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A DISCRETE MATHEMATICAL MODELING OF TRANSMISSION OF COVID-19 PANDEMIC USING OPTIMAL CONTROL

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Abstract. In this work, we present a study of optimal control strategies of a Novel Corona Virus Disease 2019 (COVID-19) spreading model in the discrete case. The targeted population is divided into six compartments $SEIC^WCR$ namely (S) susceptible, (E) exposed, (I) infected, (C^W) infected with complication, (C) infected multimorbidity with complication and (R) recovered. We also proposed an optimal strategy to fight against the spread of COVID-19. We use four controls which represent the sensitization and prevention through the media and education for the susceptible individuals, quarantined the infected at home, quarantined the infected with complication at the hospital, quarantined the infected multimorbidity with complication at the hospital with requirement breathing assistance. Theoretically, we have proved the existence of optimal controls, and a characterization of the controls in terms of states and adjoint functions principally based on Pontryagin's maximum principle. To clarify the efficiency of our theoretical results, we provide numerical simulations for numerous scenarios. Therefore, the obtained results affirm the performance of the optimization approach.

Keywords: COVID-19; pandemic; discrete mathematical models; optimal control; COVID-19 spreading model.

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1. Introduction

Infectious illnesses are produced by means of pathogenic microorganisms, such as parasites, bacteria or viruses. The disease can be transmitted, directly or indirectly, from one individual to any other. The world has known many epidemics that man could overcome either by isolation, quarantine, vaccine or treatment to avoid the ordeal that humankind has known since the time of the famous Spanish flu pandemic which took millions of lives. Thanks to evolution of science and medicine, all epidemics were taken seriously by researchers in order to limit their rate of mortality and spread. We can cite for instance the famous epidemics of the 21st century such as SARS, EBOLA or H1N1.

Nowadays, in late December 2019, China discovered a new respiratory disease called COVID-19, it was first considered by local authorities and international organizations as being controllable as all epidemics especially when it was discovered that the virus is a mutation of SARS-COV-1. The city of Wuhan and the Hubei region found out that this was not the case and the WHO took a lot of time in the face of economic challenges to declare a pandemic after the virus has spread in all directions. The World Health Organization (WHO) declared the outbreak a Public Health Emergency of International Concern on 30 January 2020, and a pandemic on 11 March 2020. COVID-19 is a respiratory infectious disease which broke out first in China and was spread all over the world right after due to the high level of contagion. This virus is known as the name of SARS-COV-2. As of 9 May 2020, more than 3.95 million cases have been reported across 187 countries and territories, resulting in more than 275,000 deaths, more than 1.31 million people have recovered [1, 2]. According to World Health Organization [3], the most common symptoms of COVID-19 are fever, tiredness and dry cough. Some patients may have pains and aches, runny nose, sore throat or nasal congestion. Eighty percentages of patients get better from the disease without using special treatment. Around 1 out of 6 patients becomes seriously ill and develops difficulty breathing. Elderly persons and those with medical problems are most probably to develop serious illness.

The COVID-19 is regarded extra lethal and dangerous comparing to different viruses, whereas the number of humans with an extreme acute respiratory syndrome (SARS) reached about 8098 humans and the death of about 774 people, which additionally commenced from Asia and China in 2002 and which researchers recommended transmission from bats to humans, as for the Middle East Syndrome (MERS) virus, 858 people died out of cases of infection. WHO numbered about 2494 humans for the reason that its appearance in 2012 [4, 5], which appeared at the start in the Kingdom of Saudi Arabia.

RELATED WORK

Mathematical models of infectious disease dynamics have a deep history of more than one hundred years. The most frequent mathematical formulations which characterize the individual transition in a community between 'compartments' which attracts the scenario of individual contamination to quite significant. In 1927 Kermack and McKendrick [6] were the first researchers on mathematical epidemiology to suggest the Susceptible-Infected-Removed (SIR) model that describes the speedy explosion of an infectious disease for a quick time. Many studies of mathematical models have been developed to simulate, analyse and understand the corona virus, in their research study, Yan and Zou [7] considered the optimal and sub-optimal control strategies associated with quarantined asymptotic individuals for a SARS SEQIJR model. In [8] Kumar and Srivastava considered the SVIR (susceptible, vaccinated, infected, recovered) epidemic model. Therefore, vaccination and treatment control strategies are used in order to contain the disease. Sari et. al. [9], considered the SVEIR epidemic model, they showed the implementation of combination control strategy in the form of vaccination and treatment to reduce the number of exposed and infected in order to fight against the spread of SARS diseases. Zhi-Qiang Xia et. al. [10] modeling the transmission of middle east respirator syndrome corona virus in the republic of Korea using a system of ordinary differential equations. A mathematical model for reproducing the stage-based transmissibility of a SARS-COV-2 is investigated through Chen et.al in [17]. In [18] the authors recommended a conceptual model for the COVID-19, which successfully catches the time line of the COVID-19 outbreak. Drosten et al [19] furnished a description of a deadly case of MERS-CoV contamination and related phylogenetic analyses. Guery et al [20] have analyzed the clinical features of the infected cases. The global problem of the outbreak has attracted the interest of researchers of different areas, giving rise to a number of proposals to analyze and predict the evolution of the pandemic [11, 12, 13].

PROBLEM STATEMENT

Control of epidemics is turning into increasingly more necessary for governments and public health officials. More precisely, it seeks to understand the dynamics of the spread of infectious diseases in order to improve prevention and intervention techniques aimed at reducing their effect on public health. Both standard and bilinear incidence rates have been applied widely in classical epidemic models (see [14]). Following a study on the spread of the epidemic of cholera in Bari in 1973, Capasso and Serio [15] have included a rate of saturated incidence $g(I) = \frac{\beta I}{1+\omega I}$, with $\omega > 0$ in epidemic models. The principal reason justifying the introduction of this functional form of incidence function is that the number of effective interactions of infective and susceptible individuals may become saturated at significant levels of infection as a result of crowding of individuals with infection or because of the preventive measures used in response to disease severity by susceptible individuals (see [16]). The aim of this work is to introduce a new approach by taking into account the spread of COVID-19 using an optimal control problem applicable to any type of populations. We will undertake the discrete time modeling data are collected at discrete moments (day, week, month, and year). So it is more direct and greater correct to describe the phenomena by using the discrete time models may avoid some mathematical complexities such as the preference of a function space and regularity of the solution. Additionally, most of the preceding research has targeted on non-stop modeling. In this work, we will study the dynamics of discrete mathematical COVID-19 model. In this epidemiological model, after the initial infection, a hat remains is latently period before becoming infectious. So the population is divided into six categories: susceptible (S), exposed (E), infected(I), infected with complication (C^W), infected multimorbidity with complication (C), and recovered (R). We consider a control goal that aims to reduce the spread of the epidemic, we seek to find the optimal strategies to minimize the number of infected, infected with complication, infected multimorbidity with implication and maximize the number of recovered by introducing a control variable into discrete $SEIC^WCR$ model.

In order to achieve this purpose, we use optimal control strategies associated with four controls: the first represents the strategy of sensitization and prevention through the media and education, and preventing gatherings through security campaigns. It can be applied by launching

campaigns to raise the citizens awareness of the cruelty of the disease and call for some preventive measures like wearing face masks, washing hands regularly, avoid kissing or giving hugs as greeting and respecting the spacing between individuals ..., etc. The second can be implemented by isolating the infected individual at home and health monitoring. The third can be interpreted as quarantined the infected with complication at the hospital and health monitoring. The last one can be interpreted as quarantined the infected multimorbidity with complication at the hospital with requirement breathing assistance. In order to fight against the disease spread, to achieve these objectives, we use theoretical results. We prove the existence of optimal control, Pontryagin's Maximum Principle in discrete time is used to characterize the optimal controls in term of states and adjoint functions. The optimality system is solved by iterative method. Other models from population dynamics and optimal controls can be found in [21, 22, 23, 24, 25, 26].
The paper is structured as follows. In section 2, we present our $SEIC^WCR$ discrete mathematical model that describes the class of COVID-19, the numerical simulation without control are given. In section 3, we present the optimal control problem for the suggested model was we provided some results regarding the existence and characterization of the optimal controls using Pontryagin's Maximum Principle in discrete time. As an application, the numerical simulations associated with our control problem are given in section 4. Finally, we conclude the paper in section 5.

2. Mathematical Model and Simulation Without Controls

2.1. Description of the Model. In this section we consider a discrete mathematical model $SEIC^WCR$ that will describe the dynamics of a population of COVID-19, our model which consists of six compartments representing the subdivision of the population that in the propagation of the spread of infectious diseases.
S: Susceptible individual.
E: Exposed (Latently Infected).
I: Infected with mild symptoms.
C^W: Infected with complication (severe symptoms).
C: Infected multimorbidity with complication (severe symptoms require breathing assistance).
And R: The individuals who have recovered from sickness.

The following graphical representation of the proposed model is shown in Figure 1. The com-

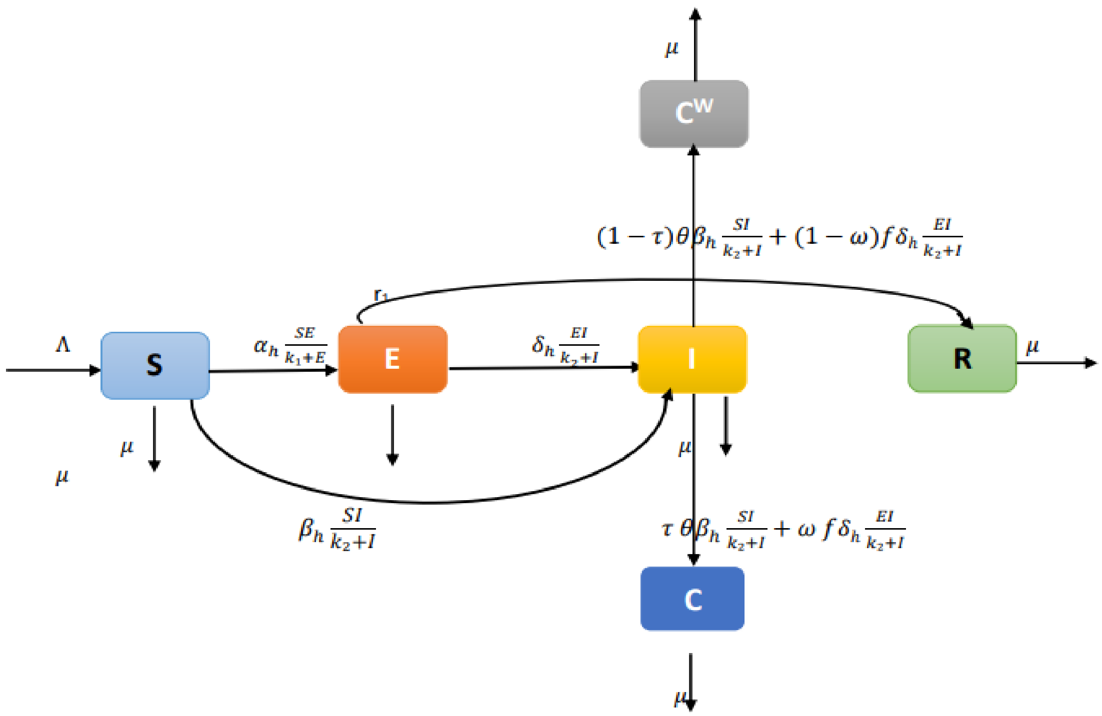

FIGURE 1. Descriptive diagram of the COVID19 dynamics.

partment S: Susceptible individuals acquire infection following contact with on active infectious individuals at rate $\beta_h \frac{I_i}{k_2+I_i}$, β_h is the probability that one susceptible becomes infected by an infectious individual, this population becomes exposed by means of the contact with an exposed individual at rate $\alpha_h \frac{E_i}{k_1+E_i}$. This population increases with the charges Λ which represent that susceptible human are requited into the population and decreases with the rate μS_i. Thus, in this compartment we have an incoming flux equal to Λ and outgoing flux equal to $\alpha_h S_i \frac{E_i}{k_1+E_i} + \beta_h S_i \frac{I_i}{k_2+I_i}$, k_1 and k_2 are the half saturated parameter.

The compartment E: Represent the number of humans exposed to the disease. Thus, we have a coming flux equal to $\alpha_h \frac{E_i}{k_1+E_i}$ which represents the proportion of the individuals come to exposed class. This population is decreasing by the following contact with on active infectious individual at rate $\delta_h E_i \frac{I_i}{k_2+I_i}$, δ_h: is the probability that one exposed becomes infected by an infectious individual, this number decreasing by μ(natural mortality) and also by amount $r_1 E_i$, r_1: an individual may recover naturally. This compartments increase at a rate $\gamma R_i \frac{I_i}{k_2+I_i}$. Thus, in this compartment we have an incoming flux equal to $\alpha_h S_i \frac{E_i}{k_1+E_i} + \gamma R_i \frac{I_i}{k_2+I_i}$ and an outgoing flux equal to $\delta_h E_i \frac{I_i}{k_2+I_i} + r_1 E_i + \mu E_i$.

The compartment I: This compartment represents the number of individuals infected with mild symptoms. This number increases at a rate $(1-\theta)\beta_h S_i \frac{I_i}{k_2+I_i} + (1-f)\delta_h E_i \frac{I_i}{k_2+I_i}$ and also decreasing by μ.

The compartment C^W: This compartment represents the number of individuals infected with complication and severe symptoms. Thus, we have an incoming flux equal to $(1-\tau)\theta\beta_h S_i \frac{I_i}{k_2+I_i} + (1-\omega) f \delta_h E_i \frac{I_i}{k_2+I_i}$ and these form the primary active cases with complication) and decreasing by μ.

The compartment C: This compartment represents the number of individuals infected multimorbidity with complication and severs symptoms require breathing assistance. This number increases at a rate $\tau\theta\beta_h S_i \frac{I_i}{k_2+I_i} + \omega f \delta_h E_i \frac{I_i}{k_2+I_i}$ which represent the portion the primary active cases with complication and require breathing assistance) and decreasing by μ.

The compartment R: This compartment represents the number of individuals who have recovered from sickness, Individuals in R are not totally immune to disease infection and are infected at rate $\gamma R_i \frac{I_i}{k_2+I_i}$ and move into E. This compartment is decreasing by μ (the natural death rate).

Hence, we present the COVID-19 mathematical model by the following nonlinear system of difference equations.

$$
(1) \qquad \begin{cases}
S_{i+1} &= \Lambda + (1-\mu) S_i - \alpha_h S_i \frac{E_i}{k_1+E_i} - \beta_h S_i \frac{I_i}{k_2+I_i} \\
E_{i+1} &= (1-\mu) E_i + \alpha_h S_i \frac{E_i}{k_1+E_i} - \delta_h E_i \frac{I_i}{k_2+I_i} - r_1 E_i + \gamma R_i \frac{I_i}{k_2+I_i} \\
I_{i+1} &= (1-\mu) I_i + (1-\theta)\beta_h S_i \frac{I_i}{k_2+I_i} + (1-f)\delta_h E_i \frac{I_i}{k_2+I_i} \\
C^W_{i+1} &= (1-\mu) C^W_i + (1-\tau)\theta\beta_h S_i \frac{I_i}{k_2+I_i} + (1-\omega) f \delta_h E_i \frac{I_i}{k_2+I_i} \\
C_{i+1} &= (1-\mu) C_i + \tau\theta\beta_h S_i \frac{I_i}{k_2+I_i} + \omega f \delta_h E_i \frac{I_i}{k_2+I_i} \\
R_{i+1} &= (1-\mu) R_i + r_1 E_i - \gamma R_i \frac{I_i}{k_2+I_i}
\end{cases}
$$

With initial values $S(0), E(0), I(0), C^W(0), C(0)$ and $R(0)$ are nonnegatives.

In order to show the effectiveness of the proposed model and the contribution of the mobility in the transmission of the disease, we give a numerical simulation of our model along a period of 50 days with the following figures to ensure that the model adapts to the reality, initial values are approximate data that we suggested after studying and researching some statistics about the population infected with the novel coronavirus 2019, the values are presented in the table.

Figures 2 present the numerical results for the numbers of susceptibles, exposed, infected with mild symptoms, infected with complication, infected multimorbidity with complication and recovered people, respectively. Susceptible individuals become exposed and after an incubation period become infected, infected with complication and infected multimorbidity with complication. Thus, the disease spreads to reach the entire population. The number of susceptible decreases sharply, in contrast there is a significant rise of infected people, infected people multimorbidity with complication. The number of populations infected with mild symptoms increased from 800 to 6500 after 30 days, as well a the increase in the number of populations with complication with mild symptoms increased as well from 300 to 3900 while the number of people with complication with mild symptoms require breathing assistance went up from 200 to 2600 as a result, of the transmission of infection from one person to another through several methods including contact with an infected person or by migrant workers, traveling, or families and relatives of the infected persons.

The remarks observed in these simulations motivate us to think of defining a suitable control strategy taking these remarks into consideration. The strategy chosen here is the introduction of four controls.

3. An Optimal Control Approach: Existence and Characterization

3.1. The model with controls. As mentioned in the last paragraph, the number of infected, infected with complication, infected multimorbidity with complication increases considerably, we introduce a control strategy into the system(1), as control measures to fight the spread of infectious, we extend our system by including four kind of controls u_1, u_2, u_3, and u_4. The first control u_1 is the proportion to be subjected to sensitization and prevention through the media and education, and preventing gatherings through security campaigns, so we note this control is the diagnosed and awareness program to susceptible people and contact prevention to exposed people. The second control u_2 can be interpreted health monitoring and have been quarantined at home, the third control u_3 can be interpreted health monitoring and have been quarantined in hospital with follow-up, the last one u_4 can be interpreted health monitoring and have been quarantined in hospital with follow-up require breathing assistance.

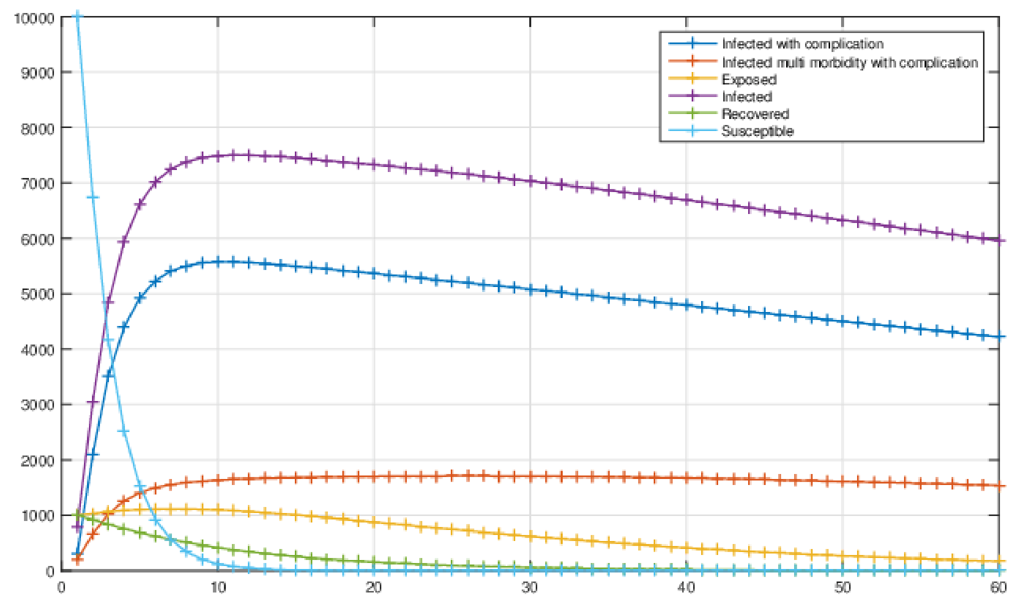

FIGURE 2. Dynamics without control strategy.

To better understand the effects of any control measure of these strategies, we introduce three new variables π_i where $i = 1,2,3,4$. $\pi_i = 0$ in the absence of control and $\pi_i = 1$ in the presence of control.

$$
(2) \quad \begin{cases}
S_{i+1} = \Lambda + (1-\mu) S_i - \alpha_h (1-\pi_1 u_{1,i}) S_i \frac{E_i}{k_1+E_i} - \beta_h S_i \frac{I_i}{k_2+I_i} \\
E_{i+1} = (1-\mu) E_i + \alpha_h (1-\pi_1 u_{1,i}) S_i \frac{E_i}{k_1+E_i} - \delta_h E_i \frac{I_i}{k_2+I_i} - r_1 E_i + \gamma R_i \frac{I_i}{k_3+I_i} \\
I_{i+1} = (1-\mu) I_i + (1-\theta) \beta_h S_i \frac{I_i}{k_2+I_i} + (1-f) \delta_h E_i \frac{I_i}{k_2+I_i} - \pi_2 u_{2,i} I_i \\
C^W_{i+1} = (1-\mu) C^W_i + \tau\theta \beta_h S_i \frac{I_i}{k_2+I_i} + \omega f \delta_h E_i \frac{I_i}{k_2+I_i} - \pi_3 u_{3,i} C^W_i \\
C_{i+1} = (1-\mu) C_i + (1-\tau) \theta \beta_h S_i \frac{I_i}{k_2+I_i} + (1-\omega) f \delta_h E_i \frac{I_i}{k_2+I_i} - \pi_4 u_{4,i} C_i \\
R_{i+1} = (1-\mu) R_i + r_1 E_i - \gamma R_i \frac{I_i}{k_3+I_i} + \pi_2 u_{2,i} I_i + \pi_3 u_3 C^W_i + \pi_4 u_{4,i} C_i
\end{cases}
$$

With initial values $S(0), E(0), I(0), C(0)$ and $R(0)$ are nonnegatives.

u_1 : Represents the proportion to be subjected to sensitization and prevention through the media and education, and preventing gatherings through security campaigns.

u_2 : Represents the rate of the individual who has been quarantined at home and health monitoring.

u_3 : Represents the rate of the individual who has been quarantined at the hospital with follow-up and health monitoring.

u_4 : Represents the rate of the individual who has been quarantined in hospital with requirement breathing assistance with follow-up and health monitoring.

3.2. Existence of an Optimal Control. The problem is to minimize the objective functional

$$\begin{aligned} J(u_1,u_2,u_3,u_4) &= MI_N + KC_N^W + JC_N + NE_N - HR_N \\ &+ \sum_{i=0}^{N-1} MI_i + KC_i^W + JC_i + NE_i - HR_i + \frac{1}{2}A\pi_1 u_{1,i}^2 + \frac{1}{2}B\pi_2 u_{2,i}^2 + \frac{1}{2}D\pi_3 u_{3,i}^2 + \frac{1}{2}F\pi_3 u_{4,i}^2. \end{aligned} \tag{3}$$

Where the parameters $M>0$, $K>0$,$N>0$, $A>0$, $B>0$, $C_i^W>0$, $C_i>0$ and $H_i>0$ for $i\in\{0,...,N\}$ are the cost coefficients. They are selected to weigh the relative importance of $I_i, C_i, C_i^W, R_i, u_{1,i}, u_{2,i}, u_{3,i}$ and $u_{4,i}$ at time i. N is the final time.

In other words, we seek the optimal controls u_1, u_2, u_3 and u_4 such that

$$J(u_1^*,u_2^*,u_3^*,u_4^*) = \min_{(u_1,u_2,u_3,u_4)\in U_{ad}^4} J(u_1,u_2,u_3,u_4)$$

where U_{ad} is the set of admissible controls defined by

$$U_{ad} = \left\{ u_j = \begin{pmatrix} u_{j,0}, & u_{j,1}, & \ldots, & u_{j,N-1} \end{pmatrix} : a_j \le u_{j,i} \le b_j \ for \ \ j=1,2,3,4, \quad i=0,1,2,\ldots,N-1 \right\}.$$

The sufficient condition for the existence of an optimal control $(u_1^*,u_2^*,u_3^*,u_4^*)$ for problem (3) comes from the following theorem.

Theorem 3.1. *There exist the optimal controls u_1^*, u_2^*, u_3^* and u_4^* such that*

$$J(u_1^*,u_2^*,u_3^*,u_4^*) = \min_{(u_1,u_2,u_3,u_4)\in U_{ad}^4} J(u_1,u_2,u_3,u_4)$$

subject to the control system (2) with initial conditions.

Proof. Since the coefficients of the state equations are bounded and there are a finite number of time steps,
$S = \begin{pmatrix} S_0, & S_1, & \ldots, & S_N \end{pmatrix}$, $E = \begin{pmatrix} E_0, & E_1, & \ldots, & E_N \end{pmatrix}$, $I = \begin{pmatrix} I_0, & I_1, & \ldots, & I_N \end{pmatrix}$,
$C^W = \begin{pmatrix} C_0^W, & C_1^W, & \ldots, & C_N^W \end{pmatrix}$, $C = \begin{pmatrix} C_0, & C_1, & \ldots, & C_N \end{pmatrix}$ and $R = \begin{pmatrix} R_0, & R_1, & \ldots, & R_N \end{pmatrix}$ are uniformly bounded for all (u_1,u_2,u_3,u_4) in the control set U_{ad}^4, and thus $J(u_1,u_2,u_3,u_4)$ is bounded for all $(u_1,u_2,u_3,u_4)\in U_{ad}^4$. Since $J(u_1,u_2,u_3,u_4)$ is bounded,
$\inf_{(u_1,u_2,u_3,u_4)\in U_{ad}^4} J(u_1,u_2,u_3,u_4)$ is finite, and there exists a sequence $\left(u_1^j,u_2^j,u_3^j,u_4^j\right)\in U_{ad}^4$ such that $\lim_{j\longrightarrow+\infty} J\left(u_1^j,u_2^j,u_3^j,u_4^j\right) = \inf_{(u_1,u_2,u_3,u_4)\in U_{ad}^4} J(u_1,u_2,u_3,u_4)$ and corresponding sequences of

states $S^j, E^j, I^j, \left(C^W\right)^j, C^j$ and R^j. Since there is a finite number of uniformly bounded sequences, there exist $\left(u_1^*, u_2^*, u_3^*, u_4^*\right)$ and $S^*, E^*, I^*, \left(C^W\right)^*, C^*$ and $R^* \in \mathbb{R}^{N+1}$ such that on a subsequence, $\left(u_1^j, u_2^j, u_3^j, u_4^j\right) \longrightarrow \left(u_1^*, u_2^*, u_3^*, u_4^*\right)$, $S^j \longrightarrow S^*$, $E^j \longrightarrow E^*$, $I^j \longrightarrow I^*$, $\left(C^W\right)^j \longrightarrow \left(C^W\right)^*$, $C^j \longrightarrow C^*$and $R^j \longrightarrow R^*$. Finally, due to the finite dimensional structure of system (2) and the objective function $J\left(u_1, u_2, u_3, u_4\right)$, $\left(u_1^*, u_2^*, u_3^*, u_4^*\right)$ is an optimal control with corresponding states $S^*, E^*, I^*, \left(C^W\right)^*, C^*$ and R^*.
Therefore $\inf\limits_{(u_1,u_2,u_3,u_4)\in U_{ad}^4} J\left(u_1, u_2, u_3, u_4\right)$ is achieved. □

3.3. Characterization of the Optimal Controls.

In order to derive the necessary condition for optimal control, the Pontryagin's maximum principle in discrete time given [27, 28] was used. The key idea is introducing the adjoint function to attach the system of difference equations to the objective functional resulting in the formation of a function called the Hamiltonian. This principle converts into a problem of minimizing a Hamiltonian at time step defined by

$$H_i = MI_i + KC_i^W + JC_i + NE_i - HR_i + \frac{1}{2}A\pi_1 u_{1,i}^2 + \frac{1}{2}B\pi_2 u_{2,i}^2 + \frac{1}{2}D\pi_3 u_{3,i}^2 + \frac{1}{2}F\pi_4 u_{4,i}^2 + \sum_{j=1}^{6} \lambda_{j,i+1} f_{j,i+1},$$

where $f_{j,i+1}$ is the right side of the system of difference equations (3) of the j^{th} state variable at time step $i+1$.

Using the Pontryagin's maximum principle in discrete time [27, 28, 29], we can say the following theorem

Theorem 3.2. *Given the optimal controls* $\left(\; u_{1,i}^*, \quad u_{2,i}^*, \quad u_{3,i}^*, \quad u_{4,i}^* \;\right)$ *and the solutions* $S^*, E^*, I^*, \left(C^W\right)^*, C^*$ *and* R^* *of the corresponding state system* (3), *there exists adjoint variables* $\lambda_{1,i}, \lambda_{2,i}, \lambda_{3,i}, \lambda_{4,i}, \lambda_{5,i}$ *and* $\lambda_{6,i}$ *satisfying:*

$$\left\{\begin{array}{lcc}
\Delta\lambda_{1,i} & = & \lambda_{1,i+1}\left((1-\mu) - \alpha_h\left(1-\pi_1 u_1\right)\frac{E_i}{k_1+E_i} - \beta_h \frac{I_i}{k_2+I_i}\right) + \lambda_{2,i+1}\left(\alpha_h\left(1-\pi_1 u_1\right)\frac{E_i}{k_1+E_i}\right) \\
 & & +\lambda_{3,i+1}\left((1-\theta)\beta_h \frac{I_i}{k_2+I_i}\right) + \lambda_{4,i+1}\left((1-\tau)\theta\beta_h \frac{I_i}{k_2+I_i}\right) + \lambda_{5,i+1}\left(\tau\theta\beta_h \frac{I_i}{k_2+I_i}\right) \\
 & & N + \lambda_{1,i+1}\left(-\alpha_h\left(1-\pi_1 u_1\right)S_i \frac{k_1}{(k_1+E_i)^2}\right) + \lambda_{3,i+1}\left(f\delta_h \frac{I_i}{k_2+I_i}\right) \\
\Delta\lambda_{2,i} & = & +\lambda_{2,i+1}\left((1-\mu) + \alpha_h\left(1-\pi_1 u_1\right)S_i \frac{k_1}{(k_1+E_i)^2} - \delta_h \frac{I_i}{k_2+I_i} - r_1\right) \\
 & & +\lambda_{4,i+1}\left(\omega f\delta_h E_i \frac{I_i}{k_2+I_i}\right) + \lambda_{6,i+1}\, r_1 + \lambda_{5,i+1}\left((1-\omega)f\delta_h \frac{I_i}{k_2+I_i}\right) \\
 & & M - \lambda_{1,i+1}\left(\beta_h S_i \frac{k_2}{(k_2+I_i)^2}\right) - \lambda_{2,i+1}\left(\delta_h E_i \frac{k_2}{(k_2+I_i)^2}\right) + \lambda_{4,i+1}\left(\tau\theta\beta_h S_i \frac{k_2}{(k_2+I_i)^2} + \omega f\delta_h E_i \frac{k_2}{(k_2+I_i)^2}\right) \\
\Delta\lambda_{3,i} & = & +\left(\gamma R_i \frac{k_3}{(k_3+I_i)^2}\right)\left(\lambda_{2,i+1} - \lambda_{6,i+1}\right) + \lambda_{3,i+1}\left((1-\mu) + (1-\theta)\beta_h S_i \frac{k_2}{(k_2+I_i)^2} + (1-f)\delta_h E_i \frac{k_2}{(k_2+I_i)^2}\right) \\
 & & -\pi_2 u_2 \lambda_{3,i+1} + \lambda_{5,i+1}\left((1-\tau)\theta\beta_h S_i \frac{k_2}{(k_2+I_i)^2} + (1-\omega)f\delta_h E_i \frac{k_2}{(k_2+I_i)^2}\right) + \lambda_{6,i+1}\left(\pi_2 u_2\right) \\
\Delta\lambda_{4,i} & = & K + \lambda_{4,i+1}\left((1-\mu) - \pi_3 u_{3;i}\right) + \lambda_{6,i+1}\left(\pi_3 u_{3,i}\right) \\
\Delta\lambda_{5,i} & = & J + \lambda_{5,i+1}\left((1-\mu) - \pi_4 u_{4;i}\right) + \lambda_{6,i+1}\left(\pi_4 u_{4,i}\right) \\
\Delta\lambda_{6,i} & = & \lambda_{6,i+1}(1-\mu) - H + \left(\gamma \frac{I_i}{k_2+I_i}\right)\left(\lambda_{2,i+1} - \lambda_{6,i+1}\right)
\end{array}\right. \tag{4}$$

with the transversality conditions at time N

$$\lambda_{1,N}=0,\quad \lambda_{2,N}=N_N,\quad \lambda_{3,N}=M_N,\quad \lambda_{4,N}=K_N,\quad \lambda_{5,N}=J_N,\quad \lambda_{6,N}=-H_N. \tag{5}$$

Furthermore, for i = $0,\ 1,\ 2,\ \ldots,\ N-1,$ *and for* $\pi_1=\pi_2=\pi_3=1$ *the optimal controls* $u_1^*, u_2^*, u_3^*,$ *and* u_4^* *are given by*

$$\begin{aligned}
u_1^* &= \min\left(0, \max\left(1, \alpha_h S_i \frac{(\lambda_{2,i+1}-\lambda_{1,i+1})}{A}\frac{E_i}{k_1+E_i}\right)\right)\\
u_2^* &= \min\left(0, \max\left(1, \frac{(\lambda_{3,i+1}-\lambda_{6,i+1})}{B} I_i\right)\right)\\
u_3^* &= \min\left(0, \max\left(1, \frac{(\lambda_{4,i+1}-\lambda_{6,i+1})}{D} C_i^W\right)\right)\\
u_4^* &= \min\left(0, \max\left(1, \frac{(\lambda_{5,i+1}-\lambda_{6,i+1})}{F} C_i\right)\right)
\end{aligned} \tag{6}$$

Proof. The Hamiltonian of the optimal problem is given by

$$\begin{aligned}
H_i &= MI_i+KC_i^W+JC_i+NE_i-HR_i+\frac{1}{2}A\pi_1u_{1,i}^2+\frac{1}{2}B\pi_2u_{2,i}^2+\frac{1}{2}D\pi_3u_{3,i}^2+\frac{1}{2}F\pi_4u_{4,i}^2\\
&+\lambda_{1,i+1}\left(\Lambda+(1-\mu)S_i-\alpha_h(1-\pi_1u_{1,i})S_i\frac{E_i}{k_1+E_i}-\beta_hS_i\frac{I_i}{k_2+I_i}\right)\\
&+\lambda_{2,i+1}\left((1-\mu)E_i+\alpha_h(1-\pi_1u_{1,i})S_i\frac{E_i}{k_1+E_i}-\delta_hE_i\frac{I_i}{k_2+I_i}-r_1E_i+\gamma R_i\frac{I_i}{k_3+I_i}\right)\\
&+\lambda_{3,i+1}\left((1-\mu)I_i+(1-\theta)\beta_hS_i\frac{I_i}{k_2+I_i}+(1-f)\delta_hE_i\frac{I_i}{k_2+I_i}-\pi_2u_{2,i}I_i\right)\\
&+\lambda_{4,i+1}\left((1-\mu)C_i^W+\tau\theta\beta_hS_i\frac{I_i}{k_2+I_i}+\omega f\delta_hE_i\frac{I_i}{k_2+I_i}-\pi_3u_{3,i}C_i^W\right)\\
&+\lambda_{5,i+1}\left((1-\mu)C_i+(1-\tau)\theta\beta_hS_i\frac{I_i}{k_2+I_i}+(1-\omega)f\delta_hE_i\frac{I_i}{k_2+I_i}-\pi_4u_{4,i}C_i\right)\\
&+\lambda_{6,i+1}\left((1-\mu)R_i+r_1E_i-\gamma R_i\frac{I_i}{k_3+I_i}+\pi_2u_{2,i}I_i+\pi_3u_{3,i}C_i^W+\pi_4u_{4,i}C_i\right),
\end{aligned}$$

We put $\Phi(N)=MI_N+KC_N^W+JC_N+NE_N-HR_N$.

For $i=0,1,\ldots,N-1$, the adjoint equations and transversality conditions can be obtained by using Pontryagin's Maximum Principle, in discrete time, given in [27, 28] such that

$$\begin{aligned}
\lambda_{1,i} &= \tfrac{\partial H_i}{\partial S_i}, \\
\lambda_{2,i} &= \tfrac{\partial H_i}{\partial E_i}, \\
\lambda_{3,i} &= \tfrac{\partial H_i}{\partial I_i}, \\
\lambda_{4,i} &= \tfrac{\partial H_i}{\partial C_i^W}, \\
\lambda_{5,i} &= \tfrac{\partial H_i}{\partial C_i}, \\
\lambda_{6,i} &= \tfrac{\partial H_i}{\partial R_i}.
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{1,N} &= \frac{\partial \Phi(N)}{\partial S_N} = 0, \\
\lambda_{2,N} &= \frac{\partial \Phi(N)}{\partial E_N} = N_N, \\
\lambda_{3,N} &= \frac{\partial \Phi(N)}{\partial I_N} = M_N, \\
\lambda_{4,N} &= \frac{\partial \Phi(N)}{\partial C_N^W} = K_N, \\
\lambda_{5,N} &= \frac{\partial \Phi(N)}{\partial C_N} = J_N, \\
\lambda_{6,N} &= \frac{\partial \Phi(N)}{\partial R_N} = -H_N.
\end{aligned}$$

For, $i = 0, 1, ..., N-1$ the optimal controls $u_{1,i}, u_{2,i}, u_{3,i}$ and $u_{4,i}$ can be solved from the optimality condition,

$$\begin{aligned}
\frac{\partial H_i}{\partial u_{1,i}} &= 0, \\
\frac{\partial H_i}{\partial u_{2,i}} &= 0, \\
\frac{\partial H_i}{\partial u_{3,i}} &= 0, \\
\frac{\partial H_i}{\partial u_{4,i}} &= 0.
\end{aligned}$$

That are

$$\begin{aligned}
\frac{\partial H_i}{\partial u_{1,i}} &= A\pi_1 u_{1,i} + \lambda_{1,i+1}\left(\pi_1 \alpha_h S_i \tfrac{E_i}{k_1+E_i}\right) - \lambda_{2,i+1}\left(\pi_1 \alpha_h S_i \tfrac{E_i}{k_1+E_i}\right) &= 0, \\
\frac{\partial H_i}{\partial u_{2,i}} &= B\pi_2 u_{2,i} - \lambda_{3,i+1}\left(\pi_2 I_i\right) + \lambda_{5,i+1}\left(\pi_2 I_i\right) &= 0, \\
\frac{\partial H_i}{\partial u_{3,i}} &= D\pi_3 u_{3,i} - \lambda_{4,i+1}\left(\pi_3 C_i^W\right) + \lambda_{5,i+1}\left(\pi_3 C_i^W\right) &= 0, \\
\frac{\partial H_i}{\partial u_{4,i}} &= F\pi_4 u_{4,i} - \lambda_{5,i+1}\left(\pi_4 C_i\right) + \lambda_{6,i+1}\left(\pi_4 C_i\right) &= 0.
\end{aligned}$$

So for $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 1$ we have

$$\begin{aligned}
u_{1,i} &= \alpha_h S_i \frac{(\lambda_{2,i+1} - \lambda_{1,i+1})}{A} \frac{E_i}{k_1+E_i} \\
u_{2,i} &= \frac{\left(\lambda_{3,i+1} - \lambda_{6,i+1}\right)}{B} I_i \\
u_{3,i} &= \frac{\left(\lambda_{4,i+1} - \lambda_{6,i+1}\right)}{D} C_i^W \\
u_{4,i} &= \frac{\left(\lambda_{5,i+1} - \lambda_{6,i+1}\right)}{F} C_i
\end{aligned}$$

By the bounds in U_{ad} of the controls, it is easy to obtain $u_{1,i}^{*} u_{2,i}^{*}, u_{3,i}^{*}$ and $u_{4,i}^{*}$ in the form of system. □

4. Numerical Simulation

In this part, we present numerical simulation to highlight the effect of our control strategy that we have developed in the framework of fight against the spread of the Coronavirus disease 2019. The initial values are the same in the Table 1, with regard to other initial values they are proposed values after a statistical study. Concerning the numerical method, we give numerical simulation to our optimality system which is formulated by state equations with initial and boundary conditions, adjoint equation with transversality conditions $(4,5)$ and optimal control characterization (6). We apply the forward-backward sweep method (FBSM) [30] to solve our optimality system in an iterative process. We start with an initial guess for the controls at the first iteration and then before the next iteration, we update the controls by using the characterization. We continue until convergence of successive iterates is achieved. The numerical solution of model (2) with the following parameter values and initial values of the state variable in Table 1 is executed using MATLAB.

TABLE 1. Rumor model parameters and values

Parameter	Description	Value
Λ	Recruitment rate	(0.05/day)
μ	Natural death rate	(0.02/day)
β_h	The proportion that one susceptible individual becomes infected	(0.4/day)
δ_h	The proportion of exposed who becomes infected after following contact with an active infectious individual.	0.025/day
α_h	The probability that one susceptible individual becomes exposed	(0.00001/day)
γ	The proportion of recovering are not totally immune to the disease and move into E.	(0.002/day)
r_1	The quarantine rate for the latently infected	(0.8/day)
τ	The proportion of individuals going to compartment C^W	(0.8 /day)
$(1-\tau)$	The proportion of infected individual become infected multimorbidity with complication	0.2 /day
ω	The probability that susceptible individuals infected will enter the compartment C^W.	0.4 /day
$(1-\omega)$	The proportion of individuals infected leave to compartment C.	0.6 /day
θ	transmission coefficient.	0.7 /day
f	transmission coefficient.	0.7 /day

Additionally, we present in this section numerical results that illustrate and reinforce the effect of our control strategy, this strategy consists in applying four kind of controls. We apply our strategy for a period of sixty days that we assume an average duration of the disease spread, where we assume that the initial susceptible, exposed, infected, infected with complication, infected multimorbidity with complication and recovered populations are given by $S_0 = 10000$, $E_0 = 1000$, $I_0 = 800$, $C_0^W = 300$, $C_0 = 200$ and $R_0 = 1000$. Also, the upper limits of the optimality conditions are considered to be $u_i^{max} = 0.9$, $i = 1,2,3,4$ and $u_i^{min} = 0.2$, $i = 1,2,3,4$. Figure 3 illustrates the founded results. As it shows in such figure, the effect of the strategy begins as early as the second day, the number of the susceptible decrease to 10000 in a fatal way until reaching almost zero. As for the number of exposed, infected, infected with complication, infected multimorbidity with complication increase in the beginning but then decreases clearly. This decrease leads to an increase in the number of recovering from the beginning and approaches a given value of 2.1×10^4.

The figure 2 represents disappearance of the recovered population in the absence of the control. In this case, susceptible individuals are transferred to the infected classes (see Figure 2), but in the second case susceptible individuals are transferred to the recovered class through the apply of our controls (see Figure 3).

In the next paragraph and in order to obtain more accurate information about the impact of each control separately, we choose to apply four scenarios. In each of these we apply separately one control, we will consider four cases of the fight against the disease spread.

Case 1: Applying only control u_1.

Case 2: Applying only control u_2.

Case 3: Applying only control u_3.

Case 4: Applying only control u_4.

Apply only control u_1. In this scenario, we simulate the case where we will be applied to the susceptible individual, we will be limited to displaying and comparing the curves of exposed and recovered in both cases with and without control strategy. The control is applied over a period of 60 days. The figure 4 shows that the number of exposed decreases clearly after the implementation of the strategy. On the other hand, the number of removed, will suddenly start

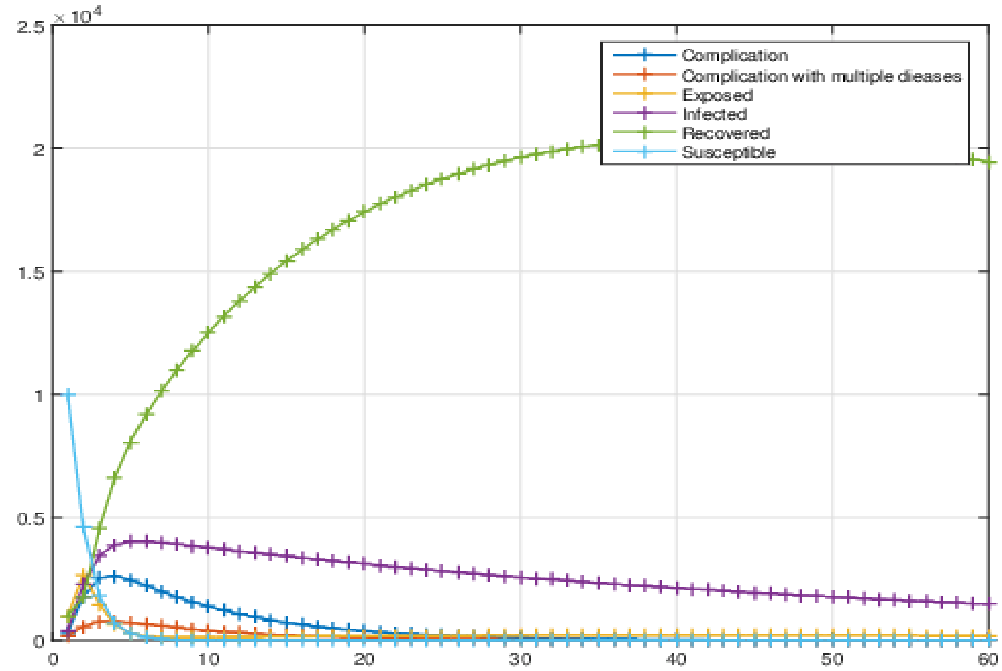

FIGURE 3. Dynamics with controls u_1, u_2, u_3 and u_4.

to rise starting from the first day. This change is probably due to the fact that the control of sensitization and prevention is aimed to take prevention measures and at telling the susceptible people through awareness-raising campaigns and mass media the seriousness of the disease. Note : to the time limit, there is no vaccine or treatment for this disease.

Apply only control u_2. In the second scenario, to protect the infected individual by applying the quarantine at home, follow medical advice remotely and inform them of the seriousness of the disease. To realize this objective, we apply only control u_2 in a period of 60 days. The figure 5 shows us the number infected people decreases from 6100 (without control u_2) to 1700 (with control u_2) at the end of the proposed control strategy. Also, we observe that the number of recovered has reached the value 16000 (with control u_2) compared to the situation when there is no control where this category tends to zero. However, there is a small decrease in number of infected with complication and infected multimorbidity with complication, we notice that the number decreases slowly. Hence, our objective has been achieved.

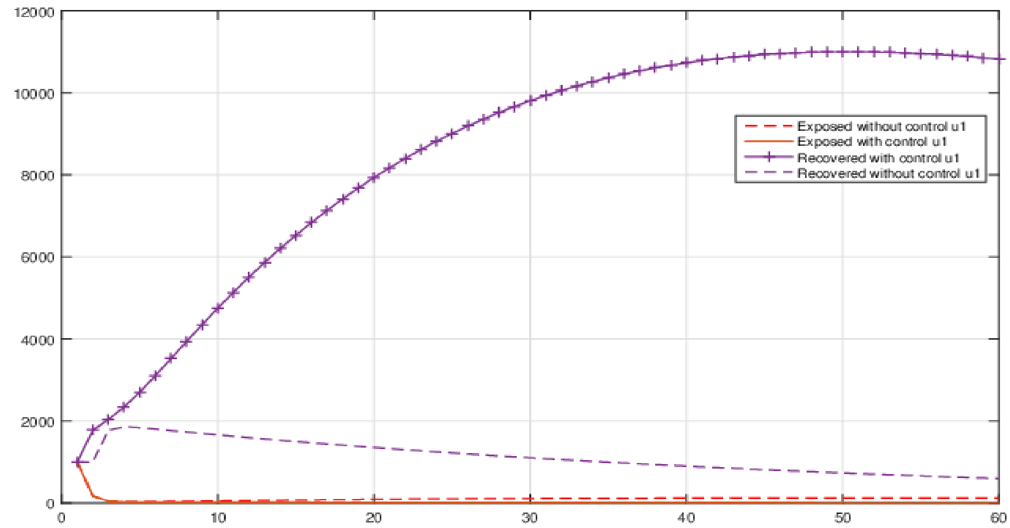

FIGURE 4. Dynamics with control u_1.

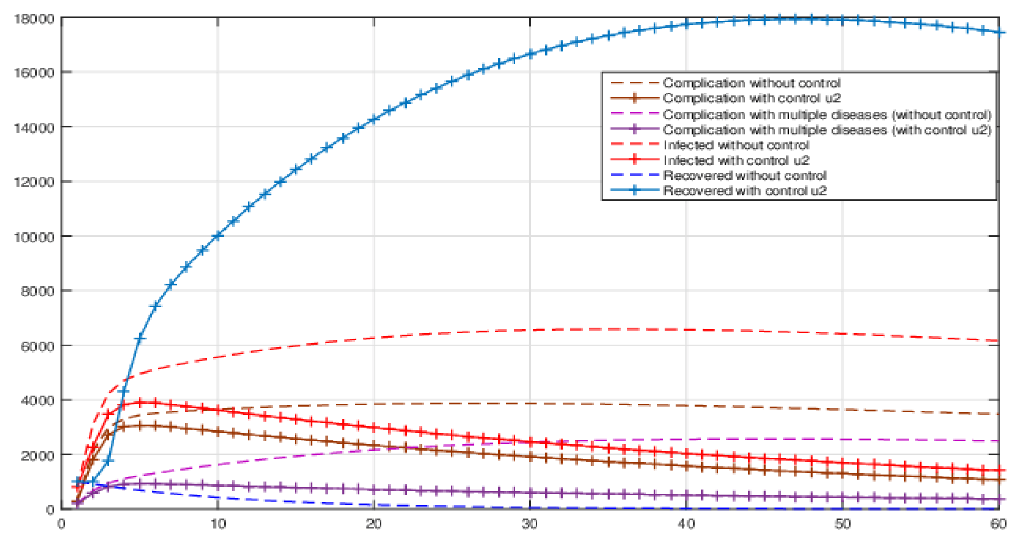

FIGURE 5. Dynamics with control u_2.

Apply only control u_3. In this case, we apply only control u_3, the protected of the infected with complication by applying the quarantine strategy at hospital and health monitoring to prevent the outbreak of the disease and avoid the spread of the disease within families. From figure 6, we remark that the effect begins immediately (after about 3 days), for the number of recovered without implementation of the control, we note that there is a gradual decline to reach zero and increases relatively weaker and is stabilised to 3000 in applying the control. However the number of infected with complication decreases since the 3^{rd} day but after 15 days we observe that the number of C^W tends to zero. These observations clearly illustrate the importance of such strategy in the fight against the spread of the disease.

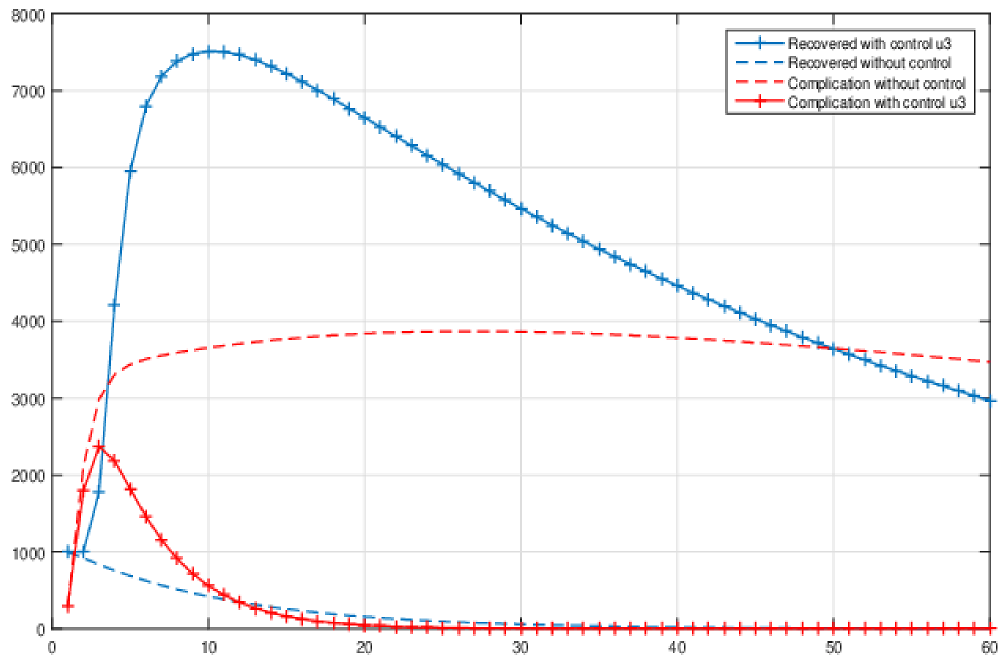

FIGURE 6. Dynamics with control u_3.

Apply only control u_4. In the last case, we apply only the remaining control u_4, effect to reduce the spread of the infection. The main objective of the whole work is to decrease the number of infected and death toll, and in this case protected the infected multimorbidity individual with serious complications by applying the quarantine at hospital strategy with the requirement breathing assistance. We deduce by simulations presented in figure 7 after 60 days that the density of the infected multimorbidity (C) is decreasing from 2500 when there is no control, to the absence of this category when applying control. However, the number of recovered increases relatively weaker and is stabilized to 2200, and that can obviously prove the effectiveness of the quarantine strategy.

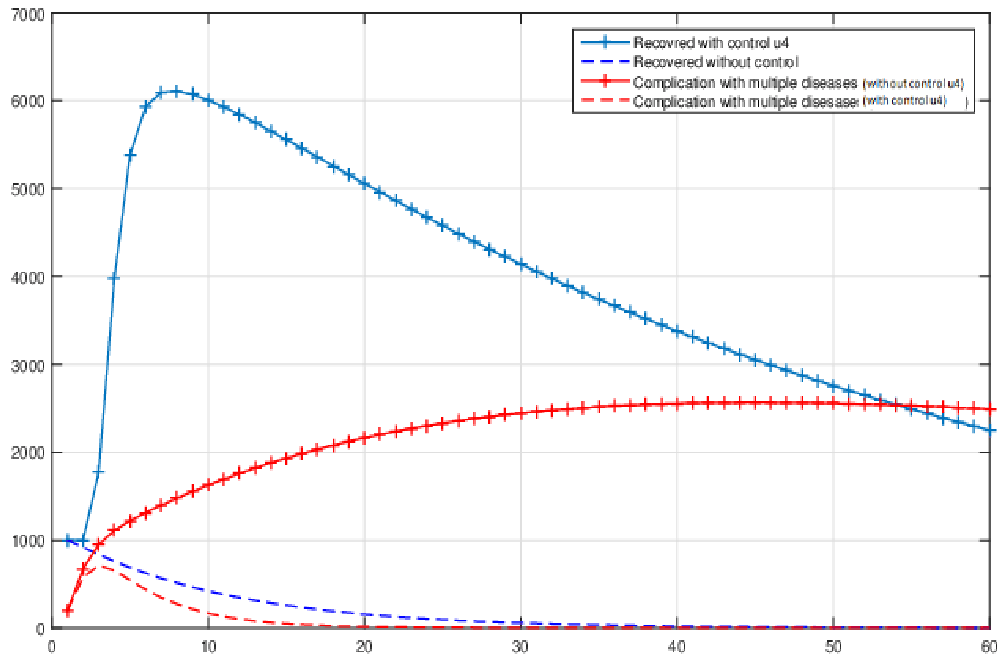

FIGURE 7. Dynamics with control u_4.

Remark 4.1. *We can also merge multiple assemblies as* (u_1,u_2)*,* (u_2,u_3)*,* (u_1,u_3) *and* (u_1,u_2,u_3) *thus get a variety of results.*

5. CONCLUSION

In this paper, we propose a new model which describes the dynamics of COVID-19 spread, we suggest also an optimal strategy in order to fight against the spread of the disease. In order to minimize the number of infected, infected with complication, infected multimorbidity with complication, four control strategies have been introduced. The first control represented the sensitization and prevention. The second control represented the quarantine at home with remote monitoring of its status. The third control represented the quarantine at the hospital. Finally, the fourth control represented the quarantine at the hospital with requirement breathing assistance, and by introduction of four new variables π_i, $i = 1,2,3,4$ we could study and combine several scenarios, in order to see the effect of each one of these controls on the reduction of the disease spread. We showed the existence of solutions to the state and an optimal control.

Pontryagin's Maximum Principle, in discrete time, is used to characterize the controls and the optimality system is solved by an iterative method. The numerical resolution of the obtained results showed the effectiveness of the proposed control strategies, as well as the numerical simulations enabled us to compare and see the difference between each scenario in a concrete way. Numerical results prove the effectiveness of our strategy and its importance in fighting the disease spread.

DATA AVAILABILITY

The disciplinary data used to support the findings of this study have been deposited in the Network Repository (http://www.networkrepository.com).

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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