e}_h = \mathbf{p}_h - \mathbf{P}_h \in X_h, \tag{11}$$

and the local error

$$\mathbf{I}_h = \Phi_h(\mathbf{p}_h) \in Y_h. \tag{12}$$

The following norms are needed to measure the errors:

$$\begin{aligned} &\|(\mathbf{V}_0, \mathbf{V}^0, \mathbf{V}^1, \cdots, \mathbf{V}^N)\|_{X_h} = \max\Big\{\|\mathbf{V}_0\|_*, \|\mathbf{V}^0\|, \|\mathbf{V}^1\|, \cdots, \|\mathbf{V}^N\|\Big\}, \\ &\|(\mathbf{U}_0, \mathbf{U}^0, \mathbf{U}^1, \cdots, \mathbf{U}^N)\|_{Y_h} = \Big\{\|\mathbf{U}_0\|^2 + (\sum_{n=0}^{N} \|\mathbf{U}^n\|^2)\Big\}^{1/2}, \end{aligned} \tag{13}$$

where $\|\mathbf{V}\|^2 =< \mathbf{V}, \mathbf{V} >, < \mathbf{V}, \mathbf{W} >= \sum_{j=1}^{J} hV_jW_j$, $\|V\|_* = \left(\sum_{j=1}^{J} {}'' h|V^j|^2\right)^{1/2}$, the double prime in the summation means that the first and last terms are halved.

Readers are referred to [4] for the definitions of consistency and convergence.

Definition 3.3[4] For $\forall h \in H$, $0 < R_h \le \infty$. The discretization (8) is said to be stable with the threshold R_h, if there exist two positive constants h_0, s(stability constant) such that, for any $h \le h_0$ and all $\mathbf{V}_h, \mathbf{W}_h$ in the open ball $B(\mathbf{p}_h, R_h)$ of X_h,

$$\|\mathbf{V}_h - \mathbf{W}_h\|_{X_h} \le s\|\Phi_h(\mathbf{V}_h) - \Phi_h(\mathbf{W}_h)\|_{Y_h}. \tag{14}$$

One of the main results in the paper is the convergence of the scheme. We need the following result:

Proposition 3.1[4] Suppose that (8) is consistent and stable with R_h, if Φ_h is continuous in $B(\mathbf{p}_h, R_h)$ and $\|\mathbf{I}_h\| = O(R_h)$ as $h \to 0$, then

(1) For h sufficiently small, the discrete equation (5) posses a unique solution in $B(\mathbf{p}_h, R_h)$;

(2) the solution of (5) is convergent and $\|\mathbf{e}_h\|_{X_h} = O(\|\mathbf{I}_h\|_{Y_h})$.

It is trivial to show the following

Proposition 3.2 Assume that β and $q_i(x)$, $i = 1,2$ are twice continuously differentiable, g and p are third continuously differentiable; furthermore, there exists P^0 such that, as $h \to 0$,

$$\left(\sum_{j=1}^{J} {}'' h (P_j^0 - p_j^0)^2 \right)^{1/2} = o(h), \tag{15}$$

then the discretization (5) posses a unique solution in the $B(\mathbf{p}_h, R_h)$.

4. Consistency and Convergence

The first step in the analysis of the numerical method (5) is to establish its consistency. If the parameters μ, β, q_i, $i = 1,2$, and the true solution p are sufficiently smooth, then the following hold:

$$\begin{aligned} &\left| Q_i(t_n) - Q_h^i(\mathbf{p}^n) \right| \leq ch^2, \, i = 1,2, \\ &\left| \int_0^l \beta(x, Q_2(t_n)) p(x,t_n) dx - Q_h^2(\beta(\mathbf{p}^n)\mathbf{p}^n) \right| \leq ch^2, \end{aligned} \tag{16}$$

where c is positive constant independent of h, which probably takes different values in different places.

For h sufficiently small, $(V^0, V^1, \cdots, V^N) \in B(\mathbf{p}_h, R_h)$, we have

$$Q_h^i(\mathbf{V}^n) \in D_i, \, 0 \leq n \leq N, \, i = 1,2, \tag{17}$$

where D_i denotes the compact area of $\left\{ \int_0^l q_i(x) p(x,t) dx, 0 \leqslant t \leqslant T \right\}$.

From the definition of $Q_h^i, i = 1,2$, and inequalities in (16), it follows that, as $h \to 0$

$$\left| Q_h^i(\mathbf{V}^n) - Q_i(t_n) \right| \leq \left| Q_h^i(\mathbf{V}^n) - Q_h^i(\mathbf{p}^n) \right| + \left| Q_h^i(\mathbf{p}^n) - Q_i(t_n) \right| \leq \|q_i\|_\infty R_h + o(1). \tag{18}$$

Theorem 4.1(Consistency) Under the hypotheses of proposition 3.2, the local error satisfies, as $h \to 0$

$$\|\Phi_h(\mathbf{p}_h)\|_{Y_h} = \left[\sum_{j=0}^{J}{}'' h(P_j^0 - p_j^0)^2 + O(h^2)^2 + O(h^2+k^2)^2 \right]^{1/2}. \tag{19}$$

Proof. With the notations in section 3, we derive that, as $h \to 0$,

$$\begin{aligned} &\left| \tfrac{p_j^n - p_j^{n-1}}{k} - p_t(x_j,t_n) \right| = O(h^2+k^2), \\ &\left| \tfrac{g_j p_j^{n-1} - g_{j-1} p_{j-1}^{n-1}}{h} - (gp)_x(x_j,t_{n-1}) \right| = O(h^2+k^2), \\ &\left| \mu_j^n(\mathbf{p}) p_j^n - \mu(x_j, Q_1(t_n)) p(x_j,t_n) \right| = O(h^2+k^2), \end{aligned} \tag{20}$$

where $\mu_j^n(\mathbf{p}) p_j^n = \mu(x_j, Q_h^1(\mathbf{p}))$ and $\mathbf{p}$ is the true solution. Thus, we arrive at

$$\left| \frac{p_j^n - p_j^{n-1}}{k} + \frac{g_j p_j^{n-1} - g_{j-1} p_{j-1}^{n-1}}{h} + \mu_j^n(\mathbf{p}) p_j^n \right| = O(h^2+k^2). \tag{21}$$

The inequality in (16) yields

$$\left| g(0)p(0,t_n) - Q_h^2(\beta(\mathbf{p}^n)\mathbf{p}^n) \right| \le ch^2. \tag{22}$$

Consequently,

$$\|\Phi_h(\mathbf{p}_h)\|_{Y_h} = \left[\sum_{j=0}^{J}{}'' h(P_j^0 - p_j^0)^2 + O(h^2)^2 + O(h^2+k^2)^2 \right]^{1/2}, \tag{23}$$

and the conclusion follows.

Next we show the stability of the discrete scheme (5). Cauchy-Schwarz inequality gives

$$\begin{aligned} \left| < \mathbf{V}, \mathbf{W} > \right| &= \left| \sum_{j=0}^{J} hW_j V_j \right| \\ &\le \left(\sum_{j=0}^{J} h(W_j)^2 \right)^{1/2} \left(\sum_{j=0}^{J} h(V_j)^2 \right)^{1/2} \\ &\le C\|\mathbf{V}\|_\infty \left(\sum_{j=0}^{J} h(W_j)^2) \right)^{1/2}, \end{aligned} \tag{24}$$

where $\|\mathbf{V}\|_\infty = \max_{0\le j\le J} |V_j|, C = \sqrt{h(J+1)}$. On the other hand, we have

$$\begin{aligned} \left| Q_h^i(\mathbf{VW}) \right| &= \left| \sum_{j=0}^{J} hW_j V_j \right| \\ &\le \left| \sum_{j=1}^{J} hW_j (V_j - V_{j-1}) \right| + \left| \sum_{j=1}^{J} hW_j V_{j-1} \right| + h|V_0 W_0| \end{aligned}$$

$$\begin{aligned}
&\leq C\max_{1\leq j\leq J}|V_{j-1}|\Big(\sum_{j=1}^{J}h(W_j)^2)\Big)^{1/2}+c\max_{1\leq j\leq J}\Big|V_j-V_{j-1}\Big| \\
&\quad\Big(\sum_{j=1}^{J}h(W_j)^2)\Big)^{1/2}+h\Big(|V_0-\max_{1\leq j\leq J}|V_{j-1}|\Big)|W_0| \\
&\leq \|\mathbf{V}\|_\infty\Big(\sum_{j=1}^{J}h(W_{j-1})^2)\Big)^{1/2}+\max_{1\leq j\leq J}\Big|V_j-V_{j-1}\Big|\Big(\sum_{j=1}^{J}h(W_j)^2)\Big)^{1/2}. \qquad (25)
\end{aligned}$$

Theorem 4.2(Stability). Under the hypotheses of the proposition 3.2, if $R_h=Rh$, $R>0$, the discretization (5) is stable with R_h.

Proof. Let

$$\begin{aligned}
&(\mathbf{V}_0,\mathbf{V}^0,\mathbf{V}^1,\cdots,\mathbf{V}^N)=\mathbf{V}_h\in B(\mathbf{p}_h,Rh), \\
&(\mathbf{W}_0,\mathbf{W}^0,\mathbf{W}^1,\cdots,\mathbf{W}^N)=\mathbf{W}_h\in B(\mathbf{p}_h,Rh), \\
&\Phi_h(\mathbf{V}_0,\mathbf{V}^0,\mathbf{V}^1,\cdots,\mathbf{V}^N)=(\mathbf{U}_0,\mathbf{U}^0,\mathbf{U}^1,\cdots,\mathbf{U}^N), \\
&\Phi_h(\mathbf{W}_0,\mathbf{W}^0,\mathbf{W}^1,\cdots,\mathbf{W}^N)=(\mathbf{R}_0,\mathbf{R}^0,\mathbf{R}^1,\cdots,\mathbf{R}^N). \qquad (26)
\end{aligned}$$

There exists positive constant C, as $h\to 0$, such that

$$\|\mathbf{V}^0\|_\infty\leq C,\|\mathbf{V}^n\|_\infty\leq C,0\leq n\leq N; \qquad (27)$$

$$\Big|Q_h^i(\mathbf{p}^n)-Q_h^i(\mathbf{V}^n)\Big|\leq Ch^{1/2},0\leq n\leq N;i=1,2. \qquad (28)$$

According to (7), we get

$$\begin{aligned}
V_j^n &= kU_j^n-\frac{k}{h}\Big(g_jV_j^{n-1}-g_{j-1}V_{j-1}^{n-1}\Big)+V_j^{n-1}-k\mu_j^{n-1}(\mathbf{V})V_j^{n-1}, \\
W_j^n &= kR_j^n-\frac{k}{h}\Big(g_jW_j^{n-1}-g_{j-1}W_{j-1}^{n-1}\Big)+W_j^{n-1}-k\mu_j^{n-1}(\mathbf{W})W_j^{n-1},
\end{aligned}$$

then

$$\begin{aligned}
V_j^n-W_j^n &= \Big(V_j^{n-1}-W_j^{n-1}\Big)+k\Big(U_j^n-R_j^n\Big) \\
&\quad -k\Big(\mu_j^{n-1}(\mathbf{V})V_j^{n-1}-\mu_j^{n-1}(\mathbf{W})W_j^{n-1}\Big) \\
&\quad -\frac{k}{h}\Big(g_j(V_j^{n-1}-W_j^{n-1})-g_{j-1}(V_{j-1}^{n-1}-W_{j-1}^{n-1})\Big). \qquad (29)
\end{aligned}$$

Let $k=rh$, $r>0$ and $\frac{1}{1-r}\leq C$, then (29) became

$$V_j^n-W_j^n = (1-rg_j)\Big(V_j^{n-1}-W_j^{n-1}\Big)+rg_{j-1}\Big(V_{j-1}^{n-1}-W_{j-1}^{n-1}\Big)$$

$$+k\left(U_j^n - R_j^n\right) - k\left(\mu_j^{n-1}(\mathbf{V})V_j^{n-1} - \mu_j^{n-1}(\mathbf{W})W_j^{n-1}\right), \tag{30}$$

where $\mu_j^{n-1}(\mathbf{V})V_j^{N-1} = \mu\left(x_j, Q_h^1(\mathbf{V}^{n-1})\right)$.

Multiplying (30) by $h(V_j^n - W_j^n)$, and summarizing j from 1 to J, we obtain that

$$\begin{aligned}\|\mathbf{V}^n - \mathbf{W}^n\|^2 &= \sum_{j=1}^{J} h(1-rg_j)\left(V_j^{n-1} - W_j^{n-1}\right)\left(V_j^n - W_j^n\right)\\ &\quad + \sum_{j=1}^{J} hrg_{j-1}\left(V_{j-1}^{n-1} - W_{j-1}^{n-1}\right)\left(V_j^n - W_j^n\right)\\ &\quad + \sum_{j=1}^{J} hk\left(V_j^n - W_j^n\right)\left(U_j^n - R_j^n\right)\\ &\quad - \sum_{j=1}^{J} hk\left(\mu_j^{n-1}(\mathbf{V})V_j^{n-1} - \mu_j^{n-1}(\mathbf{W})W_j^{n-1}\right)\left(V_j^n - W_j^n\right).\end{aligned} \tag{31}$$

For the first item on the right of (31), since $|g_j| \le M, 0 \le j \le J$, we have

$$\begin{aligned}&\sum_{j=1}^{J}\left|h(1-rg_j)\left(V_j^{n-1} - W_j^{n-1}\right)\left(V_j^n - W_j^n\right)\right|\\ \le\ & \sum_{j=1}^{J} h\left|\left(V_j^{n-1} - W_j^{n-1}\right)\left(V_j^n - W_j^n\right)\right|\\ &+ \sum_{j=1}^{J} hr|g_j|\left(V_j^{n-1} - W_j^{n-1}\right)\left(V_j^n - W_j^n\right)\\ \le\ & \frac{1}{2}\left(\sum_{j=1}^{J} h\left(V_j^{n-1} - W_j^{n-1}\right)^2 + \sum_{j=1}^{J} h\left(V_j^n - W_j^n\right)^2\right)\\ &+hr\left(\sum_{j=1}^{J}\left[g_j(V_j^{n-1} - W_j^{n-1})\right]^2\right)^{1/2}\left(\sum_{j=1}^{J}(V_j^n - W_j^n)^2\right)^{1/2}\\ \le\ & \frac{1}{2}\left(\|\mathbf{V}^{n-1} - \mathbf{W}^{n-1}\|^2 + \|\mathbf{V}^n - \mathbf{W}^n\|^2\right)\\ &+\frac{hr}{2}\left(\sum_{j=1}^{J}\left[g_j(V_j^{n-1} - W_j^{n-1})\right]^2 + \sum_{j=1}^{J}(V_j^n - W_j^n)^2\right)\\ \le\ & \frac{1}{2}\left(\|\mathbf{V}^{n-1} - \mathbf{W}^{n-1}\|^2 + \|\mathbf{V}^n - \mathbf{W}^n\|^2\right)\\ &+\frac{M^2r}{2}\|\mathbf{V}^{n-1} - \mathbf{W}^{n-1}\|^2 + \frac{r}{2}\|\mathbf{V}^n - \mathbf{W}^n\|^2\\ \le\ & \frac{1}{2}(1+M^2r)\|\mathbf{V}^{n-1} - \mathbf{W}^{n-1}\|^2 + \frac{1}{2}(1+r)\|\mathbf{V}^n - \mathbf{W}^n\|^2.\end{aligned} \tag{32}$$

A similar estimation for the second item is the following

$$\sum_{j=1}^{J} hr\left|g_{j-1}\left(V_{j-1}^{n-1} - W_{j-1}^{n-1}\right)\left(V_j^n - W_j^n\right)\right|$$

$$
\begin{aligned}
&\leq \frac{rM^2}{2}\sum_{j=1}^{J} h\Big(V_{j-1}^{n-1}-W_{j-1}^{n-1}\Big)^2+\frac{r}{2}\sum_{j=1}^{J} h\Big(V_j^n-W_j^n\Big)^2\\
&\leq \frac{rM^2}{2}\|\mathbf{V}^{n-1}-\mathbf{W}^{n-1}\|^2+\frac{r}{2}\|\mathbf{V}^n-\mathbf{W}^n\|^2+\frac{rhM^2}{2}\Big|V_0^{n-1}-W_0^{n-1}\Big|^2. \qquad (33)
\end{aligned}
$$

The third item satisfies

$$
\sum_{j=1}^{J} hk\Big|\Big(V_j^n-W_j^n\Big)\Big(U_j^n-R_j^n\Big)\Big| \leq \frac{k}{2}\Big(\|\mathbf{V}^n-\mathbf{W}^n\|^2+\|\mathbf{U}^n-\mathbf{R}^n\|^2\Big). \qquad (34)
$$

The fourth item has the property

$$
\begin{aligned}
&-\sum_{j=1}^{J} hk\Big(\mu_j^{n-1}(\mathbf{V})V_j^{n-1}-\mu_j^{n-1}(\mathbf{W})W_j^{n-1}\Big)\Big(V_j^n-W_j^n\Big)\\
=\;&-\sum_{j=1}^{J} hk\Big(\mu_j^{n-1}(\mathbf{V})V_j^{n-1}-\mu_j^{n-1}(\mathbf{V})W_j^{n-1}\\
&+\mu_j^{n-1}(\mathbf{V})W_j^{n-1}-\mu_j^{n-1}(\mathbf{W})W_j^{n-1}\Big)\Big(V_j^n-W_j^n\Big)\\
=\;&-\sum_{j=1}^{J} hk\mu_j^{n-1}(\mathbf{V})\Big(V_j^{n-1}-W_j^{n-1}\Big)\Big(V_j^n-W_j^n\Big)\\
&-\sum_{j=1}^{J} hk\Big(\mu_j^{n-1}(\mathbf{V})-\mu_j^{n-1}(\mathbf{W})\Big)\Big(V_j^n-W_j^n\Big)W_j^{n-1}\leq 0. \qquad (35)
\end{aligned}
$$

Thus, we write down

$$
\begin{aligned}
\|\mathbf{V}^n-\mathbf{W}^n\|^2\leq\;&(\tfrac{1}{2}+M^2r)\|\mathbf{V}^{n-1}-\mathbf{W}^{n-1}\|^2+(\tfrac{1+r}{2}+k)\|\mathbf{V}^n-\mathbf{W}^n\|^2\\
&+\tfrac{k}{2}\|\mathbf{U}^n-\mathbf{R}^n\|^2+\tfrac{hM^2r}{2}\Big|V_0^{n-1}-W_0^{n-1}\Big|^2. \qquad (36)
\end{aligned}
$$

For r sufficiently small and $1\leq n\leq N$, the last result can be rewritten as

$$
\begin{aligned}
\|\mathbf{V}^n-\mathbf{W}^n\|^2\leq\;&C\|\mathbf{V}^{n-1}-\mathbf{W}^{n-1}\|^2\\
&+Ck\bigg(\|\mathbf{U}^n-\mathbf{R}^n\|^2+\Big|V_0^{n-1}-W_0^{n-1}\Big|^2\bigg). \qquad (37)
\end{aligned}
$$

The second equation in (7) yields

$$
g_0\Big|V_0^n-W_0^n\Big|\leq\Big|U_0^n-R_0^n\Big|+h\Big|\beta(\mathbf{V}^n)\mathbf{V}^n-\beta(\mathbf{W}^n)\mathbf{W}^n\Big|. \qquad (38)
$$

Since $g_0=g(0)>0$ is constant, we get

$$
\begin{aligned}
\Big|V_0^n-W_0^n\Big|\leq\;&C\bigg(\Big|U_0^n-R_0^n\Big|+h\Big|\beta(\mathbf{V}^n)\mathbf{V}^n-\beta(\mathbf{W}^n)\mathbf{W}^n\Big|\bigg)\\
\leq\;&C\Big|U_0^n-R_0^n\Big|+hC\Big|\beta(\mathbf{W}^n)(\mathbf{V}^n-\mathbf{W}^n)\Big|\\
&+Ch\Big|\Big(\beta(\mathbf{V}^n)-\beta(\mathbf{W}^n)\Big)\mathbf{V}^n\Big|. \qquad (39)
\end{aligned}
$$

Notice that

$$h\Big|\beta(\mathbf{W}^n)(\mathbf{V}^n-\mathbf{W}^n)\Big| \le \|\beta(\mathbf{W}^n)\|_\infty\bigg(\sum_{j=1}^J h(V_j^n-W_j^n)^2\bigg)^{1/2} \le hC\|\mathbf{V}^n-\mathbf{W}^n\|, \tag{40}$$

and

$$\begin{aligned} h\Big|(\beta(\mathbf{V}^n)-\beta(\mathbf{W}^n))\mathbf{V}^n\Big| &\le \sum_{j=1}^J h\Big|\beta(x_j,Q_h^2(\mathbf{V}^n))-\beta(x_j,Q_h^2(\mathbf{W}^n))\Big||V_j^n| \\ &\le hC\sum_{j=1}^J\Big|\beta(Q_h^2(\mathbf{V}^n))-\beta(Q_h^2(\mathbf{W}^n))\Big| \\ &\le hC\sum_{j=1}^J hq_2(x_j)(V_j^n-W_j^n) \\ &\le h^2C\bigg(\sum_{j=1}^J h(V_j^n-W_j^n)^2\bigg)^{1/2}=h^2C\|\mathbf{V}^n-\mathbf{W}^n\|. \end{aligned} \tag{41}$$

Combining (40) and (41) with (39), we have

$$|V_0^n-W_0^n|\le C\Big[|U_0^n-R_0^n|+h(1+h)\|\mathbf{V}^n-\mathbf{W}^n\|\Big].$$

Furthermore, it follows from (38) that

$$\begin{aligned} \|\mathbf{V}^n-\mathbf{W}^n\|^2 &\le (1+Ck)\|\mathbf{V}^{n-1}-\mathbf{W}^{n-1}\|^2+Ck\Big(\|\mathbf{U}^n-\mathbf{R}^n\|^2+\Big|U_0^{n-1}-R_0^{n-1}\Big|^2\Big) \\ &\le (1+hC)^2\Big[\|\mathbf{V}^{n-2}-\mathbf{W}^{n-2}\|+h\|\mathbf{U}^n-\mathbf{R}^n\|^2+h\|\mathbf{U}^{n-1}-\mathbf{R}^{n-1}\|^2 \\ &\quad +h\Big|U_0^{n-1}-R_0^{n-1}\Big|^2+h\Big|U_0^{n-2}-R_0^{n-2}\Big|^2\Big] \\ &\le (1+hC)^n\Big[\|\mathbf{V}^0-\mathbf{W}^0\|+\sum_{j=1}^J h\|\mathbf{U}^j-\mathbf{R}^j\|^2+\sum_{j=0}^{n-1}h\Big|U_0^j-R_0^j\Big|^2\Big] \\ &= (1+hC)^n\Big[\sum_{j=1}^J h\Big(U_j^0-R_j^0\Big)^2+\sum_{j=1}^J h\|\mathbf{U}^j-\mathbf{R}^j\|^2 \\ &\quad +\sum_{j=0}^{n-1}h\Big|U_0^j-R_0^j\Big|^2\Big] \\ &\le \Big(\|\mathbf{U}_0-\mathbf{R}_0\|^2+\sum_{n=0}^N\|\mathbf{U}^n-\mathbf{R}^n\|^2\Big)^{1/2}. \end{aligned} \tag{42}$$

Now we conclude that

$$\|\mathbf{V}_h-\mathbf{W}_h\|_{X_h}\le\|\Phi_h(\mathbf{V}_h)-\Phi_h(\mathbf{W}_h)\|_{Y_h} \tag{43}$$

and the proof is completed.

Then, the following convergence result follows immediately.

Theorem 4.3(Convergence). Under the assumptions in Theorems 4.1 and 4.2, if $\|\mathbf{P}^0 - \mathbf{p}^0\| = o(h^{1/2})$, the numerical scheme (5) satisfies

$$\max_{0\le n\le N} \|\mathbf{P}^n - \mathbf{p}^n\| = O(\|\mathbf{P}^0 - \mathbf{p}^0\| + h),\ h \to 0. \tag{44}$$

5. Numerical Experiments

In this section, we present two examples to verify the feasibility of the scheme. The computations are implemented by MATLAB and used to display the stability of steady states intuitionally. We set, in the examples, size step $h = 0.025$, time step $k = 0.02$, maximum size $l = 80$.

Example 1. Zero equilibrium is asymptotically stable.

The parameters are taken as follows:

$$p_0(x) = \begin{cases} (x-8)^2(40-x)^2, & 8 \le x \le 40, \\ 24(x-40)^2(56-x)^2, & 40 \le x \le 56, \\ 8(x-56)^2(80-x)^2, & 56 \le x \le 80, \\ 0, & \text{else}; \end{cases} \tag{45}$$

$$\begin{aligned} &g(x) = (80-x)(40-x),\ q_1(x) = 80-x,\ q_2(x) = 20x, \\ &\mu(x,Q_1) = \tfrac{(x+1)(34+x)}{20} + 2Q_1,\ \beta(x,Q_2) = 2x(31+x) + Q_2. \end{aligned} \tag{46}$$

One can see from Figure 1 that the population is rapidly approaching to zero.

Example 2. Zero equilibrium is unstable.

The functional parameters of the population are as follows:

$$p_0(x) = \begin{cases} x^2(24-x)^2, & 0 \le x \le 24, \\ (x-24)^2(48-x)^2, & 24 \le x \le 48, \\ (x-48)^2(80-x)^2, & 48 \le x \le 80, \\ 0, & \text{else}; \end{cases} \tag{47}$$

$$\begin{aligned} &g(x) = 2(1-\tfrac{1}{2}e^x),\ q_1(x) = 2(1-x),\ q_2(x) = 1-x, \\ &\mu(x,Q_1) = \tfrac{x}{20} + 2Q_1,\ \beta(x,Q_2) = 6x + Q_2. \end{aligned} \tag{48}$$

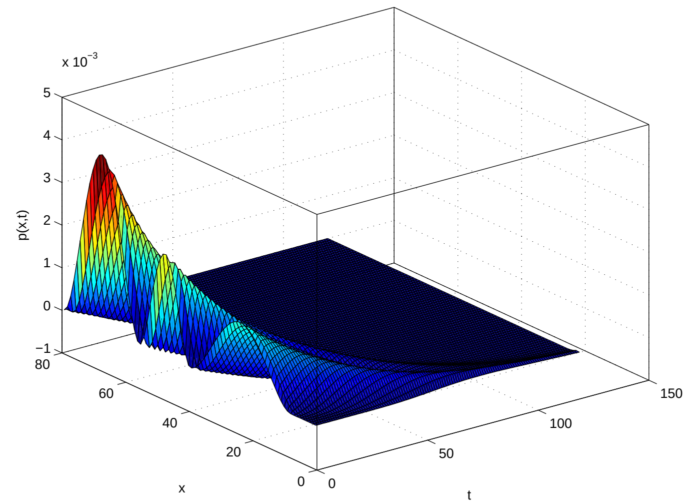

FIGURE 1. Population in extinction

In the Figure 2, the population fluctuates more and more, tending to be away from trivial state, which shows the equilibrium is unstable.

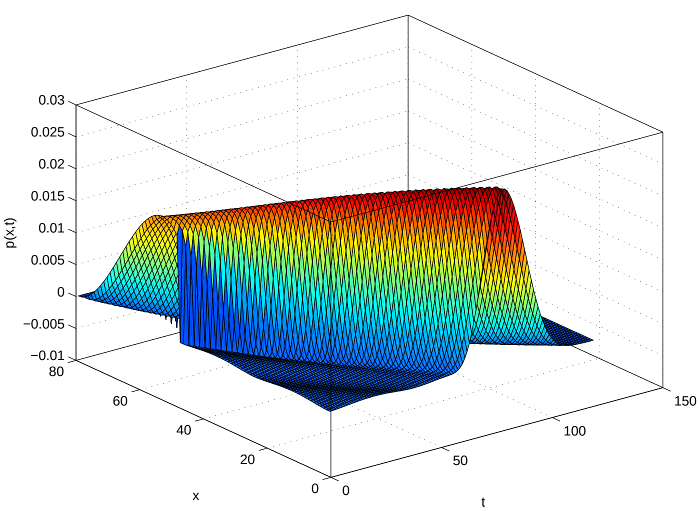

FIGURE 2. Population in more oscillations

6. Concluding remarks

In the previous sections, we have proposed an upwind difference scheme for a nonlinear size-structured population model, in which the growth of individual's size may be negative. This relaxed assumption generalizes most of the treated models. The rigid analysis of convergence supplies a solid foundation to the approximation to the solutions, which can not be derived in any closed form. Computations of solutions in two numerical examples demonstrate the effectiveness of the scheme, and display the stability of model equilibriums as well.

Conflict of Interests

The authors declare that there is no conflict of interests.

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