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OPTIMAL CONTROL ANALYSIS OF A PREDATOR-PREY MODEL WITH HARVESTING AND VARIABLE CARRYING CAPACITY

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Abstract. In this paper, we developed and fully analysed a mathematical model for the dynamics of predator and prey where the carrying capacity is considered to be a logistically increasing function of time, and both populations are under harvesting. Our results showed that if the harvesting rate is high then both populations could go to extinction. We also showed that the system undergoes Hopf bifurcation when the harvesting rate of the prey crosses a critical value; in fact the stability of the system changes with the change of the values of the prey harvesting rate. Optimal harvesting is shown to give a high yield and keep both population away from extension.

Keywords: predator-prey model; variable carrying capacity; optimal harvesting; local stability; Hopf bifurcation

2010 AMS Subject Classification: 34D20, 92B05

1. Introduction

The field of renewable natural resources contains various filed such as, forestry, fishers and agriculture. In our environment there are a lot of biologically interesting problems which are dramatically. Mathematical modelling requires to understand critical behaviour and the underlying nature of the system. Mathematical modelling and exploited analysis biological resources attract the attention of researchers from time to time. The main goal of developing the model is

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not to calculate the change in a certain population, but to study the amount of complexity that exists in the system. In the population environment generally predicts the actual size of the population according to their environment. Carrying capacity is one of the most important factors because they regulate how fast and the highest level the population can grow. Earlier research assumes the carrying capacity to be a constant quantity; however in a dynamically changing environment, carrying capacity is considered as a state variable.

In the predator-prey model presented by Leslie and Gower, it is assumed the carrying capacity of the prey population is a constant quantity k, and the predator the carrying capacity is proportional to the carrying capacity of the prey [16, 17]. With the response function of the second type of Holling model, the above model becomes Holling-Tanner model with marked dynamic behaviour [4, 6, 13, 14].

Harvesting of different species has a strong effect on predatory environmental prey the system. After harvest, population density may be much lower from the previous time. Harvesting can lead to positive extinction of the population likely. Extinction generally occurred whenever it was exploitable and exploitable the resource is harvested continuously more than the desired limit for subsequent preservation. Some works with predator harvesting have already been reported in the literature [1, 2, 3, 7, 8, 10, 11, 12, 15]. Recently, Huang et al. [9] established analytically that the model is subject to Bogdanov-Takens bifurcation (cusp case) of coding 3 gave a deep insight into various bifurcation scenarios, including the presence from two reduction cycles, Hopf dendrites are supercritical and subcritical, and coexistence between a stable homoclinic ring and an unstable reduction cycle, the homoclinic bifurcation of co-dimension.

Another important parameter is needed besides the enrichment parameter is the harvesting parameter. Over-fishing on fisheries has become an acute crisis that can affect human daily life. Ganguly et al conducted a recent research on harvesting in an intraguild model [18]. Both groups are appended to harvesting efforts using the hunting hypothesis per unit. The virtue of what we know, there is limited literature examining the impact of harvest on system 1 through independent harvest strategy. Most studies on System (1) in [8] confirmed on Implications of resource enrichment but not harvest. In addition, this paper aims to investigate both the optimal thresholds for harvesting prey provide maximum monetary interest while preserving fishery resources. In our model, both predatory fish and fish prey obey logistics growth, encounter different harvesting ranges and cause toxic at different rates.

2. Model Building and Analysis

To build our model, we consider a predator-prey interaction model with Holling type II functional response assuming that there is a harvesting on both predator and prey populations which is proportional to the size of each population. We assume that the carrying capacity is not constant rather it is takes the form of a logistic function of time. The model is then given by the following set of differential equations.

$$
\begin{aligned}
\frac{dN}{dt} &= rN(t)\left(1-\frac{N(t)}{\kappa(t)}\right)-\frac{aN(t)P(t)}{1+\gamma N(t)}-h_1N(t)\\
\frac{d\kappa}{dt} &= \alpha\left(\kappa(t)-\kappa_1\right)\left(1-\frac{\kappa(t)-\kappa_1}{\kappa_2}\right)\\
\frac{dP}{dt} &= \frac{\varepsilon aN(t)P(t)}{1+\gamma N(t)}-(c+h_2)P(t)
\end{aligned}
\tag{1}
$$

subject to the initial conditions: $N(0)=N_0$, $k(0)=k_0$ and $P(0)=P_0$, where N,P denote prey and predator population densities, respectively, and $\kappa(t)$ denotes the carrying capacity. r represents prey's per capita growth rate, c is the death rate of the predator, h_1 and h_2 represents the harvesting rates on the prey and the predator, respectively. Note that κ, the carrying capacity that increases sigmoidally between an initial value $k_0>k_1$ and a final value k_1+k_2 with a growth rate α.

2.1. Mathematical Analysis.

The system (1) has the following equilibrium points: $E_1=(0,\kappa_1,0)$, $E_2=(0,\kappa_1+\kappa_2,0)$, $E_3=\left(\frac{\kappa_1(r-h_1)}{r},\kappa_1,0\right)$, $E_4=\left(\frac{(\kappa_1+\kappa_2)(r-h_1)}{r},\kappa_1+\kappa_2,0\right)$, $E_5=\left(\frac{c+h_2}{a\varepsilon-\gamma(c+h_2)},\kappa_1,\frac{\varepsilon(a\varepsilon-(c+h_2)\gamma)(r-h_1)\kappa_1}{(a\varepsilon-\gamma(c+h_2))^2\kappa_1}\right)$, and

$$E_6=\left(\frac{(c+h_2)}{a\varepsilon-\gamma(c+h_2)},\kappa_1+\kappa_2,\frac{\varepsilon(r(a\varepsilon(\kappa_1+\kappa_2)-(c+h_2))-a\varepsilon h_1(\kappa_1+\kappa_2)-(\kappa_1+\kappa_2)(r-h_1)(c+h_2)\gamma)}{(a\varepsilon-\gamma(c+h_2))^2(\kappa_1+\kappa_2)}\right)$$

Note that $\kappa(t)\to(\kappa_1+\kappa_2)$ as $t\to\infty$; therefore the equilibrium points E_1,E_3 and E_5 are always unstable. The stability of the remaining equilibrium points given by the following Theorems.

Theorem 1:

The Local stability of the equilibrium points E_2, E_4 and E_6 of system (1) is given by:

(i) E_2 is locally asymptotically stable if $h_1 > r$.

(ii) E_4 is locally asymptotically stable if $\gamma(c+h_2) > a\varepsilon$.

(iii) E_6 is locally asymptotically stable if

 (1) $a\varepsilon - \gamma(c+h_2) > 0$

 (2) $(\kappa_1+\kappa_2)(r-h_1)[a\varepsilon - \gamma(c+h_2)] > r(c+h_2)$

 (3) $r[a\varepsilon + \gamma(c+h_2)] > \gamma(\kappa_1+\kappa_2)(r-h_1)[a\varepsilon - \gamma(c+h_2)$

Proof:

The Jacobian matrix of the system (1) is:

$$J = \begin{bmatrix} r\left(1-\frac{N}{K}\right) - \frac{rN}{K} - \frac{aP}{\gamma N+1} + \frac{aNP\gamma}{(\gamma N+1)^2} - h_1 & \frac{rN^2}{K^2} & -\frac{aN}{\gamma N+1} \\ 0 & \alpha\left(1-\frac{K-\kappa_1}{\kappa_2}\right) - \frac{\alpha(K-\kappa_1)}{\kappa_2} & 0 \\ \frac{\varepsilon aP}{\gamma N+1} - \frac{\varepsilon aNP\gamma}{(\gamma N+1)^2} & 0 & \frac{\varepsilon aN}{\gamma N+1} - c - h_2 \end{bmatrix}$$

(i) Evaluating the Jacobin matrix at E_2 gives:

$$J_2 = \begin{bmatrix} r-h_1 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -c-h_2 \end{bmatrix}$$

The eigenvalues are $-\alpha$, $-(c+h_2)$, and $r-h_1$. Clearly for this point to be locally asymptotically stable we should have $h_1 > r$.

(ii) Evaluating the Jacobin matrix at E_4 gives:

$$J_4 = \begin{bmatrix} -(r-h_1) & \frac{(r-h_1)^2}{r} & -\frac{a(\kappa_1+\kappa_2)(r-h_1)}{(\kappa_1+\kappa_2)(r-h_1)\gamma+r} \\ 0 & -\alpha & 0 \\ 0 & 0 & \frac{(\kappa_1+\kappa_2)(r-h_1)[a\varepsilon-\gamma(c+h_2)]-r(c+h_2)}{(\kappa_1+\kappa_2)(r-h_1)\gamma+r} \end{bmatrix}$$

with eigenvalues $-(r-h_1)$, $-\alpha$, and $\frac{(\kappa_1+\kappa_2)(r-h_1)[a\varepsilon-\gamma(c+h_2)]-r(c+h_2)}{(\kappa_1+\kappa_2)(r-h_1)\gamma+r}$. For the existence of this point we should have $r > h_1$; therefore the point E_4 is locally asymptomatically stable if $\gamma(c+h_2) > a\varepsilon$.

(iii) Evaluating the Jacobin matrix at E_6 gives $-\alpha$ to be one of the eigenvalues, and the other two are the roots of the polynomial

$$\lambda^2 + A\lambda + B = 0$$

where

$$\begin{aligned} A &= \frac{(c+h_2)}{a\varepsilon(\kappa_1+\kappa_2)[a\varepsilon-\gamma(c+h_2)]}\left[r[a\varepsilon+\gamma(c+h_2)]-\gamma(\kappa_1+\kappa_2)(r-h_1)[a\varepsilon-\gamma(c+h_2)]\right] \\ B &= \frac{(c+h_2)}{a\varepsilon(\kappa_1+\kappa_2)}\left[(\kappa_1+\kappa_2)(r-h_1)[a\varepsilon-\gamma(c+h_2)]-r(c+h_2)\right] \end{aligned}$$

For E_6 to be locally stable we need $A < 0$, and $B > 0$; therefore the stability conditions for E_6 are

(1) $a\varepsilon - \gamma(c+h_2) > 0$

(2) $(\kappa_1+\kappa_2)(r-h_1)[a\varepsilon-\gamma(c+h_2)] > r(c+h_2)$

(3) $r[a\varepsilon+\gamma(c+h_2)] > \gamma(\kappa_1+\kappa_2)(r-h_1)[a\varepsilon-\gamma(c+h_2)$

Which concludes the proof.

Theorem 2:

The global stability of the steady states is given by:

(i) E_2 is globally asymptomatically stable whenever it is locally stable.

(ii) E_4 is globally asymptomatically stable whenever it is locally stable.

(iii) E_6 is globally asymptomatically stable if

$$(\kappa_1+\kappa_2)(r-h_1)[a\varepsilon-\gamma(c+h_2)]-r(c+h_2) < a\varepsilon r$$

Proof:

The global stability of stability points will be analysed by transforming the system of equations (1) into a linear system and then choosing a suitable Lyapunov function.

By letting $N = N^* + n$, $\kappa = \kappa^* + k$ and $P = P^* + p$, where n, k and p are small perturbations about the general equilibrium point (N^*, κ^*, P^*).

The system of equations (1) is turned into a linear system which is of the form $\dot{x} = J(E)x$, where $J(E)$ is the Jacobian matrix of equations (1). Thus, the linear system of equations (1) is,

$$\frac{dn}{dt} = \left(\frac{-rN^*}{\kappa^*} + \frac{aN^*P^*}{(1+\gamma N^*)^2}\right)n - \frac{rN^{*2}}{\kappa^{*2}}k - \frac{aN^*}{(1+\gamma N^*)}p$$

$$\frac{dk}{dt} = -\alpha\kappa \tag{2}$$

$$\frac{dp}{dt} = \left(\frac{\varepsilon a P^*}{(1+\gamma N^*)} - \frac{\varepsilon a\gamma N^* P^*}{(1+\gamma N^*)^2}\right)n$$

(i) To prove the global stability of $E_2(0, \kappa^*, 0)$, we define the following Lyapunov function

$$V(n,k,p) = \frac{n^2}{2} + \frac{k^2}{2} + \frac{p^2}{2}$$

Clearly $V(n,k,p)$ is positive definite, and $V(n^*, k^*, p^*) = 0$. Now we have

$$\begin{aligned} \dot{V}(n,k,p) &= n\dot{n} + \kappa\dot{\kappa} + p\dot{p} \\ &= -\alpha\kappa^2 \quad \leq 0 \end{aligned}$$

therefore the point E_2 is globally asymptotically stable.

(ii) Define a Lyapunov function as

$$L(n,k,p) = \frac{n^2}{2N^*} + \frac{k^2}{2} + \frac{p^2}{2}$$

It is obvious that $L(n,k,p)$ is a positive definite function. Differentiating L with respect to time t we get,

$$\begin{aligned} \dot{L}(n,k,p) &= \frac{n\dot{n}}{N^*} + k\dot{k} + p\dot{p} \\ &= -\left[\frac{r}{\kappa^*}n + \frac{rN^*}{\kappa^{*2}}\kappa + \frac{a}{(1+\gamma N^*)}p\right]n - \alpha\kappa^2 \end{aligned}$$

Therefore, $E_4(N^*, \kappa^*, 0)$ is globally asymptotically stable.

(ii) To prove the global stability of $E_6(N^*, \kappa_*, P^*)$. We define a Lyapunov function as

$$Q(n,k,p) = \frac{n^2}{2aN^*} + \frac{k^2}{2\kappa^*} + \frac{p^2}{2\varepsilon a P^*}$$

It is obvious that $Q(n,k,p)$ is a positive definite function. Differentiating Q with respect to time t we get,

$$\begin{aligned}
\dot{Q}(n,k,p) &= \frac{n\dot{n}}{aN^*} + \kappa\dot{\kappa} + \frac{p\dot{p}}{\varepsilon a P^*} \\
&= \left[\left(\frac{-r}{a\kappa^*} + \frac{P^*}{(1+\gamma N^*)^2}\right)n - \frac{rN^*}{a\kappa^{*2}}\kappa - \frac{1}{(1+\gamma N^*)}p\right]n - \alpha\kappa^2 \\
&\quad + \left[\left(\frac{1}{(1+\gamma N^*)} - \frac{N^*}{(1+\gamma N^*)^2}\right)n\right]p \\
&= \left[\left(\frac{-r}{a\kappa^*} + \frac{P^*}{(1+\gamma N^*)^2}\right)n - \frac{rN^*}{a\kappa^{*2}}k\right]n - \alpha\kappa^2 - \frac{N^*}{(1+\gamma N^*)^2}np
\end{aligned}$$

which is negative semi-definite if

$$\frac{-r}{a\kappa^*} + \frac{P^*}{(1+\gamma N^*)^2} < 0$$

Which is equivalent to

$$(\kappa_1+\kappa_2)(r-h_1)[a\varepsilon - \gamma(c+h_2)] - r(c+h_2) < a\varepsilon r$$

Therefore, $E_6(N^*,\kappa^*,P^*)$ is globally asymptotically stable if the above mentioned condition satisfied. This concludes our proof.

The following Theorem discuss the possibility of the existence of Hopf Bifurcation.

Theorem 3:

System (1) undergoes a Hopf bifurcation at the positive equilibrium E_6 when h_1, if chosen as the bifurcation parameter, passes throw $h_1^* = \frac{r\left((\kappa_1+\kappa_2)(c+h_2)\gamma^2+(c+h_2-a\varepsilon(\kappa_1+\kappa_2))\gamma+\varepsilon a\right)}{\gamma((c+h_2)\gamma-\varepsilon a)(\kappa_1+\kappa_2)}$

Proof:

The eigenvalues of the linearized system around the equilibrium point E_6 are $-\alpha$ and $\mu_{1,2} = \alpha(h_1) \pm i\beta(h_1)$

where

$$\begin{aligned}
\alpha(h_1) &= \frac{1}{2}trac(J^*) \\
\beta(h_1) &= \sqrt{det(j^*) - (\alpha(h_1))^2}
\end{aligned}$$

Now, at h_1^*,

$$\begin{aligned}\alpha(h_1^*) &= 0\\ \beta(h_1^*) &= \sqrt{\frac{r(c+h_2)}{(\kappa_1+\kappa_2)\gamma}}\end{aligned}$$

Also:

$$\frac{d\alpha(h_1)}{dh_1}|_{h=h_1^*} = -\frac{(c+h_2)\gamma}{\varepsilon a}$$

Therefore the proof is concluded.

3. Optimal Control

In commercial exploitation of renewable resources the fundamental problem from the economic point of view, is to determine the optimal trade-off between present and future harvests. The emphasis of this section is on the profit-making aspect of fisheries. It is a thorough study of the optimal harvesting policy and the profit earned by harvesting, focusing on quadratic costs and conservation of fish population by constraining the latter to always stay above a critical threshold. The prime reason for using quadratic costs is that it allows us to derive an analytical expression for the optimal harvest; the resulting solution is different from the bang-bang solution which is usually obtained in the case of a linear cost function. It is assumed that price is a function which decreases with increasing biomass. Thus, to maximize the total discounted net revenues from the fishery, the optimal control problem can be formulated as

$$J(h_1,h_2) = \int_{t_0}^{t_1} e^{-\delta t}\left(p_1(h_1+h_2) - v_1(h_1^2+h_2^2) - \frac{c_1(h_1+h_2)}{q_1N+q_2P}\right) \quad dt$$

where c_1 be the constant fishing cost per unit effort, p_1 is the constant price per unit biomass of harvested population, v_1 is an economic constant and δ is the instantaneous annual discount rate.

Suppose h_1 is an optimal control with corresponding states N^*, K^* and P^*. We are seeking to derive optimal controls h_1^*, h_2^* such that

$$J(h_1^*,h_2^*) = max\{J(h_1,h_2) : h_1,h_2 \in U\}$$

where U is the control set. Applying Pontryagin's maximum principle this problem is solvable. Now the current value Hamiltonian of this control problem is

$$\begin{aligned} H &= \left(p_1(h_1+h_2)-v_1(h_1^2+h_2^2)-\frac{c_(h_1+h_2)}{q_1N+q_2P}\right)+\lambda_1\left(rN(t)\left(1-\frac{N(t)}{\kappa(t)}\right)-\frac{aN(t)P(t)}{1+\gamma N(t)}-h_1N(t)\right) \\ &\quad +\lambda_2\alpha\left(\kappa(t)-\kappa_1\right)\left(1-\frac{\kappa(t)-\kappa_1}{\kappa_2}\right)+\lambda_3\left(\frac{\varepsilon aN(t)P(t)}{1+\gamma N(t)}-(c+h_2)P(t)\right) \end{aligned}$$

On the control set we have:

$$\begin{aligned} \frac{\partial H}{\partial h_1} &= p_1-2v_1h_1-\frac{c_1}{q_1N+q_2P}-\lambda_1 \\ \frac{\partial H}{\partial h_2} &= p_1-2v_1h_2-\frac{c_1}{q_1N+q_2P}-\lambda_2 \end{aligned}$$

Which implies:

$$\begin{aligned} h_1^* &= \frac{1}{2v_1}\left(p_1-\frac{c_1}{q_1N^*+q_2P^*}-\lambda_1\right) \\ h_2^* &= \frac{1}{2v_1}\left(p_1-\frac{c_1}{q_1N^*+q_2P^*}-\lambda_3\right) \end{aligned} \tag{3}$$

Now, the autonomous set of equations of the control problem are

$$\begin{aligned} \frac{d\lambda_1}{dt} &= \delta\lambda_1-\frac{\partial H}{\partial N} = \delta\,\lambda_1-\frac{c_1\,(h_1+h_2)\,q_1}{(q_1\,N+q_2\,P)^2}-\lambda_3\left(\frac{\varepsilon\,aP}{\gamma N+1}-\frac{\varepsilon\,aNP\gamma}{(\gamma N+1)^2}\right) \\ &\quad -\lambda_1\left(r\left(1-\frac{N}{K}\right)-\frac{rN}{K}-\frac{aP}{\gamma N+1}+\frac{aNP\gamma}{(\gamma N+1)^2}-h_1\right) \\ \frac{d\lambda_2}{dt} &= \delta\lambda_2-\frac{\partial H}{\partial K} = \delta\,\lambda_2-\delta\,\lambda_2-\frac{\lambda_1\,rN^2}{K^2}-\lambda_2\,\alpha\left(1-\frac{K-\kappa_1}{\kappa_2}\right)+\frac{\lambda_2\,\alpha\,(K-\kappa_1)}{\kappa_2} \\ \frac{d\lambda_3}{dt} &= \delta\lambda_3-\frac{\partial H}{\partial P} = \delta\,\lambda_3-\delta\,\lambda_3-\frac{c_1\,(h_1+h_2)\,q_2}{(q_1\,N+q_2\,P)^2}+\frac{\lambda_1\,aN}{\gamma N+1}-\lambda_3\left(\frac{\varepsilon\,aN}{\gamma N+1}-c-h_2\right) \end{aligned} \tag{4}$$

Therefore, we arrive to the following theorem:

Theorem 4: There exists an optimal control set $\{h_1^*, h_2^*\}$ and corresponding solution N^*, K^* and P^* that maximizes $J(h_1,h_2)$ over U. Furthermore, there exists adjoint functions, λ_1,λ_2 and λ_3 satisfying equation (4) with transversality conditions $\lambda_i(t_1)=0, i=1,2,3$. Moreover, the set of optimal control is given by equation (3).

4. Numerical Simulation and Discussion

In this section, we will perform some numerical simulations in order to verify our analytic results.

For simulation purpose, we will take the parameters from the following table

Parameter	Value
r	0.25
a	0.06
γ	0.005
α	0.01
ε	0.8
c	0.22
K_1	300
K_2	500
h_1	variable
h_2	0.1

TABLE 1. Parameter Values

Using the parameters shown in Table 1 above, we have the following graphs.

As discussed earlier in Theorem 2, the system exhibits the phenomena of Hopf bifurcation as the bifurcation parameter h_1 passes through a critical value. This is very clear from figures 1 - 8. In figure 1, the system has a stable limit cycle, for $0.001 < h_1 < 0.1$, which looses its stability as h_1 increases and the coexistence equilibrium became stable, as shown in figure 3. When h_1 increases further, the coexistence equilibrium point losses its stability and now the predator-free equilibrium point became stable, as shown in figure 5. When h_1 increases above 0.3, the predator-free equilibrium point looses its stability and the trivial equilibrium (i.e. the equilibrium point where both populations extent) became stable, which is very clear from figure 7. The same storyline is clear when looking at the phase-diagrams instead of the

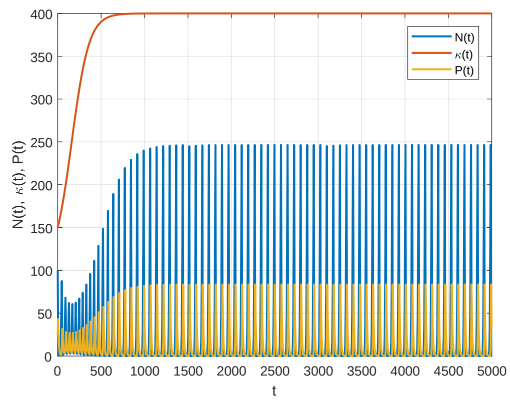

FIGURE 1. Stable limit cycle appears when $0.001 < h_1 < 0.1$

time-series solution, which is shown in figures 2,4,6 and 8.

To study the optimal control problem numerically, we use the forward-backward Rung-Kutta sweep method. The results are given in the following graphs. Note that all the parameters are taken from table 1.

Figures 9 and 10 show that when the constant effort harvesting on the prey (i.e. h_1) is very low (i.e. $h_1 = 0.001$), then both prey population and predator populations with constant harvesting effort exhibit oscillations with very long period and the densities of both populations is very high; however under optimal harvesting both populations are kept in a low density, but away from extension, with periodic solutions which have very short periods. Whilst when the constant harvesting on the prey is medium (i.e. $h_1 = 0.17$), both prey and predator populations with constant harvesting efforts keep oscillating but now the density decreases a little bit, and also the period decreases in a drastic manner; and the prey and predator populations have a similar behaviour as the previous case; which can be shown from figures 11 and 12. When the constant harvesting on prey increases (i.e. $h_1 = 0.31$) then both prey and predator populations with constant effort harvesting go extinct; however both the populations exhibit periodic solutions under optimal effort harvesting, as could be seen from 13 and 14. Actually, the optimal effort harvesting might reach a maximum as high as 0.6; which could never been achieved through constant effort harvesting, which is clear from figures 15 and 16.

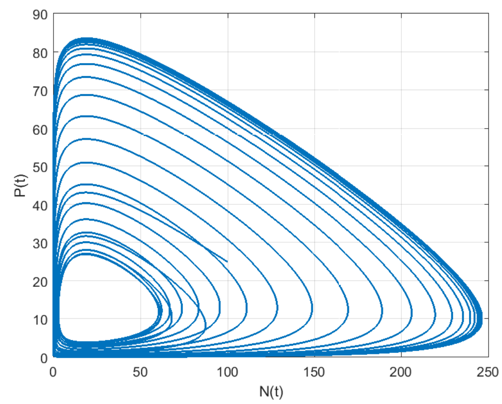

FIGURE 2. Phase digram showing the limit cycle when $0.001 < h_1 < 0.1$

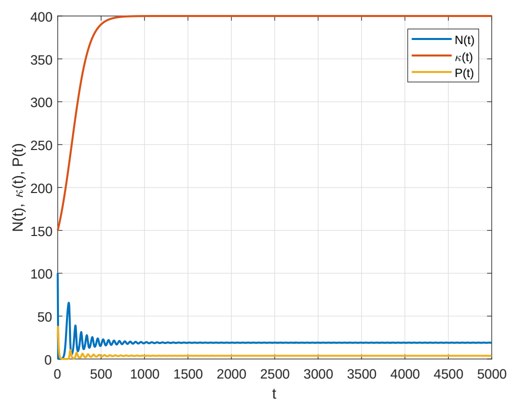

FIGURE 3. Stable coexistence equilibrium appears when $0.17 < h_1 < 0.25$

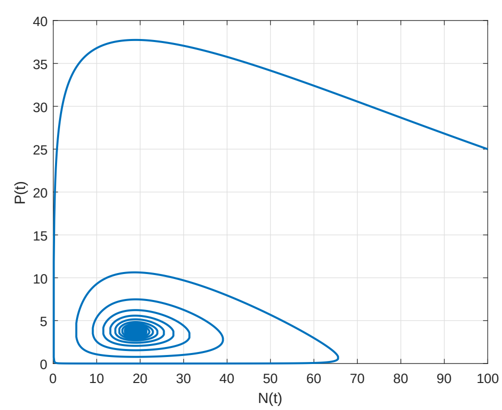

FIGURE 4. Phase digram showing the appearance of stable coexistence equilibrium when $0.17 < h_1 < 0.25$

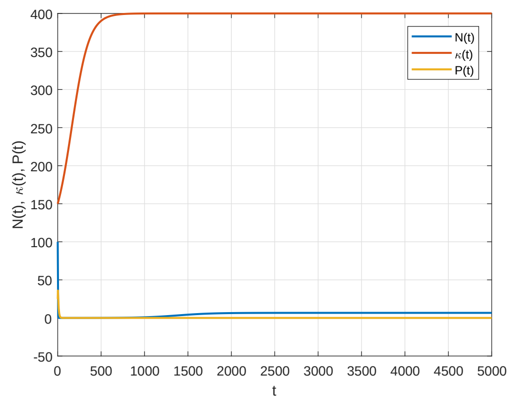

FIGURE 5. The stability of the predator-free equilibrium for $0.285 < h_1 < 0.295$

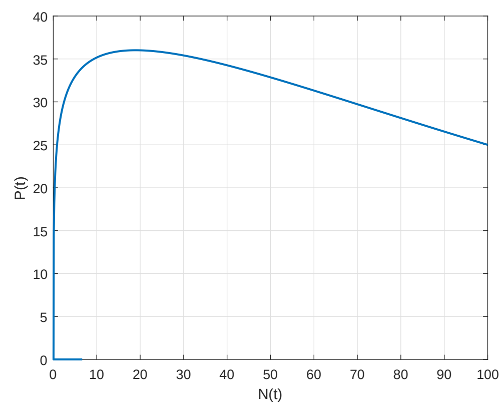

FIGURE 6. Phase digram showing the stability of the predator-free equilibrium when $0.285 < h_1 < 0.295$

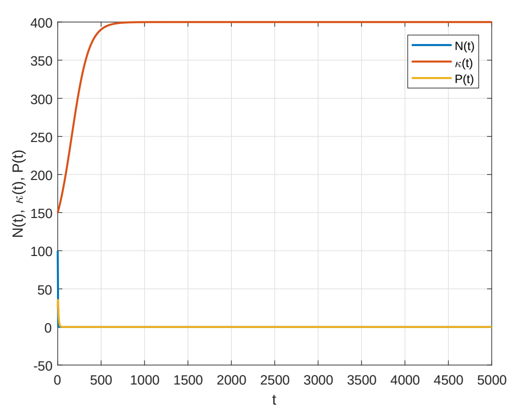

FIGURE 7. The stability of the trivial equilibrium when $h_1 > 0.3$

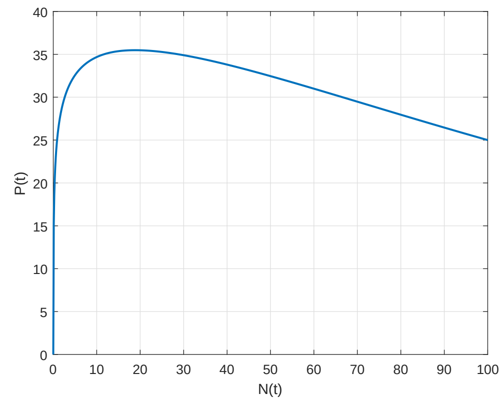

FIGURE 8. Phase digram showing that the trivial equilibrium is stable $h_1 > 0.3$

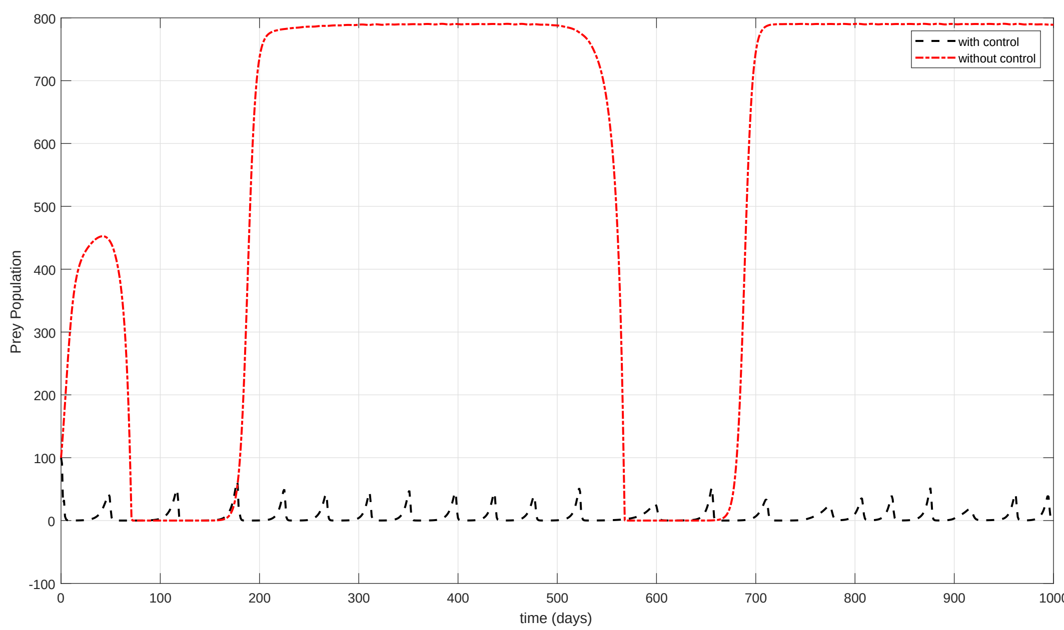

FIGURE 9. Prey population with effort harvesting and optimal harvesting, with very low efforts

5. CONCLUSION

In this paper, we present and analysed a mathematical model describing a prey-predator interaction under harvesting when the carrying capacity is variable. Our results show that the system has six steady-states 3 of which are locally and globally stable under some conditions. It is also shown that the system undergoes Hopf bifurcation when the bifurcation parameter h_1

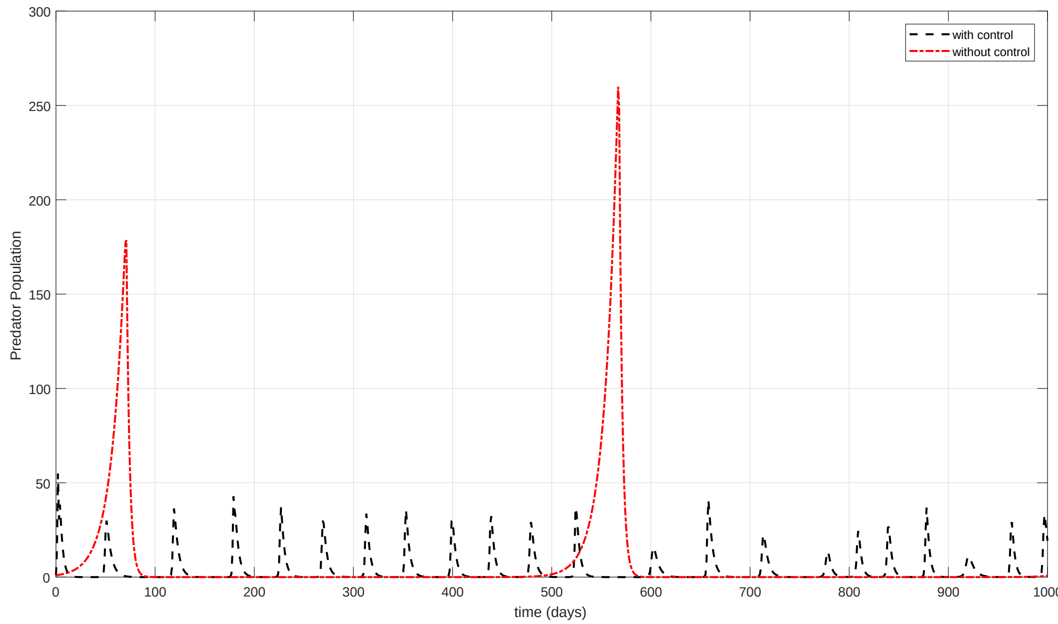

FIGURE 10. Predator population with effort harvesting and optimal harvesting, with very low efforts

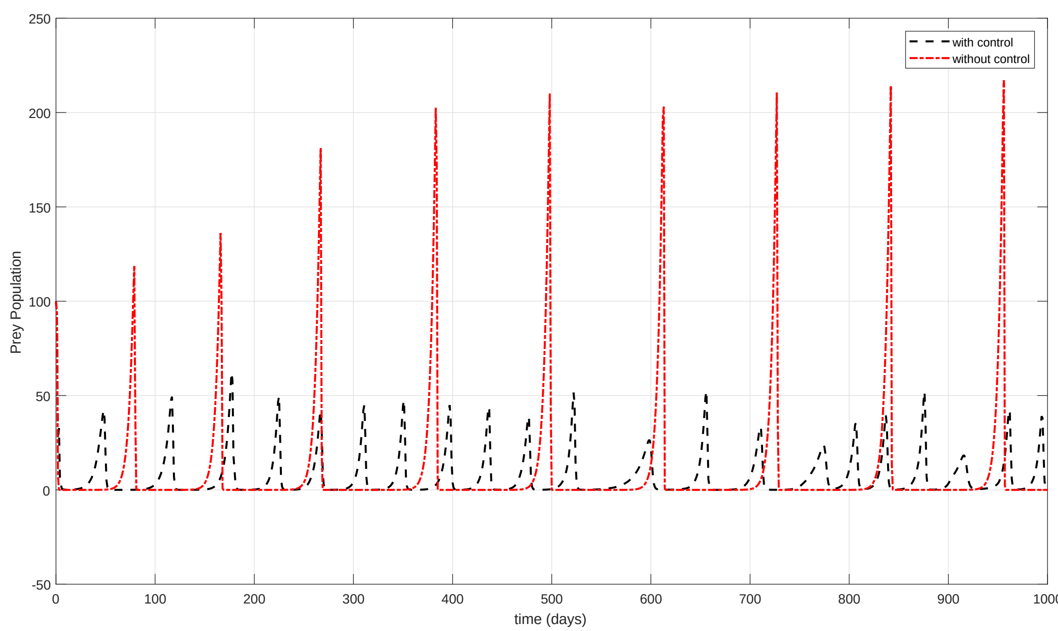

FIGURE 11. Prey population with effort harvesting and optimal harvesting, with medium efforts

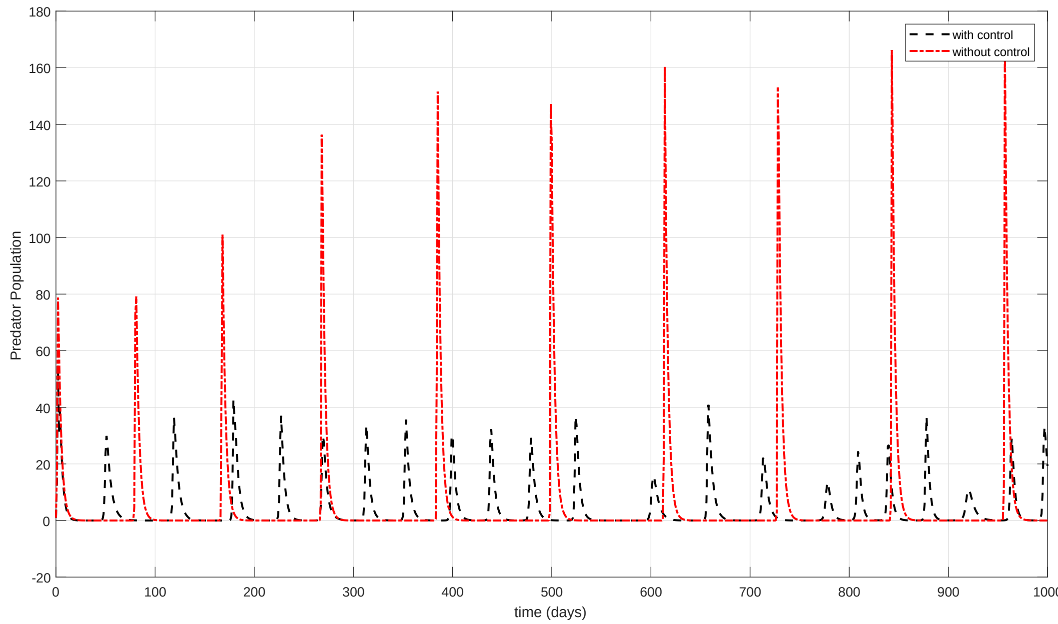

FIGURE 12. Predator population with effort harvesting and optimal harvesting, with medium efforts

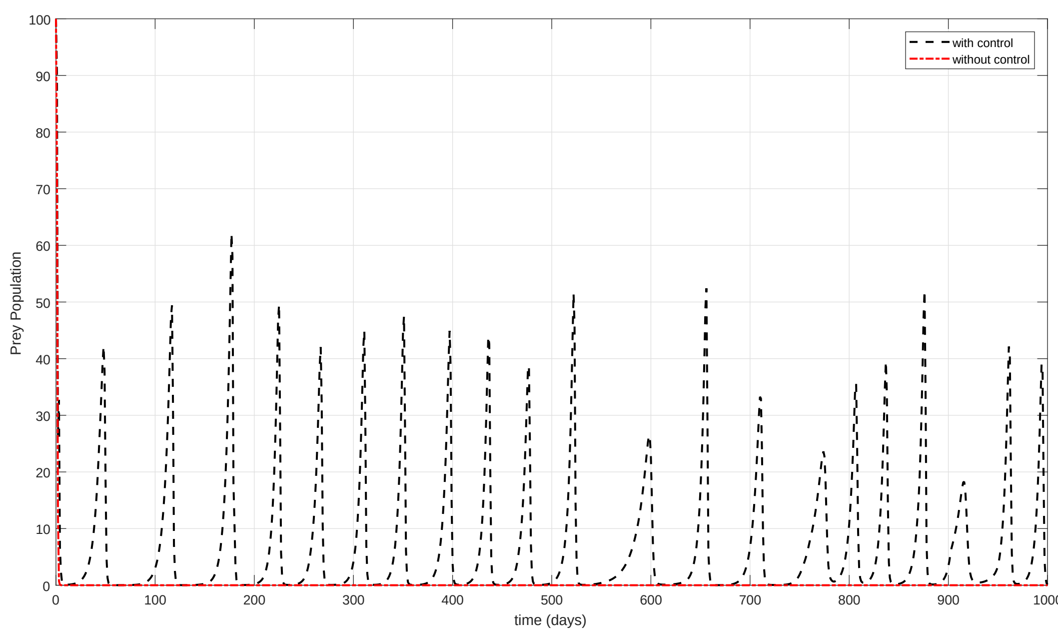

FIGURE 13. Prey population with effort harvesting and optimal harvesting, with high efforts

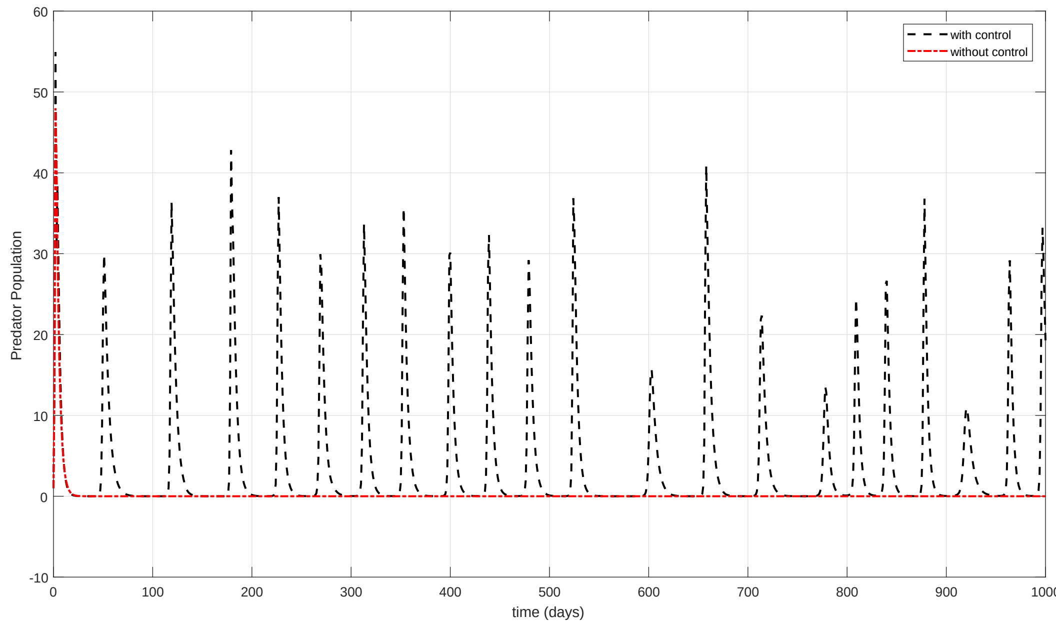

FIGURE 14. Predator population with effort harvesting and optimal harvesting, with high efforts

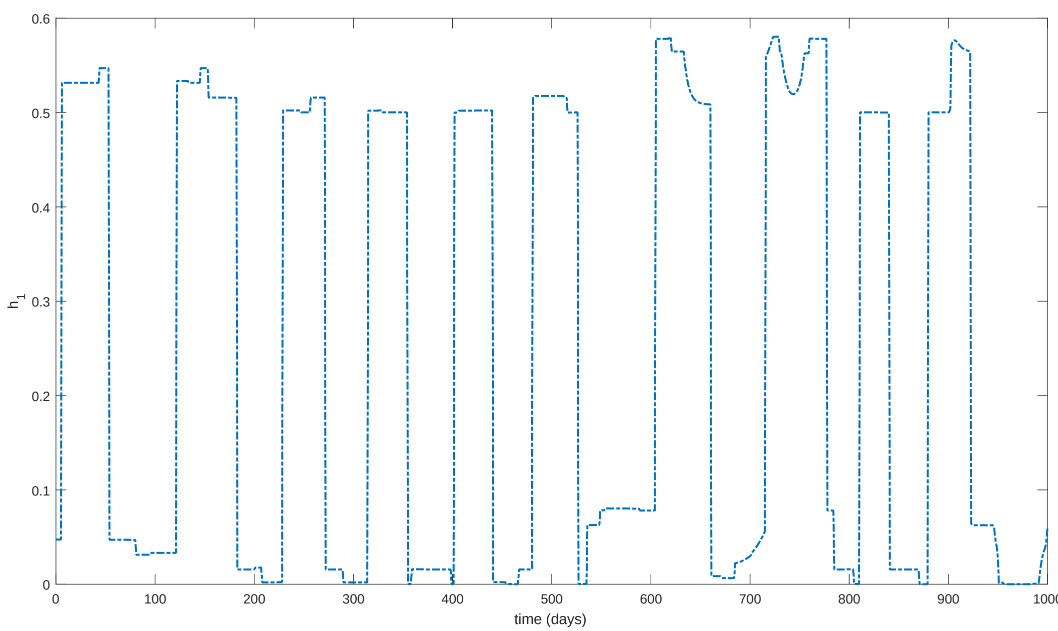

FIGURE 15. The control profile for the optimal effort h_1

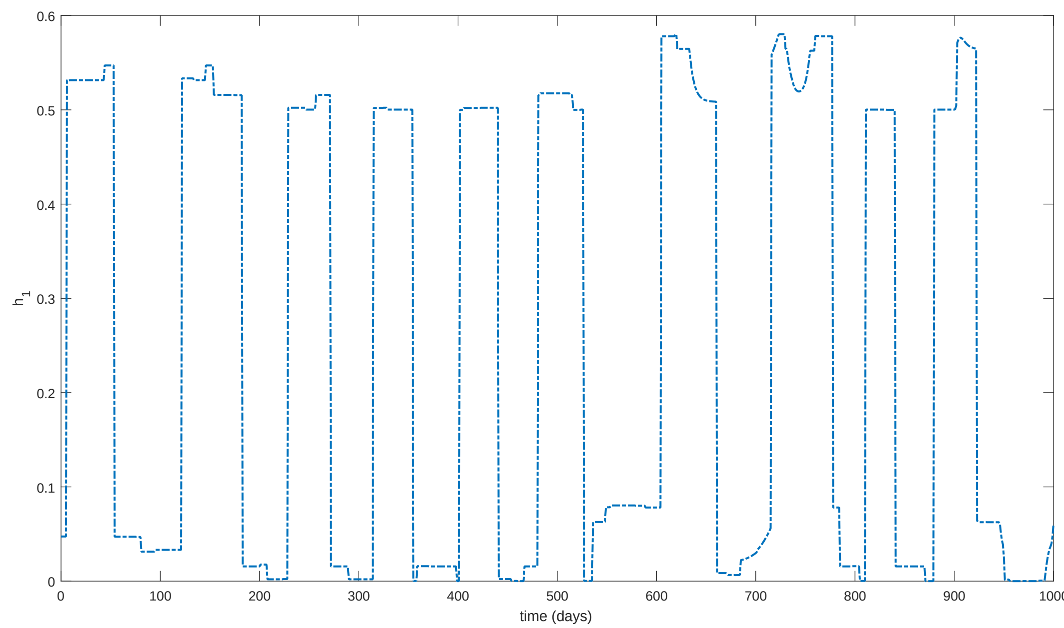

FIGURE 16. The control profile for the optimal effort h_2

passes a critical value. Numerical simulations show that the stability of the equilibrium points changes when from one range of h_1 to another. It is also shown that the following the optimal harvesting, one is capable of achieving a high yield and maintaining the both population away from extension; which is what is desired.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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