Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2019, 2019:32
https://doi.org/10.28919/cmbn/4236
ISSN: 2052-2541

BIO-ELECTRIC POTENTIAL FIELD AND CURRENT SOURCE DISTRIBUTION ABOUT FEASIBLE AUTOMATIC VOLUME CONDUCTION SYSTEM

NZEREM FRANCIS EGENTI* AND UGORJI HYCINTH CHIMEZIE

Department of Mathematics and Statistics, University of Port Harcourt, Choba, Nigeria

Abstract: Bioelectric sources are the bastion of action potentials that pervade abutting cells and their regions. Such sources are characterized by impressed current density which originate from a nonelectrical source of energy. In the CCS under consideration current density emanates from the movement of ions by reason of concentration gradient, which is the main cause of the creation of an electric field. The pertinent issue is the determination of the quantity of ionic current that exits a source to a sink, which in effect drives a potential field in a dipole environment. Therefore, the quantity of current that transmits through dipoles and the concomitant potential field about the cardiac conduction system were sought. The equation satisfying the potential field in the region of interest was found by solving Laplace's equation in cylindrical coordinate. It is hoped that a good knowledge of the quantity of current and the potential field may qualitatively inform cardiac procedures.

Keywords: boundary conditions; bio-electric; cardiac; current density; dipole.

2010 AMS Subject Classification: 94C15, 47N70, 62P10

1. INTRODUCTION

Volume conduction models are of essence in analysing bioelectric phenomena. They describe the geometry and conductivity of any tissue within which electric current flows, and also describe the current sources in the tissue. Sources are produced by the passage of current across the membrane of active excitable cells. The heart is a known bioelectric source and volume conductor, with most

*Corresponding author
E-mail address: frankjournals@yahoo.com
Received July 26, 2019

of its nodal structures as electric sources and sinks. It is a veritable site for the propagation of autonomous and automatic action potential (AP). The eventuation of such APs depends on the effective conduction of electrical current from an excited cell to a contiguous quiescent cell. This excited cell to quiescent cell mode of transmission is required to effectuate cardiac cycle. In the event of a cell delivering a charge to an adjoining quiescent cell, the beneficent cell becomes the *source* while the recipient cell becomes the *sink*. In fine, virtually every cell has the dual role of a source and a sink. The source-sink balance is a crucial element of cardiac system. The consummation of the conduction system presupposes the generating of the required quantity of current that would be sufficient for the local sink. This is one possible way of avoiding source-sink mismatch which is implicated in deleterious cardiac events. In this regard, Boyle and Vigmond [1], Kleber and Rudy [2] stressed the safety need and, therefore, the quantification of the source-sink balance. A physiological heart is herein referred to as *feasible*. This term is used in the sense of bio-communication network. Automaticity is a known characteristic of the heart in the initiation of action potentials.

In analysing source-field models, Malmivuo and Plonsey [3] described bioelectric sources as surface/volume distributions of two types of source element, namely the monopole and/or dipole. There the source was treated as a bundle of parallel muscle fibres and therefore, a one-dimensional problem. Studies on the spread of electric currents arising from sources in the cardiac cells, through surrounding tissues are of immense interest. Sources and sinks are spatially localized phenomena that are the physical grounds of field potentials. It is believed that impressed currents relate to electrical activity at the membranes of excitable cells [4]. This paper treats bio-electric sources and sinks phenomena from the standpoint of the cardiac conduction system, CCS.

In the CCS electrical sources and fields may be described by (monopole and) dipole fields. In current flow fields a monopole refers to a point source or sink of current within a conducting medium. It was the view of Plonsey [5] that the monopole is a rarity in bioelectricity since sources emanate from active tissues consisting of distinctively spaced source and sink combinations. It is instructive to note that pairs of equal and unlike abutting monopoles constitute a central source unit in electrophysiology. A combination of current source and current sink is a dipole. A distribution of dipoles is found in an active membrane source.

The heart's structure, as a conductor, is considered *finite, inhomogeneous and anisotropic*. In general, conductivity within the tissue differs from point to point (inhomogeneity). Usually, the extracellular space is considered a volume conductor when only the electro-migration current emanating from the presence of the electric field is put into consideration. Nonetheless, diffusion or advection could prompt the migration of ions in the interstitial fluid even in the absence of an electric field [6]. Fibrous tissues, such as muscle and nerve, demonstrate anisotropic property; the conductivity in the direction of the fibres exceeds the conductivity perpendicular to the fibres [7]. The cell geometry, cell size, the directional distribution of gap junctions and membrane ion channels constitute the structural determinants of anisotropic conduction. A consequence of anisotropy is that the current density may not be in the same direction as the electric field and therefore, it (anisotropy) mediates the propagation of action potential [8]. This lends credence to the notion that rotational anisotropy accelerates the spread of electrical excitation in the heart [9]. In order to ease the cumbersomeness of inhomogeneity, the delineation of volume conductor into compartments was perceived. This way, compartmental homogeneity is assumed and therefore constant conductivity is enhanced. For the purpose of approximating the cardiac tissue as homogeneous and isotropic the heart, as a volume conductor, is assumed made up of layers of dipole source elements, which are located in the isochronal activation surfaces.

2. FUNDAMENTAL ELECTRICAL PROPERTIES OF TISSUES

A universal volume conductor may define a region of volume, Ω, which has conductivity, σ, and permittivity, ϵ, wherein there exists a source current, I_{ω}. The current sources stem from bio-excitable cells undergoing an activation process. In biological volume conductors the central question is to acquire the accurate location and strength of the sources inside the conductor by knowledge of an electrical potential measured on a point/region of interest. [10]. In the cardiac tissue activation can be perceived as the process in which cells experience rapid depolarization. A regular problem in bioelectric theory is the determining of the potential distribution, $\varphi(.)$, arising from sources, all through a volume conductor. In cardiac physiology, calculation of the scalar potential, φ, is essential in cardiac pacing and defibrillation. In bioelectric problems it is assumed to be quasi-static since it frequently changes slowly enough [3]. Therefore, capacitive and inductive effects are ignored. In the cardiac conduction system in which source fields exist and where source–sink mismatch is seen, the much needed problem shall be to determine the source

distribution that give rise to potentials**.** For instance how does the relatively small SAN cope with the electrical exigencies of the much larger atrial tissue? (see Joyner and Capelle [11]). How do the Purkinje cells satisfy the ventricular mass to cope with source-sink mismatch? (see Morley *et al*. [12]). In a resistive volume conductor the equation for the potential field is of the form

$$\nabla.\sigma\nabla\varphi = \nabla.\,\mathbf{J}_{\omega}, \quad \text{in } \Omega, \tag{1}$$

where σ is the conductivity of the medium, φ is the electric potential, and $\mathbf{J}_{\omega}$ is the source current density. A boundary condition for the potential may be found by confining every volume current inside the volume, Ω.

$$\left(\sigma\nabla\varphi(\mathbf{r})\right).\mathbf{n} = 0, \ \mathbf{r} \in \partial\Omega, \tag{2}$$

which is the Neuman condition, where **n** is the outward unit normal vector of the surface $\partial\Omega$.

In any region where conductivity is homogeneous and isotropic Poisson's equation is obtained from equation (1) as

$$\nabla^2\varphi = \lambda \tag{3}$$

where $\lambda = \nabla.\mathbf{J\omega}/\sigma$. In the absence of bioelectric current source $\lambda = 0$.Thus, divergence-free conducting regions are better described by the potential (Laplace's) equation. In regions distal from the site of an action potential the current density, **J**(A/m^2), is linearly related to the electric field intensity, **E** (*V/m*). This relationship, in line with the microscopic form of Ohm's law, given by the current density

$$\mathbf{J} = \sigma\mathbf{E}\ . \tag{4}$$

The above is an estimation of a region devoid of bioelectric sources, with assigned uniform bulk conductivity, σ. The current density, **J,** arises secondary to the existence of the aforesaid electric

field **E**. In the absence of electromagnetic wave the electric field at each instant is derivable from the gradient of an electric scalar potential, φ.

$$\mathbf{E} = -\nabla\varphi. \tag{5}$$

Substituting (4) and (5) into (1) gives

$$\nabla\cdot\mathbf{J} = -\nabla\cdot\mathbf{J}_{\boldsymbol{\omega}} \tag{6}$$

These properties of body tissues suggest that the instantaneous currents depend purely on the sources at that instant.

2.1 Divergence of current density in a volume

The concept of divergence is crucial to the analysis of issues on volume conduction. It relates the deviation of the current density in a volume, Ω, to the current density through the surface, $\partial\,\Omega$, of the volume. In effect we have,

$$\iiint_{\Omega} \nabla.\mathbf{J}\,d\Omega = \oiint_{\partial\Omega} \mathbf{J}.\mathbf{n}\,dS \tag{7}$$

where **n** is the outward normal vector on $\partial\,\Omega$. From (5) we have

$$\iiint_{\Omega} \nabla.\mathbf{J}_{\omega}\,d\Omega = -\oiint_{\partial\Omega} \mathbf{J}.\mathbf{n}\,dS \quad . \tag{8}$$

The left hand side of (8) encodes the net current source generated inside the volume, whereas the right hand side is the total conduction current flowing through the surface $\partial\,\Omega$ (leavingΩ). For a monopole conveying a current I_{ω} and with an infinitesimal small volume Ω_{ε} about the source, equation (8) reads

$$\lim_{\Omega_\varepsilon \to 0} \iiint_{\Omega\varepsilon} \nabla . J_\omega d\Omega = -I_\omega , \tag{9}$$

and thus,

$$\nabla . \mathbf{J_\omega} = -I_\omega \delta(\tau) \tag{10}$$

where $\delta(\tau)$ is a three dimensional Dirac-delta function [7]. If tissue capacitance is neglected then the sources vary, charges on boundaries and interfaces are self-redistributed in a relatively short time such that equation (6) reads

$$\nabla \cdot \mathbf{J} = \mathbf{0}. \tag{11}$$

The above is in line with divergence-free region of the CCS, such as along the internodal pathways (see [13]). This is also the case when each node is a sink. For instance, the atrioventricular node (AVN) serves as a sink prior to depolarization. It only becomes a source when impulse is transmitted to it from the SAN via the SAN-AVN internodal pathways. The CCS is an agglomeration of current sources (sinks) connected by conduction pathways. It is marked by various phases of action potentials (depolarization) wherewith resting phases apply. The heart contains about 5×10^{10} cells of which possibly 5% are active at any instant during depolarization [5]. In this regard, quiescent cells and nodes are only electrically active when they are activated by abutting active nodal cells. The quiescent cells are in such a state because an abutting sink has not been furnished to act as a source for the next phase of action potential. From the physical electricity standpoint, a steady current density, $\mathbf{J}(x, y, z)$ presupposes that $\nabla \cdot \mathbf{J} = \mathbf{0}$ at all directions, x, y, z. The conducting bioelectric medium extends continuously with distributed resistances, capacitances, and batteries [3].

3. CARDIAC CURRENT SOURCES AND DISTRIBUTION

The cardiac conduction system comprises five elements [13, 14, 15, 16]: the sino-atrial node (SAN), the atrio-ventricular node (AVN), the bundle of His, the bundle branches, the Purkinje fibres, all of which are connected by the conduction pathways. Source-sink process determine the electrical flow in the system. Fig. 1 below is the CCS. Fig.2, an extract from Fig. 1, shows the nodal points among other indications. The nodal points represent the source and sink points.

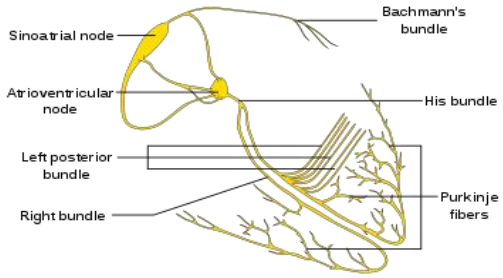

Fig.1 Electrical conduction system of the heart [17]

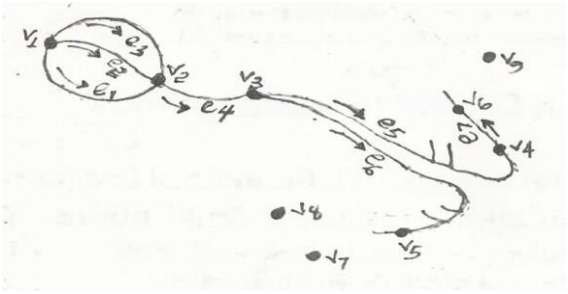

Fig.2 Nodal point schematic of the CCS [13]

In Fig. 2 the nodes are represented by v_i (i = 1, 2 ,…, 7). The node: v_1 stands for the SAN, v_2 stands for the AVN, v_3 stands for the point of bifurcation of the bundle of His, v_5 stands for the left bundle branch, v_6 stands for the right bundle branch. The individual bundle branch has attachments to the Purkinje fibres, which have infinitely many miniscule branches and nodes. The arrow heads (Fig. 2) indicate flow direction.

3.1 From monopole to dipole

From the perspective of the CCS, the primary pacemaker, the SAN is the pivotal current source of finite extent. At times is beneficial to describe sources of finite extent as a continuum of point sources (monopoles). Since the SAN is within the cardiac volume, the left hand side of equation (8) holds. The SAN may be considered a point source carrying a current I_{ω}, with an infinitesimal small volume Ω_{ε} about the source. Equation (9) holds well for such current source. If I_{ω} is a point current source, such as the SAN centre, lying in a conducting medium and conductivity σ, then the resulting flow lines must be uniform and directed radially. Therefore, the current density J crossing a concentric spherical uniform surface of arbitrary radius r *is*

$$J = \frac{I_{\omega}}{4\pi r^2} \tag{12}$$

By reason of the current being everywhere in the radial direction we express the current density as a vector in the form

$$\mathbf{J} = \frac{I_{\omega}}{4\pi r^2}\mathbf{e}_r\,, \tag{13}$$

where $\boldsymbol{e}_r$ is the unit vector in the radial direction, where the origin is at the point source. Associated with the current flow field defined by equation (13) is a scalar potential field φ to which the electric field **E** is related as shown in (5). There is variation of potential along a transverse direction in virtue of the fact that the field is everywhere radial. Therefore, the isopotential surfaces are, expectedly, a series of concentric spheres surrounding the point source, with the origin at the monopole source, whose potentials diminish for increasing values of r. In connection with monopole, the potential φ_m is given by

$$\varphi_m = \frac{I_\omega}{4\pi\sigma r}. \tag{14}$$

The CCS comprises several sources, to wit SAN, AVN, His bundle branch point, among others each of which may be considered a monopole on individual basis. It is therefore not at all times suitable to place the coordinate system origin at the point source. So the coordinates of the point source(s) have to be distinguished from that of the field point. This way we have

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\ , \tag{15}$$

with each monopole located at (x, y, z) and the field point is at (x', y', z'). In the feasible conducting system in context the respective *dipole* elements are to wit (see Fig.2): the SAN(v_1) → AVN (v_2), AVN → Bundle of His branch point (v_3), both of which are in series arrangement, $v_3 \rightarrow v_4$, $v_3 \rightarrow v_5$, $v_4 \rightarrow v_6$ **(**the source point of v_4 and v_5 is v_3**).** Each arrow represents source-sink direction. If d is a small distance separating each dipole**,** the dipole moment is $p = I_\omega d$. Thus, **d** is the displacement from negative to positive point source and $\mathbf{e}_d$ is a unit vector in that direction, and the dipole vector, **p**, is related to **d** by

$$\boldsymbol{p} = I_\omega \boldsymbol{d} = I_\omega d \boldsymbol{e}_d \tag{16}$$

The exact expression for a well-defined dipole is [5]

$$\varphi_d = \frac{p}{4\pi\sigma}\nabla\left(\frac{1}{r}\right)\cdot\mathbf{e}_d \,. \qquad (17)$$

Now we represent the sources by impressed current density, $\mathbf{J}^c$, which do not emanate from electric field; they originate in a nonelectrical source of energy. In the CCS the current density emanates from the movement of ions by reason of concentration gradients; it is the primary cause for the establishment of an electric field [5]. With this, equation (4) modifies to

$$\mathbf{J} = \sigma\mathbf{E} + \mathbf{J}^c \qquad (18)$$

Thus, the total current density is the sum of the *Ohmic* component, which gives a linear current – voltage relationship, and the impressed current density.

Equation (11) combine with (5) and (18) to give

$$\nabla.\nabla\sigma\mathbf{V} = \nabla.\mathbf{J}^c \;. \qquad (19)$$

In regions containing sources the source term can be estimated using equation (19). Such estimates would furnish information about the divergence of $\mathbf{J}^c$ whose dimension is current per unit volume.

3.2 Ionic current through CCS dipoles

As said earlier, $\mathbf{J}^c$ represents the sources which emanate from the movement of ions by reason of concentration gradients. Let us take a look at the SAN→AVN dipole, and by extension all other dipoles listed here earlier.

.

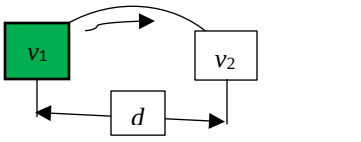

Fig.3. A dipole separated by a distance, d

In Fig 3. above the green box (v_1) represents the active node (source), while the blank box (v_2) represents the inactive node (sink); the arc indicates the source-to-sink path. and the arrow indicates flow direction.

The governing differential equation of the action potential of coupled cells is given by [18]

$$\frac{\mathrm{d}V_{\mathrm{m}}}{\mathrm{d}t} = -\frac{1}{C_{\mathrm{m}}} I_{\Sigma} - \sum_{i} g_{\mathrm{j}}(V_{\mathrm{m}} - V_{\mathrm{m,i}}), \tag{20}$$

where V_{m} is the membrane potential, I_{Σ} is total ionic current, $V_{\mathrm{m},i}$ is the membrane potential of cells connected to the cell of interest, C_m is the membrane capacitance of an individual cell and g_{j} is the coupling conductance. In the absence of coupling, $g_{\mathrm{j}} = 0$, and the second term of the right hand side of equation (20) vanishes. Assume the membrane potential is constant over some time interval. Then, a quasi-static current that flows across the dipole is,

$$I_{\Sigma} = -C_{\mathrm{m}} \sum_{\mathrm{i}} g_{\mathrm{j}}(V_{\mathrm{m}} - V_{\mathrm{m,i}}) \ . \tag{21}$$

(It may be of note that the ionic current I_{Σ} is in the form of impressed current density, $\mathbf{J}^{\mathrm{c}}$). The flow of current from the SAN (source) via the atrial conduction pathways to the AVN (sink), Fig. (2) can be obtained from (21) as

$$I_{\Sigma} = I = -C_{\mathrm{m}} g(V_{\mathrm{m(SAN)}} - V_{\mathrm{m(AVN)}}) \ . \tag{22}$$

Suppose these ions move autonomously from concentration c_1 to concentration c_2 (i.e. SAN→AVN). The current they carry has the form (Peskin [19])

$$I = -q(l_1(V)c_1 - l_2(V)c_2) \ ,$$

where, for an ionic species, q is the charge per molecule, V is the voltage across the two nodal cells, and l_1, l_2 are the voltage-dependent rate constants. Both $l_1(V)$ and $l_2(V)$ are constrained by ionic thermodynamics. In thermodynamic consideration, the work done in moving the ion molecules from concentration c_1 to concentration c_2 can be calculated as

$$qV = lT\log\frac{c_2}{c_1} \tag{23}$$

From (23) above we get

$$\frac{c_2}{c_1} = e^{\xi V}, \tag{24}$$

(where $\xi = \frac{q}{lT}$) At equilibrium ($I = 0$), we have

$$\frac{l_1(V)}{l_2(V)} = e^{\xi V}. \tag{25}$$

Therefore the current reads

$$I = q l_1(V) c_1 \left(1 - \frac{c_2}{c_1} e^{-\xi V}\right). \tag{26}$$

The conservation law requires that concentrations c_1, c_2 be in equilibrium. Let V_0 (c_1, c_2) be the potential at which equilibrium holds. Then

$$\frac{c_2}{c_1} = e^{\xi V_0}, \tag{27}$$

$$I = q l_1(V) c_1 \left(1 - e^{-\xi(V - V_0)}\right). \tag{28}$$

The issue of linear current-voltage relationship is quite salient since the voltage across the resistor is linearly proportional to the current through it- the basis of Ohm's Law. There is no choice of $l_1(V)$ independent of V_0 that may furnish linearity for all concentrations. To this effect a recourse to Peskin [19] for a linear relationship by constructing such $l_1(V)$ for a particular V_0 is considered.

Let

$$l_1(V) = \frac{H(V - V')}{1 - e^{-\xi(V - V')}} \tag{29}$$

$$l_2(V) = \frac{H(V - V')e^{-\xi V}}{1 - e^{-\xi(V - V')}} \tag{30}$$

where H is some value that depends on its argument.

Therefore,

$$I = \frac{qH(V-V')}{1-e^{-\xi(V-V')}} c_1 \left(1-e^{-\xi(V-V')}\right) . \tag{31}$$

With $V_0 = V'$ we get

$$I = qc_1 H(V- V') . \tag{32}$$

The above is a linear current-voltage relations which holds for all c_1, c_2 such that

$$\frac{c_2}{c_1} = e^{V'} \tag{33}$$

From equation (31) we see that the equation

$$I = \frac{qH(V-V')}{1-e^{-\xi(V-V')}} c_* \left(1-e^{-\xi(V-V')}\right) \tag{34}$$

holds well for any dipole current, where c_* is the ionic concentration at each dipole source. The details supplied by the SAN→AVN dipole shall therefore by analogous to those of other diploes of the CCS.

3.3 Potential field and current in the dipole path

Potential field in the region between each dipole nodal cells is considered in this section. The analysis in this section takes from the seminal work by Heppner and Plonsey ([20], it also may interest one to see [21]). The abutting dipole cells are assumed cylindrical, each of radius r_o, one assumed active and the other assumed inactive, and are separated by a distance d.

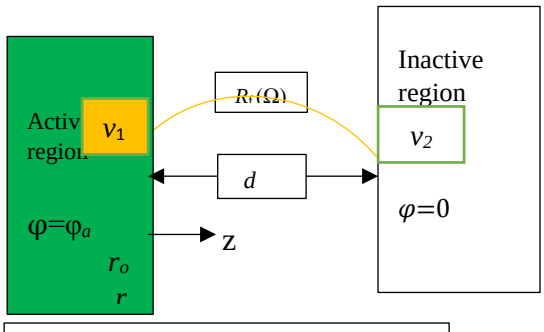

Fig.4. Resistances across dipole path

Assume the potential field $\varphi(r, z)$ dominate a region, $0 < z < d$; $0 \le r \le r_o$ about the dipole v_1, v_2. It is assumed that the connecting fibres are swathed with non-excitable membranes, with specific resistance R_f (Ω-cm^2) and the flow of current depends only on the voltage drop across the gap. In the specified region the potential, φ, will satisfy Laplace's equation $\nabla^2\varphi(r,z) = 0$. With the cylinder assumed axisymmetric we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+\frac{\partial^2\varphi}{\partial z^2} = 0 \tag{35}$$

The associated boundary conditions in relation to the current density satisfies the Neumann condition at $z = 0$ and $z = d$, and together with the behaviour of φ in the radial direction we get [20]:

$$\left.\begin{array}{ll} \text{(i)} & \dfrac{\varphi(r,0^+)-\varphi_a}{R_f} = \sigma\left.\dfrac{\partial\varphi}{\partial z}\right|_{z=0^+}, \\ \text{(ii)} & \dfrac{\varphi(r,d^-)}{R_f} = -\sigma\left.\dfrac{\partial\varphi}{\partial z}\right|_{z=d^-,}, \\ \text{(iii)} & \varphi \text{ is finite at } r = 0, \\ \text{(iv)} & \varphi = 0 \text{ at } r = r_o. \end{array}\right\} \tag{36}$$

Conditions (iii) and (iv) of (36) are imposed to have a solution of equation (35) is in the form

$$\varphi(r,z) = \sum_{n=1}^{\infty} J_0(k_n r)[A_n\cosh(k_n z) + B_n\sinh(k_n z)], \tag{37}$$

where k_n are the roots of $J_o(k, r) = 0$. Applying boundary conditions (i) and (ii) of (36) gives

$$\sum_{n=1}^{\infty} A_n J_0(k_n r) - \varphi_a = R_d\sum_{n=1}^{\infty} k_n B_n(k_n r), \tag{38}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} J_0(k_n r)[A_n\cosh(k_n d) + B_n\sinh(k_n d)] \\ = R_f\sigma\sum_{n=1}^{\infty} J_0(k_n r)k_n[A_n\cosh(k_n d) + B_n\sinh(k_n d)].\end{aligned} \tag{39}$$

The coefficients A_n and B_n are obtained by multiplying both sides of the above two equations by $rJ_0(k_m r)$, integrating with respect to r over $[0,r_0]$ and applying the orthogonality property of the Bessel functions. Substitute the resulting expressions into equation (36) and get

$$\varphi(r,z)=\varphi_a\sum_{n=1}^{\infty}\frac{2J_0(k_n r)}{\left(f_n+\sigma R_f k_n\right)k_n r_0 J_1(k_n r_0)}.[f_n\cosh(k_n z)-\sinh(k_n z)], \tag{40}$$

where

$$f_n=[\sigma R_f k_n+\tanh(k_n d)]/[1+\sigma R_f k_n\tanh(k_n d)].$$

The gap region accommodates current density $-\sigma\nabla\varphi$, with φ given by equation (40) above. Three notable currents come to play. They are the currents crossing the active and quiescent disc surfaces and the current across the cylindrical wall at $r = r_0$ $(0 < z < d)$. Denote these currents by I_A, I_Q and I_W respectively. The total current flowing into the gap from I_A is

$$I_A=-2r\sigma\int_0^{r_0} r\left.\frac{\partial\varphi}{\partial z}\right|_{z=0} dr.$$

The current flowing into the quiescent cell is

$$I_Q=-2r\sigma\int_0^{r_0} r\left.\frac{\partial\varphi}{\partial z}\right|_{z=d} dr.$$

The residual current, I_W, across the cylindrical gap boundary is

$$I_W=-2r\sigma\int_0^{r_0} r\left.\frac{\partial\varphi}{\partial r}\right|_{r=r_0} dz.$$

Substitute equation (37) into the above three equations and interchange the order of integration to get

$$I_A=4\pi\sigma r_0\varphi_a\sum_{n=1}^{\infty}\frac{1}{e_n}, \tag{41}$$

$$I_Q=-4\pi\sigma r_0\varphi_a\sum_{n=1}^{\infty}\frac{1}{e_n}[f_n\sinh(k_n d)-\cosh(k_n d)], \tag{42}$$

$$I_W=4\pi\sigma r_0\varphi_a\sum_{n=1}^{\infty}\frac{1}{e_n}[f_n\sinh(k_n d)-\cosh(k_n d)+1], \tag{43}$$

where

$$e_n=\left(f_n+\sigma R_f k_n\right)k_n r_0,$$

$$f_n=\frac{\sigma R_f k_n+\tanh(k_n d)}{1+\sigma R_f k_n+\tanh(k_n d)},$$

and φ_a is the constant which denotes the potential assigned to the active cell relative to the resting potential. The quiescent cell has zero potential.

We have come to an important expression for source/sink in the milieu of a potential field. When dipole current is expressed as the current crossing the active and quiescent nodal cells via the pathways, which are assumed cylindrical, we get

$$I = \frac{qH(V-V')}{1-e^{-\xi(V-V')}} c_* \left(1-e^{-\xi(V-V')}\right) = I_A \ , \quad \text{(see equation (41)).} \tag{44}$$

By continuity law,

$$I_A = I_Q + I_W. \tag{45}$$

Thus, the current across each dipole satisfies

$$I_\Sigma = \frac{qH(V-V')}{1-e^{-\xi(V-V')}} c_* \left(1-e^{-\xi(V-V')}\right) = I_Q + I_W \tag{46}$$

4. SUMMARY AND DISCUSSION

A feasible automatic volume conductor is, in this context, a bio-electric medium within which electrical impulse is generated and transmitted as a network. The term *feasible* indicates that the current flow is through a physiological conductor- the cardiac conduction system, as was treated here. In the next work, an inverse problem is being treated to analyse possible pathologies that affect the sources in order to propose the basis for their diagnostic decisions. This work treated each active cell as a current source and the quiescent cell as a sink. It is of note, however that every quiescent cell becomes a source when depolarised. Each intuitive cell centre is considered a node. The arcs (paths) connecting two nodes are the appurtenances of the of the network structure.

A combination of source and sink, which are equal but opposite, is a dipole. In a physiological state, a dipole obeys the continuity principle. This is the bastion of *feasibleness* of electric current. The independent flow of ions was considered. In standard physics of electricity, **J** obeys Ohm's law. However, in bio-electric volume conduction sources are represented by impressed current

density, $\mathbf{J}^c$, which do not derive from electric field. In the CCS the current density emanates basically from the movement of ions as a sequel to concentration gradients.

A potential field $\varphi(r,z)$ is created in the region between each dipole nodal cells. In any region of interest the potential field is assumed to satisfy Laplace's equation in cylindrical coordinates with axial symmetry. If φ is known, usually by solving the Laplace's equation, the currents leaving the active node (source) to the inactive node (sink) may be evaluated. This was done here. For avoidance of clumsy repetitions only the SAN-AVN dipole was used; the analysis of the other dipoles of the CCS is similar to the aforesaid dipole.

We now turn to the divergence of the current density, J, and its implication as regards the CCS. From the physical electricity standpoint, a steady current density, $J(x, y, z)$ presupposes that $\nabla\cdot\mathbf{J} = \mathbf{0}$ at all directions, *x, y, z.* This encodes a solenoidal vector field, typical of region of the CCS such as the dipole distance, *d*, where a current source does not exist within the cardiac volume. By this, the heart fails to be autonomous and automatic in the creation of electricity

.

The intrinsic of the CCS, and perhaps isolated bio-electric systems with sources, does oppugn the presumption of physical electricity. So long as sources exist and quiescent cells await impulse, divergence condition must prevail! This means that the divergence of current density must be nonzero (i.e.$\nabla\cdot\mathbf{J} \neq \mathbf{0}$) at source points. The existence of divergence in the CCS entrenches the benefit of sovereign and adequate energy creation, which may be niggardly supplied from an outside source in critical cases of demand.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] P.M. Boyle and E.J. Vigmond, An intuitive safety factor for cardiac propagation, Biophys. J. 98(2010), L57–L59.

[2] A. G. Kleber, Y. Rudy,Basic mechanisms of cardiac impulse propagation and associated arrhythmias. Physiol. Rev. 84 (2004), 431–488.

[3] J. Malmivuo, R. Plonsey, Bioelectromagnetism Principles and Applications of Bioelectric and Biomagnetic Fields, Oxford University Press. (1995**).**

[4] S. L. Gratiy, G. Halnes, D. Denman, M.J. Hawrylycz, C. Koch, G. T. Einevoll, C. A. Anastassiou, From Maxwell's equations to the theory of current-source density analysis, Eur. J. Neurosci. 45(2017), 1013–1023.

[5] R. Plonsey, C. B. Roger, Bioelectricity: A quantitative approach, Springer-Verlag, (2007).

[6] R.F. Probstein, Physicochemical Hydrodynamics: An Introduction. John Wiley & Sons. (2005).

[7] J. J. Struijk, Ken Yoshida, Volume conduction, https://person.hst.aau.dk/jjs/Bioelektrisk%20Modellering/Volume%20conduction.pdf. (2004).

[8] M. Valderrábano, Influence of anisotropic conduction properties in the propagation of the cardiac action potential, Prog. Biophys. Mol. Biol. 94(2007), 144–168.

[9] F. P. Sergey, H. Dierckx, L. B. Katsnelson, S.Olga, S. M. Vladimir, V. P. Alexander, Electrical Wave Propagation in an Anisotropic Model of the Left Ventricle Based on Analytical Description of Cardiac Architecture, PLOS ONE 9(6)(2014), e101611.

[10] M. Doschoris and F. Kariotou, Mathematical Foundation of Electroencephalography, Electroencephalography, IntechOpen, DOI: 10.5772/68021. (2017).

[11] R. W. Joyner, F. J. L. van Capelle, Propagation through electrically coupled cells, How a Small SA Node Drives a Large Atrium, Biophys. J. 50(1986), 1157-1164.

[12] G.E.Morley, S.B. Danik, Bernstein, S. Sun, Y. Rosner, G.Gutstein, D.E.,etal. Reduced intercellular coupling leads to paradoxical propagation across the Purkinje-ventricular junction and aberrant myocardial activation. Proc. Natl. Acad. Sci. U.S.A. 102(2005), 4126– 4129.

[13] F. E. Nzerem, H. C. Ugorji, Cardiac conduction system: the graph theoretic approach J. Math. Comput. Sci. 9 (2019), 303-326.

[14] Cardiac Conduction System, https://www.nottingham.ac.uk/nursing/practice/resources/cardiology/function/conduction.php

[15] H. Jan van Weerd, V. M. Christoffels, The formation and function of the cardiac conduction system, Development, 143(2016), 197-210.

[16] J.H., T.B. Craig, Specification of the Cardiac Conduction System by transcription factors, Circ. Res., 105(7) (2009), 620–630.

[17] Electrical conduction system of the heart, https://en.wikipedia.org/wiki/Electrical_conduction_system_of_the_heart

[18] S. Inada, H. Zhang, JO Tellez, N. Shibata, K. Nakazawa, K. Kamiya, et al. Importance of Gradients in Membrane Properties and Electrical Coupling in Sinoatrial Node Pacing. PLoS ONE 9(4) (2014), e94565.

[19] C.S. Peskin, & Courant Institute of Mathematical Sciences. Mathematical aspects of heart physiology. Courant Institute of Mathematical Sciences, New York University, New York. (1975).

[20] D. B. Heppner, Robert Plonsey, Simulation of electrical interaction of cardiac cells, Biophys. J. 10 (1970), 1057-1075.

[21] J.W. Woodbury, W.E. Crill, The potential in the gap between two abutting cardiac muscle cells a closed solution. Biophys. J. 10 (1970), 1076-1083.