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A COINFECTED MODELING OF ANTHRAX AND LISTERIOSIS WITH POWER LAW

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Abstract. Coinfections are difficult to manage and control due to the different complexities of the diseases. In this study, we analyse the fractional order version of the Anthrax-Listeriosis coinfection model formulated by Osman and Makinde in [1]. We investigate the positivity, existence and uniqueness of the model solutions and use the Liouville-Caputo operator to perform stability analysis of the model. The Adams-type predictor corrector method is also employed to analyse the trajectories of the model. We numerically simulate the theoretical results by varying fractional orders to see how changes in the fractional order derivative affects the coinfection dynamics. Our results indicate that the model notably depends on the fractional order derivative and model parameters. It also suggests that the fractional order is the main driver of the coinfection. The results we obtained are hand in glove with the results obtained in the corresponding integer model by Osman and Makinde, however, our results reveal that the use of the Liouville Caputo operator and fractional operators in general enhance a better description of biological processes than integer order operators.

Keywords: anthrax; listeriosis; Liouville-Caputo operator; fractional derivative; coinfection.

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1. Introduction

Listeriosis is a serious food-borne bacteria infection in humans, plants and animals, caused by the bacterium *Listeria monocytogene (L. monocytogene)* [2]. There are five strains of the bacterium, namely: Listeria innocua, Listeria seeligeri, Listeria welshimeri, Listeria ivanovii and Listeria monocytogenes which is the only strain that is pathogenic to both humans and animals. The food-borne pathogen L. Monocytogene survives and grows in environments with temperatures ranging from $30°C$ to $37°C$ and even in refrigerated environments with temperatures of as low as $4°C$ [3]. Its primary habitats are soil and water, and it is transmitted to humans and animals by consumption of contaminated ready-to-eat (RTE) food products such as polony, cheese, ham, meat, raw milk, processed meats, fresh and frozen poultry, fresh produce including fruits and vegetables. After consumption, the bacteria enter the gastrointestinal tract and counteract changes in acidity, osmolarity, oxygen tension, or the challenging effects of antimicrobial peptides and bile [4]. It then crosses the epithelium barrier of the infected individual through the transcytosis and invades the mesenteric lymph nodes into the blood [5]. In pregnant women, the bacteria are transmitted to the foetus either during delivery or through the placenta, which can lead to premature labour, death of new born babies, serious illness, meningitis, miscarriages, and stillbirth [6]. Listeriosis is a relatively rare but preventable disease. Data from European countries in 2016 suggests that there were about $2,555$ reported cases with high rates of 1.3 per $100,000$ population found in children below 1 year and a rate of 1.6 per $100,000$ population among ages over 64 years [7]. These cases pose serious public health concerns in Europe and in the USA, it is estimated to be the cause of over 80% percent of deaths related to food consumption. [8, 9].

Anthrax is a zoonotic disease that can be found in all continents except Antarctica [10, 11]. Anthrax is caused by the obligate in vivo pathogen *Bacillus anthracis*. Infected animal host leaves their bacilli in the soil, the spores which can survive in the soil for decades are picked up by another host through germination. Humans contract the disease via their exposure to the bones, hides and carcasses of dead animals. *Bacillus anthracis* is a potential biological weapon, when exposed to it, the lessions are found on the exposed regions of the body. It has an incubation period ranging from few hours to 3 weeks, often 2 to 6 days. In 2001 the disease became more pronounced when 5 people died after receiving letters contaminated with anthrax spores in the USA [12, 13]. A study carried out in Georgia to see how the disease has evolved from the year $2000-2013$ saw an increase in human anthrax from 0.7 cases per $100,000$ in 2000 to 3.7 cases per $100,000$ by 2013, by the end of the compulsory annual livestock anthrax vaccination, [14].

The use of mathematical models to study the dynamics and transmission of infectious diseases has become prevalent as seen in the existing mathematical epidemiological models. These models use medical data, and continuous surveillance information and other mathematical epidemiological tools to prove and predict the disease transmission dynamics [15, 16, 17]. The Authors of [18] considered a model to investigate an outbreak in Listeriosis in a community. They considered neonatal infections and a compartment to represent the reservoir for bacteria infection. They used existing data to predict the possible future outbreaks as well as the disease eradication using the reproductive number. A Listeriosis transmission model which considers transmissions through ready-to eat foods in a food processing plant was also studied in [19]. Anthrax transmission models were considered by the authors of [20, 21] but their emphasis was on disease transmissions in the animal population only. A mathematical model developed to consider both Anthrax and Listeriosis transmission in both human and vector populations was considered by authors of [1]. They considered the transmission dynamics in the human population using stability theory and differential equations. They also looked at the sensitivity analysis of the coinfection model, and the effects of the contact rate on the disease transmission. Our model is an improved fractional order version of theirs, which considers the effect of vaccination on the coinfection transmission dynamics.

Fractional calculus deals with classical concepts of differential and integral operators. It has a non local property that considers that the future state depends on the current and the past states [22]. The authors of [23] derived the composition of the fractional derivatives with Shukla function and also examined the difference between the Riemann-Louville and Caputo derivatives of the generalized Mittag-Leffler function to determine the reason for their differences. Also, authors of [23] considered the Mittag-Leffler fractional HIV-TB coinfection model. They derived a new model with nonsingular Mittag-Leffler kernel which considers anomalous spread

like that of coinfection biological models. In [24], the authors integrated some properties of the Caputo derivatives and generalized the factional derivative defined in the real line to the partial fractional derivatives in higher space dimensions.
The use of mathematical power law in recent years by researchers to analyse real life problems has become more pronounced [25, 26, 27, 28, 29]. It has been widely used in the fields of life sciences, mathematics, physics Economics and Engineering . Researchers who in the past studied mathematical biology problems in the framework of integer order are in recent times looking at problems more from the fractional order view point. [30, 31].

We propose an Anthrax-Listeriosis coinfection model which is an improvement of the model by Osman and Makinde [1]. Our model considers the effects of vaccination on the coinfection dynamics. We analyse the model from a fractional order view point using the Liouville-Caputo operator and simulate our results by investigating the effects of the fractional order derivative on the disease transmission dynamics.

2. Mathematical Preliminaries

This section presents a few imperative mathematical concepts that are needed to carry out the model analysis.

Definition 2.1. *The Riemann-Louisville (RL) integral of arbitrary real order $\omega > 0$ of a function $f(t)$ is defined by the following integral:*

$$P^{\omega}_{0,t} f(t) = \frac{1}{\Lambda(\omega)} \int_0^t (t-\kappa)^{\omega-1} f(t)\, d\kappa, t, \omega > 0.$$

Definition 2.2. *For a given well-defined absolutely continuous function $f(t) \in E^n[0,T]$ with $\omega > 0$, the Caputo fractional derivative of $f(t)$ is defined by the following integral:*

$$ {}^{C}D^{\omega}_{0} f(t) = \frac{1}{\Lambda(m-\omega)} \int_0^t (t-\kappa)^{m-\omega-1} f^{(m)} d\kappa, \tag{1}$$

Where $m-1 < \omega \leq m, m \in N$. Note that if $\omega \to 1$, then ${}^{C}D^{\omega}_{0,t} f(t)$ approaches $f'(t)$.

Theorem 1. *(Generalized mean value theorem) Let $h(x) \in C[0,T]$ and ${}^{C}D^{\omega}_{0,t} h(t) \in (0,T]$*

$$g(t) = g(0) + \frac{1}{\Lambda(\omega)} \left[{}^{C}D^{\omega}_{0,t} g \right](\kappa)\, t^{\omega}, \tag{2}$$

With $0 \leq \kappa \leq t$, $\forall t \in (0,T]$.

Corollary 2.1. *Considering that* $g(x) \in C[0,T]$, ${}^{C}D^{\omega}_{0,t} g(t) \in (0,T)$, *where* $\omega \in (0,1]$. *Then if*

(1) ${}^{C}D^{\omega}_{0,t} g(x) \geq 0$, $\forall x \in (0,T]$, *then* $g(x)$ *is non-decreasing.*

(2) ${}^{C}D^{\omega}_{0,t} g(x) \leq 0$, $\forall x \in (0,T)$, *then* $g(x)$ *is non-decreasing.*

3. Mathematical Model Formulation and Analysis

The total human population (N_h) is subdivided into susceptible humans (S_h), individuals that are infected with Anthrax (I_a), individuals that are infected with Listeriosis (I_l), individuals that are infected with both Anthrax and Listeriosis (I_{al}), individuals vaccinated (V_h) and those recoverd from Anthrax, Listeriosis, and both Anthrax and Listeriosis

respectively, (R_a), (R_l) and (R_{al}). The total vector population is represented by (N_v) and subdivided into susceptible animals (S_v) and animals infected with Anthrax (I_v), where (C_p) is the compartment for the pathogen infested animal carcasses in the soil. Carcasses of animals which may have not been properly disposed have the tendency of generating pathogens. The total human and vector populations respectively are:

$$N_h = S_h + I_a + I_l + I_{al} + V_h + R_a + R_l + R_{al},\ N_v = S_v + I_v,$$

where $\pi = \frac{C_p \nu}{\kappa + C_p}$ and κ and V denote the concentration of pathogen infested carcasses and ingestion rate respectively. Listeriosis related death rates are m and η respectively, and Anthrax related death rates are ϕ and n respectively. Waning immunity rates are given by ω, k, and ψ. α, δ and σ are the recovery rates, respectively, and $\tau(1-\sigma)$ is the rate of recovery of the coinfected persons from Anthrax only. The natural death rates of human and vector populations are μh and μV respectively, and the modification parameter is given by θ. The rate of recovery of coinfected persons from Listeriosis is denoted by $(1-\tau)(1-\sigma)$. This implies that $\sigma + \tau(1-\sigma) + (1-\tau)(1-\sigma) = 1$. The following differential equations were obtained from the flowchart diagram of the coinfection model in Figure 1. The model under investigation in equation (3) has been extensively investigated in integer order by Osman and Makinde [1]. The system of non-linear ordinary equations is given equation (3).

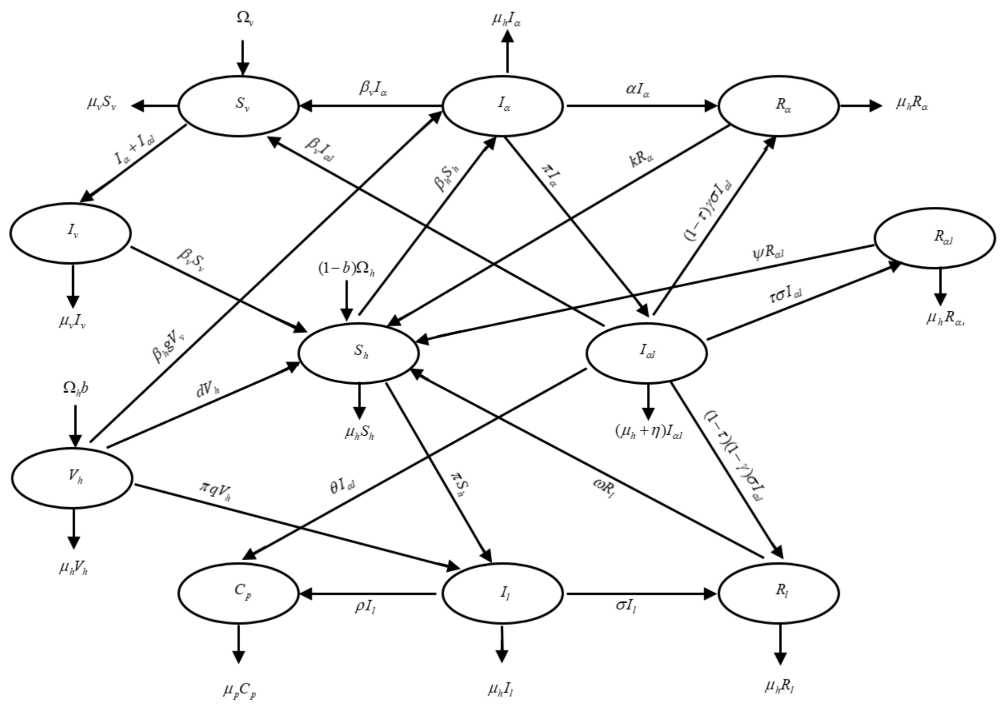


FIGURE 1. Flowchart for the coinfection model

(3)

$$
\begin{aligned}
\frac{dS_h}{dt} &= (1-b)\,\Omega_h + kR_a(t) + wR_l(t) + \psi R_{al}(t) - \beta_h I_v(t)\, S_h(t) \\
&\quad -\pi S_h(t) + dV_h(t) - \mu_h S_h(t), \\
\frac{dI_a}{dt} &= \beta_h I_v(t)\, S_h(t) - \pi I_a(t) + \beta_h g I_v(t)\, V_h(t) - (\alpha + \mu_h + \phi)\, I_a(t), \\
\frac{dI_l}{dt} &= \pi S_h(t) - \beta_l I_v(t)\, I_l(t) + \pi q V_h(t) - (\sigma + \mu_h + m + \rho)\, I_l(t), \\
\frac{dI_{al}}{dt} &= \beta_h I_v(t)\, I_l(t) + \pi I_a(t) - (\sigma + \mu_h + \eta + \theta)\, I_{al}(t), \\
\frac{dV_h}{dt} &= \Omega_h b - (d + \mu_h)\, V_h(t) - \beta_h g I_v(t)\, V_h(t) - \pi q V_h(t), \\
\frac{dR_a}{dt} &= \alpha I_a(t) - (k + \mu h)\, R_a(t) + (1-\tau)\,\gamma\sigma I_{al}(t),
\end{aligned}
$$

$$\begin{aligned}
\frac{dR_l}{dt} &= \sigma I_l(t) - (w+\mu h) R_l(t) + (1-\tau)(1-\gamma)\sigma I_{al}(t), \\
\frac{dR_{al}}{dt} &= \tau\sigma I_{al}(t) - (\psi+\mu h) R_{al}(t), \\
\frac{dC_p}{dt} &= \rho I_l(t) + \theta I_{al}(t) - \mu_b C_p(t), \\
\frac{dS_v}{dt} &= \Omega_v - \beta_v (I_a(t) + I_{al}(t)) S_v(t) - \mu_v S_v(t), \\
\frac{dI_v}{dt} &= \beta_v (I_a(t) + I_{al}(t)) S_v(t) - \mu_v I_v(t).
\end{aligned}$$

3.1. Coinfection Model with Liouville-Caputo (LC) Fractional Order (FO) Derivative.

In this subsection we give the model in the context of Caputo derivative as follows:

$$\begin{aligned}
{}_0^{\mathrm{C}}\mathrm{D}_t^{\varpi} S_h(t) &= (1-b)\Omega_h + kR_a(t) + wR_l(t) + \psi R_{al}(t) - \beta_h I_v(t) S_h(t) \\
&\quad - \pi S_h(t) + dV_h(t) - \mu_h S_h(t), \\
{}_0^{\mathrm{C}}\mathrm{D}_t^{\varpi} I_a(t) &= \beta_h I_v(t) S_h(t) - \pi I_a(t) + \beta_h g I_v(t) V_h(t) - (\alpha+\mu_h+\phi) I_a(t), \\
{}_0^{\mathrm{C}}\mathrm{D}_t^{\varpi} I_l(t) &= \pi S_h(t) - \beta_l I_v(t) I_l(t) + \pi q V_h(t) - (\sigma+\mu_h+m+\rho) I_l(t), \\
{}_0^{\mathrm{C}}\mathrm{D}_t^{\varpi} I_{al}(t) &= \beta_h I_v(t) I_l(t) + \pi I_a(t) - (\sigma+\mu_h+\eta+\theta) I_{al}(t), \\
{}_0^{\mathrm{C}}\mathrm{D}_t^{\varpi} V_h(t) &= \Omega_h b - (d+\mu_h) V_h(t) - \beta_h g I_v(t) V_h(t) - \pi q V_h(t), \\
{}_0^{\mathrm{C}}\mathrm{D}_t^{\varpi} R_a(t) &= \alpha I_a(t) - (k+\mu h) R_a(t) + (1-\tau)\gamma\sigma I_{al}(t), \\
{}_0^{\mathrm{C}}\mathrm{D}_t^{\varpi} R_l(t) &= \sigma I_l(t) - (w+\mu h) R_l(t) + (1-\tau)(1-\gamma)\sigma I_{al}(t), \\
{}_0^{\mathrm{C}}\mathrm{D}_t^{\varpi} R_{al}(t) &= \tau\sigma I_{al}(t) - (\psi+\mu h) R_{al}(t),
\end{aligned} \tag{4}$$

$$
\begin{aligned}
{}_{0}^{\mathrm{C}}\mathrm{D}_{t}^{\varpi} C_{p}(t) &= \rho I_{l}(t)+\theta I_{al}(t)-\mu_{b} C_{p}(t), \\
{}_{0}^{\mathrm{C}}\mathrm{D}_{t}^{\varpi} S_{v}(t) &= \Omega_{v}-\beta_{v}\left(I_{a}(t)+I_{al}(t)\right) S_{v}(t)-\mu_{v} S_{v}(t), \\
{}_{0}^{\mathrm{C}}\mathrm{D}_{t}^{\varpi} I_{v}(t) &= \beta_{v}\left(I_{a}(t)+I_{al}(t)\right) S_{v}(t)-\mu_{v} I_{v}(t),
\end{aligned}
$$

where ${}_{0}^{\mathrm{C}}\mathrm{D}_{t}^{\varpi}$ is the FO in LC sense, $0<\varpi \leq 1$ is the fractional order and the associated initial conditions are given by:

$$
\begin{aligned}
& S_{h(0)}=S_{h}(0), \quad I_{a(0)}=I_{a}(0), \quad I_{l(0)}=I_{l}(0), \quad I_{al(0)}=I_{al}(0), \quad V_{h(0)}=V_{h}(0), \\
& R_{a(0)}=R_{a}(0), \quad R_{l(0)}=R_{l}(0), \quad R_{al(0)}=R_{al}(0), \quad C_{p(0)}=C_{p}(0), \quad S_{v(0)}=S_{v}(0), \\
& I_{v(0)}=I_{v}(0).
\end{aligned} \tag{5}
$$

3.2. Positivity of the Coinfection Model Solutions. In order to establish the positivity of solutions and invariant region of the system (4) in the hyperoctant R_{+}^{11} We consider that

$$
\mathbb{R}_{+}^{11}=\{u \in \mathbb{R}^{11} | u \geq 0\}.
$$

Additionally,

$$
u=\left(S_{h}(t), I_{a}(t), I_{l}(t), I_{al}(t), V_{h}(t), R_{a}(t), R_{l}(t), R_{al}(t), C_{p}(t), S_{v}(t), I_{v}(t)\right).
$$

It is prudent to show that on every hyperplane bounding , the non-negative hyperoctant of the vector field points into $\mathbb{R}_{+}^{11}$. From the Caputo fractional model, we have,

$$
\begin{aligned}
& {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} S_{h}(t)=(1-b)\Omega_{h}+kR_{a}(t)+wR_{l}(t)+\psi R_{al}(t) \geq 0, \\
& {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} I_{a}(t)=\beta_{h} I_{v}(t) S_{h}(t) \geq 0, \\
& {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} I_{l}(t)=\pi S_{h}(t) \geq 0, \\
& {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} I_{al}(t)=\beta_{h} I_{v}(t) I_{l}(t)+\pi I_{a}(t) \geq 0,
\end{aligned} \tag{6}
$$

$$ {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} V_h(t) = \Omega_h b \geq 0, $$

$$ {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} R_a(t) = \alpha I_a(t) + (1-\tau)\gamma\sigma I_{al}(t) \geq 0, $$

$$ {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} R_l(t) = \sigma I_l(t) + (1-\tau)(1-\gamma)\sigma I_{al}(t) \geq 0, $$

$$ {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} R_{al}(t) = \tau\sigma I_{al} \geq 0, $$

$$ {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} C_p(t) = \rho I_l(t) + \theta I_{al}(t) \geq 0, $$

$$ {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} S_v(t) = \Omega_v \geq 0, $$

$$ {}_{0}^{\mathrm{CF}}\mathrm{D}_{t}^{\varpi} I(t)_v = \beta_v (I_a(t) + I_{al}(t)) S_v(t) \geq 0. $$

Using Corollary 2.1 it can be established that the solution will stay in $\mathbb{R}_{+}^{11}$. This ends the proof of the positivity of the model solutions.

Making use of the concept of Laplace transforms and the inverse characteristics of Laplace transforms on both side of the system (4), the connected equation is transformed into the following solutions:

(7)

$$ S_h(t) = S_h(0) + L^{-1}\left\{ \frac{1}{s^{\varpi}} L \begin{bmatrix} (1-b)\Omega_h + kR_a(t) + wR_l(t) + \psi R_{al}(t) \\ -\beta_h I_v(t) S_h(t) - \pi S_h(t) + dV_h(t) - \mu_h S_h(t) \end{bmatrix} (s) \right\}(t), $$

$$ I_a(t) = I_a(0) + L^{-1}\left\{ \tfrac{1}{s^{\varpi}} L[\beta_h I_v(t) S_h(t) - \pi I_a(t) + \beta_h g I_v(t) V_h(t) - (\alpha + \mu_h + \phi) I_a(t)](s) \right\}(t), $$

$$ I_l(t) = I_l(t) + L^{-1}\left\{ \tfrac{1}{s^{\varpi}} L[\pi S_h(t) - \beta_l I_v(t) I_l(t) + \pi q V_h(t) - (\sigma + \mu_h + m + \rho) I_l(t)](s) \right\}(t), $$

$$ I_{al}(t) = I_{al}(0) + L^{-1}\left\{ \tfrac{1}{s^{\varpi}} L[\beta_h I_v(t) I_l(t) + \pi I_a(t) - (\sigma + \mu_h + \eta + \theta) I_{al}(t)](s) \right\}(t), $$

$$\begin{aligned}
V_h(t) &= V_h(0) + L^{-1}\left\{\tfrac{1}{s^{\varpi}} L\left[\Omega_h b - (d+\mu_h) V_h(t) - \beta_h g I_v(t) V_h(t) - \pi q V_h(t)\right](s)\right\}(t), \\
R_a(t) &= R_a(0) + L^{-1}\left\{\tfrac{1}{s^{\varpi}} L\left[\alpha I_a(t) - (k+\mu h) R_a(t) + (1-\tau)\gamma\sigma I_{al}(t)\right](s)\right\}(t), \\
R_l(t) &= R_l(0) + L^{-1}\left\{\tfrac{1}{s^{\varpi}} L\left[\sigma I_l(t) - (w+\mu h) R_l(t) + (1-\tau)(1-\gamma)\sigma I_{al}(t)\right](s)\right\}(t), \\
R_{al}(t) &= R_{al}(0) + L^{-1}\left\{\tfrac{1}{s^{\varpi}} L\left[\tau\sigma I_{al}(t) - (\psi+\mu h) R_{al}(t)\right](s)\right\}(t), \\
C_p(t) &= C_p(0) + L^{-1}\left\{\tfrac{1}{s^{\varpi}} L\left[\rho I_l(t) + \theta I_{al}(t) - \mu_b C_p(t)\right](s)\right\}(t), \\
S_v(t) &= S_v(0) + L^{-1}\left\{\tfrac{1}{s^{\varpi}} L\left[\Omega_v - \beta_v (I_a(t) + I_{al}(t)) S_v(t) - \mu_v S_v(t)\right](s)\right\}(t), \\
I_v(t) &= I_v(0) + L^{-1}\left\{\tfrac{1}{s^{\varpi}} L\left[\beta_v (I_a(t) + I_{al}(t)) S_v(t) - \mu_v I_v(t)\right](s)\right\}(t).
\end{aligned}$$

Considering the connected iterated scheme in equation (7), we obtain the following:

$$\begin{aligned}
S_{h(u)}(t) &= S_h(0) + L^{-1}\Big\{\tfrac{1}{s^{\varpi}} L\Big[(1-b)\Omega_h + kR_{a(u-1)}(t) + \omega R_{l(u-1)}(t) \\
&\quad + \psi R_{al(u-1)}(t) - \beta_h I_{v(u-1)}(t) S_{h(u-1)}(t) - \pi S_{h(u-1)}(t) \\
&\quad + dV_{h(u-1)}(t) - \mu_h S_{h(u-1)}(t)(s)\Big]\Big\}(t), \\
I_{a(u)}(t) &= I_a(0) + L^{-1}\Big\{\tfrac{1}{s^{\varpi}} L\Big[\beta_h I_{v(u-1)}(t) S_{h(u-1)}(t) - \pi I_{a(u-1)}(t) \\
&\quad + \beta_h g I_{v(u-1)}(t) V_{h(u-1)}(t) - (\alpha + \mu_h + \phi) I_{a(u-1)}(t)(s)\Big]\Big\}(t), \\
I_{l(u)}(t) &= I_l(0) + L^{-1}\Big\{\tfrac{1}{s^{\varpi}} L\Big[\pi S_{h(u-1)}(t) - \beta_l I_{v(u-1)}(t) I_{l(u-1)}(t)
\end{aligned} \tag{8}$$

$$
\begin{aligned}
& +\pi q V_{h(u-1)}(t)-(\sigma+\mu_h+m+\rho)\, I_{l(u-1)}(t)(s)\Big]\Big\}(t), \\
I_{al(u)}(t) &= I_{al}(0)+L^{-1}\Big\{\tfrac{1}{s^{\varpi}}L\Big[\beta_h I_{v(u-1)}(t)\, I_{l(u-1)}(t)+\pi I_{a(u-1)}(t) \\
& -(\sigma+\mu_h+\eta+\theta)\, I_{al(u-1)}(t)(s)\Big]\Big\}(t), \\
V_{h(u)}(t) &= V_h(0)+L^{-1}\Big\{\tfrac{1}{s^{\varpi}}L\Big[\Omega_h b-(d+\mu_h)\, V_{h(u-1)}(t) \\
& -\beta_h g I_{v(u-1)}(t)\, V_{h(u-1)}(t)-\pi q V_{h(u-1)}(t)(s)\Big]\Big\}(t), \\
R_{a(u)}(t) &= R_a(0)+L^{-1}\Big\{\tfrac{1}{s^{\varpi}}L\Big[\alpha I_{a(u-1)}(t)-(k+\mu_h)\, R_{a(u-1)}(t) \\
& +(1-\tau)\,\gamma\sigma I_{al(u-1)}(t)(s)\Big]\Big\}(t), \\
R_{l(u)}(t) &= R_l(0)+L^{-1}\Big\{\tfrac{1}{s^{\varpi}}L\Big[\sigma I_{l(u-1)}(t)-(w+\mu_h)\, R_{l(u-1)}(t) \\
& +(1-\tau)(1-\gamma)\,\sigma I_{al(u-1)}(t)(s)\Big]\Big\}(t), \\
R_{al(u)}(t) &= R_{al}(0)+L^{-1}\Big\{\tfrac{1}{s^{\varpi}}L\Big[\tau\sigma I_{al(u-1)}(t)-(\psi+\mu_h)\, R_{al(u-1)}(t)(s)\Big]\Big\}(t), \\
C_{p(u)}(t) &= C_p(0)+L^{-1}\Big\{\tfrac{1}{s^{\varpi}}L\Big[\rho I_{l(u-1)}(t)+\theta I_{al(u-1)}(t)-\mu_b C_{p(u-1)}(t)(s)\Big]\Big\}(t), \\
S_{v(u)}(t) &= S_v(0)+L^{-1}\Big\{\tfrac{1}{s^{\varpi}}L\Big[\Omega_v-\beta_v\left(I_{a(u-1)}(t)+I_{al(u-1)}(t)\right) S_{v(u-1)}(t) \\
& -\mu_v S_{v(u-1)}(t)(s)\Big]\Big\}(t), \\
I_{v(u)}(t) &= I_v(0)+L^{-1}\Big\{\tfrac{1}{s^{\varpi}}L\Big[\beta_v\left(I_{a(u-1)}(t)+I_{al(u-1)}(t)\right) S_{v(u-1)}(t) \\
& -\mu_v I_{v(u-1)}(t)(s)\Big]\Big\}(t),
\end{aligned}
$$

where

$$S_{h(0)} = S_h(0), \quad I_{a(0)} = I_a(0), \qquad I_{l(0)} = I_l(0), \qquad V_{h(0)} = V_h(0), \quad R_{a(0)} = R_a(0),$$

(9)

$$R_{l(0)} = R_l(0), \quad R_{al(0)} = R_{al}(0), \quad C_{p(0)} = C_p(0), \quad S_{v(0)} = S_v(0), \quad I_{v(0)} = I_v(0).$$

In this regard, the approximate solution is attained as the limits at u approach infinity so that,

$$S_h(t) = \lim_{x\to\infty} S_{h(u)}(t), \qquad I_a(t) = \lim_{x\to\infty} I_{a(u)}(t), \qquad I_l(t) = \lim_{x\to\infty} I_{l(u)}(t),$$

$$\mathrm{V}_h(t) = \lim_{x\to\infty} V_{h(u)}(t), \qquad R_a(t) = \lim_{x\to\infty} R_{a(u)}(t), \quad R_l(t) = \lim_{x\to\infty} R_{l(u)}(t),$$

(10)

$$R_{al}(t) = \lim_{x\to\infty} R_{al(u)}(t), \quad C_p(t) = \lim_{x\to\infty} C_{p(u)}(t), \quad S_v(t) = \lim_{x\to\infty} S_{v(u)}(t),$$

$$I_v(t) = \lim_{x\to\infty} I_{v(u)}(t).$$

Stability Analysis of Coinfection Model with LC Operator. In order for one to investigate the stability of the system (8), eleven positive constraints a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11} are assumed such that for all $0 \le t \le T \le \infty$,

$$\|S_h(t)\| < a_1, \quad \|I_a(t)\| < a_2, \quad \|I_l(t)\| < a_3, \qquad \|I_{al}(t)\| < a_4, \quad \|V_h(t)\| < a_5,$$

(11) $$\|R_a(t)\| < a_6, \quad \|R_l(t)\| < a_7, \quad \|R_{al}(t)\| < a_8, \quad \left\|C_p(t)\right\| < a_9, \quad \|S_v(t)\| < a_{10},$$

$$\|I_v(t)\| < a_{11}$$

We consider a subset of the form $Q_2((c,d)(0,T))$ expressed as

$$P = \left\{ \Phi : (c,d)(0,T) \mapsto R, \frac{1}{T(0)} \int (t-\Phi)^{\varpi-1} v(\Phi) S(\Phi)\, d\Phi < \infty \right\}, \tag{12}$$

where Φ is given by:

$$
\Phi(X) = \begin{cases}
((1-b)\Omega_h + kR_a(t) + wR_l(t) + \psi R_{al}(t) - \beta_h I_v(t) \\
S_h(t) - \pi S_h(t) + dV_h(t) - \mu_h S_h(t)), \\
\beta_h I_v(t) S_h(t) - \pi I_a(t) + \beta_h g I_v(t) V_h(t) \\
-(\alpha + \mu_h + \phi) I_a(t), \\
\pi S_h(t) - \beta_l I_v(t) I_l(t) + \pi q V_h(t) - (\sigma + \mu_h + m + \rho) I_l(t), \\
\beta_h I_v(t) I_l(t) + \pi I_a(t) - (\sigma + \mu_h + \eta + \theta) I_{al}(t), \\
\Omega_h b - (d + \mu_h) V_h(t) - \beta_h g I_v(t) V_h(t) - \pi q V_h(t), \\
\alpha I_a(t) - (k + \mu h) R_a(t) + (1-\tau)\gamma\sigma I_{al}(t), \\
\sigma I_l(t) - (w + \mu h) R_l(t) + (1-\tau)(1-\gamma)\sigma I_{al}(t), \\
\tau\sigma I_{al}(t) - (\psi + \mu h) R_{al}(t), \\
\rho I_l(t) + \theta I_{al}(t) - \mu_b C_p(t), \\
\Omega_v - \beta_v (I_a(t) + I_{al}(t)) S_v(t) - \mu_v S_v(t), \\
\beta_v (I_a(t) + I_{al}(t)) S_v(t) - \mu_v I_v(t),
\end{cases} \tag{13}
$$

where $X = (S_h, I_a, I_l, I_{al}, V_h, R_a, R_l, R_{al}, C_p, S_v, I_v)$. Then,

$$
(14)\quad \begin{cases}
\Big\langle \Phi\Big(S_h, I_a, I_l, I_{al}, V_h, R_a, R_l, R_{al}, C_p, S_v, I_v\Big) - \Phi\Big(M_1, M_2, M_3, M_4, M_5, M_6, \\
M_7, M_8, M_9, M_{10}, M_{11}\Big), (\mathrm{S_h} - \mathrm{M_1}) ., \mathrm{I_a} - \mathrm{M_2}, \mathrm{I_l} - \mathrm{M_3}, \mathrm{I_{al}} - \mathrm{M_4}, \mathrm{V_h} - \mathrm{M_5}, \mathrm{R_a} \\
-M_6, R_l - M_7, R_{al} - M_8, C_p - M_9, S_v - M_{10}, I_{\ v} - M_{11}) \Big\rangle, \\
\Big\langle (1-b)\,\Omega_h + k\,(R_a(t) - M_6) + w\,(R_l(t) - M_7) + \psi\,(R_{al}(t) - M_8) - \\
\beta_h\,(I_v(t) - M_{11})\,(S_h(t) - M_1) - \pi\,(S_h(t) - M_1) + d\,(V_h(t) - M_5) \\
-\mu_h\,(S_h(t) - M_1) \Big\rangle \\
\\
\Big\langle \beta_h\,(I_v(t) - M_{11})\,(S_h(t) - M_1) - \pi\,(I_a(t) - M_2) + \beta_h g\,(I_v - M_{11})\,(V_h - M_5) \\
-(\alpha + \mu_h + \phi)\,(I_a(t) - M_1) \Big\rangle \\
\\
\Big\langle \pi\,(S_h(t) - M_1) - \beta_l\,(I_v(t) - M_{11})\,(I_l(t) - M_3) + \pi q\,(V_h(t) - M_5) \\
-(\sigma + \mu_h + m + \rho)\,(I_l(t) - M_3) \Big\rangle \\
\\
\Big\langle \beta_h\,(I_v(t) - M_{11})\,(I_l(t) - M_3) + \pi\,(I_a(t) - M_2) \\
(\sigma + \mu_h + \eta + \theta)\,(I_{al}(t) - M_4) \Big\rangle \\
\\
\Big\langle \Omega_h b - (d + \mu_h)\,(V_h(t) - M_5) - \beta_h g\,(I_v(t) - M_{11}) \\
(V_h(t) - M_5) - \pi q\,(V_h(t) - M_5) \Big\rangle \\
\\
\Big\langle \alpha\,(I_a(t) - M_2) - (k + \mu_h)\,(R_a(t) - M_6) + (1-\tau)\,\gamma\sigma\,(I_{al}(t) - M_4) \Big\rangle \\
\\
\Big\langle \sigma\,(I_l(t) - M_3) - (w + \mu_h)\,(R_l(t) - M_7) + (1-\tau)\,(1-\gamma)\,\sigma\,(I_{al}(t) - M_4) \Big\rangle \\
\\
\Big\langle \tau\sigma\,(I_{al}(t) - M_4) - (\psi + \mu_h)\,(R_{al}(t) - M_8) \Big\rangle \\
\\
\Big\langle \rho\,(I_l(t) - M_2) + \theta\,(I_{al}(t) - M_4) - \mu_v\,\big(C_p(t) - M_9\big) \Big\rangle \\
\\
\Big\langle \Omega_v - \beta_v\,((I_a(t) - M_2) + (I_{al}(t) - M_4))\,(S_v(t) - M_1) - \mu_v\,(S_v(t) - M_{10}) \Big\rangle \\
\\
\Big\langle \beta_v\,((I_a(t) - M_2) + (I_{al}(t) - M_4))\,(S_v(t) - M_{10}) - \mu_v\,(I_v(t) - M_{11}) \Big\rangle
\end{cases}
$$

Taking the absolute values and the norm on both sides, we have:

$$
(15) \qquad < \begin{cases}
\left\{(1-b)\,\Omega_n + \frac{k\|R_a(t)-M_6(t)\|}{\|S_h(t)-M_1(t)\|^2} + \frac{w\|R_l(t)-M_7(t)\|}{\|S_h(t)-M_1(t)\|^2} + \frac{\psi\|R_{al}(t)-M_8(t)\|}{\|S_h(t)-M_1(t)\|^2} \right. \\
-\frac{\beta_h((\|I_v(t)-M_{11}(t)\|)(\|S_h(t)-M_1(t)\|))}{\|S_h(t)-M_1(t)\|^2} - \pi\left(\frac{1}{\|S_h(t)-M_1(t)\|}\right) \\
\left. -d\left(\frac{\|V_h-M_5\|}{\|S_h(t)-M_1(t)\|^2}\right) - \mu_h\left(\frac{1}{\|S_h(t)-M_1(t)\|}\right)\right\} \|S_h(t)-M_1(t)\|^2, \\
\left\{\frac{\beta_h(\|I_v(t)-M_{11}(t)\|)(\|S_h(t)-M_1(t)\|)}{\|I_a(t)-M_2(t)\|^2} - \frac{\pi(\|I_a(t)-M_2(t)\|)}{\|I_a(t)-M_1(t)\|^2} + \frac{\beta_h g((\|I_v(t)-M_{11}\|)(\|V_h(t)-M_5\|))}{\|I_a(t)-M_1(t)\|^2}\right. \\
\left. -\frac{(\alpha+\mu_h+\phi)}{\|I_a(t)-M_2(t)\|}\right\} \|I_a(t)-M_2(t)\|^2, \\
\left\{\frac{\pi(\|S_h(t)-M_1(t)\|)}{\|I_l(t)-M_3(t)\|^2} - \frac{\beta_l(\|(I_v(t)-M_{11}(t))\|\|(I_l(t)-M_3(t))\|)}{\|I_l(t)-M_3(t)\|^2} + \frac{\pi q(\|V_h(t)-M_5\|)}{\|I_l(t)-M_3(t)\|^2}\right. \\
\left. -\frac{(\sigma+\mu_h+m+\rho)}{\|I_l(t)-M_3(t)\|}\right\} \|I_l(t)-M_3(t)\|^2, \\
\left\{\frac{\beta_h((\|I_v(t)-M_{11}(t)\|)(\|I_l(t)-M_3(t)\|))}{\|I_{al}(t)-M_4(t)\|^2} + \frac{\pi(\|I_a(t)-M_2(t)\|)}{\|I_{al}(t)-M_4(t)\|^2},\right. \\
-\frac{(\sigma+\mu_h+\eta+\theta)(\|I_{al}(t)-M_4(t)\|)}{\|I_{al}(t)-M_4(t)\|} \|I_{al}(t)-M_4(t)\|^2, \\
\left\{\Omega_h b + \frac{(d+\mu_h)(V_h(t)-M_5(t))}{\|V_h(t)-M_5(t)\|^2} + \frac{\beta_h g(\|I_v(t)-M_{11}(t)\|)(\|V_h(t)-M_5(t)\|)}{\|V_h(t)-M_5(t)\|^2}\right. \\
-\frac{\pi q(\|V_h(t)-M_5(t)\|)}{\|V_h(t)-M_5(t)\|^2} \|V_h(t)-M_5(t)\|^2, \\
\left\{\frac{\alpha(\|I_a(t)-M_2(t)\|)}{\|R_a(t)-M_6(t)\|^2} - \frac{(k+\mu h)(\|R_a(t)-M_6(t)\|)}{\|R_a(t)-M_6(t)\|} + \frac{(1-\tau)\gamma\sigma(\|I_{al}(t)-M_4(t)\|)}{\|R_a(t)-M_6(t)\|^2}\right\} \\
\|R_a(t)-M_6(t)\|^2, \\
\left\{\frac{\sigma(\|I_l(t)-M_3(t)\|)}{\|R_l(t)-M_7(t)\|^2} - \frac{(w+\mu h)(\|R_l(t)-M_7(t)\|)}{\|R_l(t)-M_7(t)\|^2} + \frac{(1-\tau)(1-\gamma)\sigma(\|I_{al}(t)-M_4(t)\|)}{\|R_l(t)-M_7(t)\|^2}\right\} \\
\|R_l(t)-M_7(t)\|^2, \\
\left\{\frac{\tau\sigma(\|I_{al}(t)-M_4(t)\|)}{\|R_{al}(t)-M_8(t)\|^2} - \frac{(\psi+\mu h)(\|R_{al}(t)-M_8(t)\|)}{\|R_{al}(t)-M_8(t)\|^2}\right\} \|R_{al}(t)-M_8(t)\|^2, \\
\left\{\frac{\rho(\|I_l(t)-M_3(t)\|)}{\left\|C_p(t)-M_9(t)\right\|^2} + \frac{\theta(\|I_{al}(t)-M_4(t)\|)}{\left\|C_p(t)-M_9(t)\right\|^2} - \frac{\mu_b\left(\left\|C_p(t)-M_9(t)\right\|\right)}{\left\|C_p(t)-M_9(t)\right\|^2}\right\} \left\|C_p(t)-M_9(t)\right\|^2, \\
\left\{\Omega_v - \frac{\beta_v(\|(I_a(t)-M_2(t))\|+(\|I_{al}(t)-M_4(t)\|))(\|S_v(t)-M_{10}(t)\|)}{\|S_v(t)-M_{10}(t)\|^2} - \frac{\mu_v(\|S_v(t)-M_{10}(t)\|)}{\|S_v(t)-M_{10}(t)\|^2}\right\} \\
\|S_v(t)-M_{10}(t)\|^2, \\
\left\{\frac{\beta_v((\|I_a(t)-M_2(t)\|)+(\|I_{al}(t)-M_4(t)\|))(\|S_v(t)-M_{10}(t)\|)}{\|I_v(t)-M_{11}(t)\|^2} - \frac{\mu_v(I_v(t)-M_{11}(t))}{\|I_v(t)-M_{11}(t)\|^2}\right\} \\
\|I_v(t)-M_{11}(t)\|^2.
\end{cases}
$$

Where

$$\Big\langle \Phi\Big(S_h, I_a, I_l, I_{al}, V_h, R_a, R_l, R_{al}, C_p, S_v, I_v\Big) - \Phi\Big(M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}, M_{11}\Big), (\mathrm{S_h} - \mathrm{M_1}).,\mathrm{I_a} - \mathrm{M_2}, \mathrm{I_l} - \mathrm{M_3}, \mathrm{I_{al}} - \mathrm{M_4}, \mathrm{V_h} - \mathrm{M_5}, \mathrm{R_a} - M_6, R_l - M_7, R_{al} - M_8, C_p - M_9, S_v - M_{10}, I_v - M_{11})\Big\rangle, \tag{16}$$

$$\begin{cases} B_1 \left\|S_h(t) - M_1(t)\right\|^2, \\ B_2 \left\|I_a(t) - M_2(t)\right\|^2, \\ B_3 \left\|I_l(t) - M_3(t)\right\|^2, \\ B_4 \left\|I_{al}(t) - M_4(t)\right\|^2, \\ B_5 \left\|V_h(t) - M_5(t)\right\|^2, \\ B_6 \left\|R_a(t) - M_6(t)\right\|^2, \\ B_7 \left\|R_l(t) - M_7(t)\right\|^2, \\ B_8 \left\|R_{al}(t) - M_8(t)\right\|^2, \\ B_9 \left\|C_p(t) - M_9(t)\right\|^2, \\ B_{10} \left\|S_v(t) - M_{10}(t)\right\|^2, \\ B_{11} \left\|I_v(t) - M_{11}(t)\right\|^2, \end{cases} \tag{17}$$

with

$$
\begin{aligned}
B_1 &= \Bigg\{(1-b)\,\Omega_h + \frac{k\|R_a(t)-M_6(t)\|}{\|S_h(t)-M_1(t)\|^2} + \frac{w\|R_l(t)-M_7(t)\|}{\|S_h(t)-M_1(t)\|^2} + \frac{\psi\|R_{al}(t)-M_8(t)\|}{\|S_h(t)-M_1(t)\|^2} \\
&\quad -\frac{\beta_h((\|I_v(t)-M_{11}(t)\|)(\|S_h(t)-M_1(t)\|))}{\|S_h(t)-M_1(t)\|^2} + \frac{d|V_h(t)-M_5(t)|}{\|S_h(t)-M_1(t)\|^2} - \pi\left(\frac{1}{\|S_h(t)-M_1(t)\|}\right) \\
&\quad -\mu_h\left(\frac{1}{\|S_h(t)-M_1(t)\|}\right)\Bigg\}\|S_h(t)-M_1(t)\|^2, \\
B_2 &= \Bigg\{\frac{\beta_h(\|I_v(t)-M_{11}(t)\|)(\|S_h(t)-M_1(t)\|)}{\|I_a(t)-M_2(t)\|^2} - \frac{\pi(\|I_a(t)-M_2(t)\|)}{\|I_a(t)-M_1(t)\|^2} \\
&\quad +\frac{\beta_h g(\|I_v(t)-M_{11}(t)\|)(\|V_h(t)-M_5(t)\|)}{\|I_a(t)-M_1(t)\|^2} - \frac{(\alpha+\mu_h+\phi)}{\|I_a(t)-M_2(t)\|}\Bigg\}\|I_a(t)-M_2(t)\|^2, \\
B_3 &= \Bigg\{\frac{\pi(\|S_h(t)-M_1(t)\|)}{\|I_l(t)-M_3(t)\|^2} - \frac{\beta_l(\|(I_v(t)-M_{10}(t))\|\|(I_l(t)-M_3(t))\|)}{\|I_l(t)-M_3(t)\|^2} + \frac{\pi q\|V_h(t)-M_5(t)\|}{\|I_l(t)-M_3(t)\|^2} \\
&\quad -\frac{(\sigma+\mu_h+m+\rho)}{\|I_l(t)-M_3(t)\|}\Bigg\}\|I_l(t)-M_3(t)\|^2, \\
B_4 &= \Bigg\{\frac{\beta_h((\|I_v(t)-M_{11}(t)\|)(\|I_l(t)-M_3(t)\|))}{\|I_{al}(t)-M_4(t)\|^2} + \frac{\pi(\|I_a(t)-M_2(t)\|)}{\|I_{al}(t)-M_4(t)\|^2} \\
&\quad -\frac{(\sigma+\mu_h+\eta+\theta)(\|I_{al}(t)-M_4(t)\|)}{\|I_{al}(t)-M_4(t)\|}\Bigg\}\|I_{al}(t)-M_4(t)\|^2, \\
B_5 &= \Bigg\{\Omega_h b - \frac{(d+\mu_h)\|V_h(t)-M_5(t)\|}{\|V_h(t)-M_5(t)\|^2} - \frac{\beta_h g I_v(t)\|V_h(t)-M_5(t)\|}{\|V_h(t)-M_5(t)\|^2} - \frac{\pi q\|V_h(t)-M_5(t)\|}{\|V_h(t)-M_5(t)\|^2}\Bigg\} \\
&\quad \|V_h(t)-M_5(t)\|^2, \\
B_6 &= \Bigg\{\frac{\alpha(\|I_a(t)-M_2(t)\|)}{\|R_a(t)-M_6(t)\|^2} - \frac{(k+\mu h)(\|R_a(t)-M_6(t)\|)}{\|R_a(t)-M_6(t)\|} + \frac{(1-\tau)\gamma\sigma(\|I_{al}(t)-M_4(t)\|)}{\|R_a(t)-M_6(t)\|^2}\Bigg\} \\
&\quad \|R_a(t)-M_6(t)\|^2,
\end{aligned} \tag{18}
$$

$$B_7 = \left\{ \frac{\sigma(\|I_l(t)-M_3(t)\|)}{\|R_l(t)-M_7(t)\|^2} - \frac{(w+\mu h)(\|R_l(t)-M_7(t)\|)}{\|R_l(t)-M_7(t)\|^2} + \frac{(1-\tau)(1-\gamma)\sigma(\|I_{al}(t)-M_4(t)\|)}{\|R_l(t)-M_7(t)\|^2} \right\} \|R_l(t)-M_7(t)\|^2,$$

$$B_8 = \left\{ \frac{\tau\sigma(\|I_{al}(t)-M_4(t)\|)}{\|R_{al}(t)-M_8(t)\|^2} - \frac{(\psi+\mu h)(\|R_{al}(t)-M_8(t)\|)}{\|R_{al}(t)-M_8(t)\|^2} \right\} \|R_{al}(t)-M_8(t)\|^2,$$

$$B_9 = \left\{ \frac{\rho(\|I_l(t)-M_3(t)\|)}{\left\|C_p(t)-M_9(t)\right\|^2} + \frac{\theta(\|I_{al}(t)-M_4(t)\|)}{\left\|C_p(t)-M_9(t)\right\|^2} - \frac{\mu_b\left(\left\|C_p(t)-M_9(t)\right\|\right)}{\left\|C_p(t)-M_9(t)\right\|^2} \right\} \left\|C_p(t)-M_9(t)\right\|^2,$$

$$B_{10} = \left\{ \Omega_v - \frac{\beta_v(\|(I_a(t)-M_2(t))\|+(\|I_{al}(t)-M_4(t)\|))(\|S_v(t)-M_{10}(t)\|)}{\|S_v(t)-M_{10}(t)\|^2} - \frac{\mu_v(\|S_v(t)-M_{10}(t)\|)}{\|S_v(t)-M_{10}(t)\|^2} \right\} \|S_v(t)-M_{10}(t)\|^2,$$

$$B_{11} = \left\{ \frac{\beta_v((\|I_a(t)-M_2(t)\|)+(\|I_{al}(t)-M_4(t)\|))(\|S_v(t)-M_{10}(t)\|)}{\|I_v(t)-M_{11}(t)\|^2} - \frac{\mu_v(I_v(t)-M_{11}(t))}{\|I_v(t)-M_{11}(t)\|^2} \right\} \|I_v(t)-M_{11}(t)\|^2.$$

If we consider the Ortain null-rector $(S_h, I_a, I_l, I_{al}, V_h, R_a, R_l, R_l, R_{al}, C_p, S_v, I_v)$ and take some iterations in equation (18), we obtain the following:

$$\left\langle \Phi\Big(S_h, I_a, I_l, I_{al}, V_h, R_a, R_l, R_{al}, C_p, S_v, I_v\Big) - \Phi\Big(M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}, M_{11}\Big), \begin{pmatrix} S_h - M_1, I_a - M_2, I_l - M_3, I_{al} - M_4, V_h - M_5, R_a - M_6, R_l - M_7, R_{al} - M_8, \\ C_p - M_9, S_v - M_{10}, I_v - M_{11} \end{pmatrix} \right\rangle, \tag{19}$$

$$
\begin{cases}
B_1 \left\| S_h(t) - M_1(t) \right\| \left\| S_h(t) \right\|, \\
B_2 \left\| I_a(t) - M_2(t) \right\| \left\| I_a(t) \right\|, \\
B_3 \left\| I_l(t) - M_3(t) \right\| \left\| I_l(t) \right\|, \\
B_4 \left\| I_{al}(t) - M_4(t) \right\| \left\| I_{al}(t) \right\|, \\
B_5 \left\| V_h(t) - M_5(t) \right\| \left\| V_h(t) \right\|, \\
B_6 \left\| R_a(t) - M_6(t) \right\| \left\| R_a(t) \right\|, \\
B_7 \left\| R_l(t) - M_7(t) \right\| \left\| R_l(t) \right\|, \\
B_8 \left\| R_{al}(t) - M_8(t) \right\| \left\| R_{al}(t) \right\|, \\
B_9 \left\| C_p(t) - M_9(t) \right\| \left\| C_p(t) \right\|, \\
B_{10} \left\| S_v(t) - M_{10}(t) \right\| \left\| S_v(t) \right\|, \\
B_{11} \left\| I_v(t) - M_{11}(t) \right\| \left\| I_v(t) \right\|.
\end{cases} \tag{20}
$$

Following the results obtained in Eq (16) and Eq (19) one can conclude that the iterative approach is stable.

3.2.1. *Predictor-Corrector Adams-Bashforth-Mouton Method.*

$$\Theta(t)=\sum_{k=0}^{n-1}\Theta^{k(t)}\frac{t^k}{k!}+\frac{1}{T(\Theta)}\int_0^t (t-v)^{\Theta-1} f(v,\Theta(v))\,dv,\ t<T. \tag{21}$$

The equation (21) is considered to have a singular solution expressed in the interval $t\in[0,1]$ the equation (21) holds the Volterra integral equation given by equation (22):

$$\Theta(t)=\sum_{k=0}^{n-1}\Theta^{k(t)}\frac{t^k}{k!}+\frac{1}{T(\Theta)}\int_0^t (t-v)^{\Theta-1} f(v,\Theta(v))\,dv,\ t<T, \tag{22}$$

where $\Theta>0$ and ${}_{o}^{C}\mathrm{D}_t^{\varpi}$ represents the Liouvile-Caputo Operator.

The scheme for this work is Predictor-Conector Adom-Bashforth-Moulton iterated [32]. The associated general iterative solution is given by

$$\begin{aligned} g_{k+1}^{Q} &= \sum_{j=0}^{n-1}\tfrac{t_{k+1}^{j}}{j!}g_0^{(j)}+\tfrac{1}{T(\Theta)}\sum_{j=0}^{k}P_j,k+1g\left(t_j,\Theta_j\right),\\ \Theta_{k+1} &= \sum_{j=0}^{n-1}\tfrac{t_{k+1}^{j}}{j!}g_0^{(j)}+\tfrac{1}{T(0)}\left(\sum_{j=0}^{k}r_j,k+1\left(t_j,\Theta_j\right)+r_{k+1},k+1g\left(t_{k+1},g_{k+1}^{Q}\right)\right), \end{aligned} \tag{23}$$

$$\begin{aligned} r_j,k+1 &= \tfrac{h^{\Theta}}{\Theta(\Theta+1)}\begin{cases} k^{\Theta+1}-(k-\Theta)^{\Theta} & j=0,\\ \left((k-j+2)^{\Theta+1}(k-j)^{\Theta+1}-2(k-j+1)^{\Theta+1}\right) & 1\le j\le k,\\ 1 & j=k+1, \end{cases}\\ p_{j,k+1} &= \tfrac{h^{\Theta}}{\Theta}\left((k+1-j)^{\Theta}-2(k-j+1)^{\Theta}\right). \end{aligned}$$

Making use of this iterative scheme the context of Adams-Bashford Technique as follows:

$$S_h(t) = \sum_{k=0}^{n-1}S_h(0)^k\tfrac{t^k}{k!}+\tfrac{1}{T(\Theta)}\int_0^t (t-v)^{\Theta-1}\left[\begin{array}{l}(1-b)\Omega_h+kR_a(t)+wR_l(t)\\ +\psi R_{al}(t)-\beta_h I_v(t)S_h(t)\\ +dV_h(t)-\pi S_h(t)-\mu_h S_h(t)\end{array}\right]dv$$

(24)

$$I_l(t) = \sum_{k=0}^{n-1} I_a(0)^k \frac{t^k}{k!} + \frac{1}{T\Theta}\int_0^t (t-v)^{\Theta-1} \begin{bmatrix} \beta_h I_v(t) S_h(t) - \pi I_a(t) + \beta_h g I_v(t) V_h(t) \\ -(\alpha+\mu_h+\phi) I_a(t) \end{bmatrix} dv$$

$$I_a(t) = \sum_{k=0}^{n-1} I_a(0)^k \frac{t^k}{k!} + \frac{1}{T\Theta}\int_0^t (t-v)^{\Theta-1} \begin{bmatrix} \pi S_h(t) - \beta_l I_v(t) I_l(t) + \pi q V_h(t) \\ -(\sigma+\mu_h+m+\rho) I_l(t) \end{bmatrix} dv$$

$$I_{al}(t) = \sum_{k=0}^{n-1} I_{al}(0)^k \frac{t^k}{k!} + \frac{1}{T\Theta}\int_0^t (t-v)^{\Theta-1} \begin{bmatrix} \beta_h I_v(t) I_l(t) + \pi I_a(t) \\ -(\sigma+\mu_h+\eta+\theta) I_{al}(t) \end{bmatrix} dv$$

$$V_h(t) = \sum_{k=0}^{n-1} I_{al}(0)^k \frac{t^k}{k!} + \frac{1}{T\Theta}\int_0^t (t-v)^{\Theta-1} \begin{bmatrix} \Omega_h b - (d+\mu_h) V_h(t) \\ -\beta_h g I_v(t) V_h(t) - \pi q V_h(t) \end{bmatrix} dv$$

$$R_a(t) = \sum_{k=0}^{n-1} R_a(0)^k \frac{t^k}{k!} + \frac{1}{T\Theta}\int_0^t (t-v)^{\Theta-1} \begin{bmatrix} \alpha I_a(t) - (k+\mu_h) R_a(t) \\ +(1-\tau)\gamma\sigma I_{al}(t) \end{bmatrix} dv$$

$$R_l(t) = \sum_{k=0}^{n-1} R_l(0)^k \frac{t^k}{k!} + \frac{1}{T\Theta}\int_0^t (t-v)^{\Theta-1} \begin{bmatrix} \sigma I_l(t) - (w+\mu_h) R_l(t) \\ +(1-\tau)(1-\gamma)\sigma I_{al}(t) \end{bmatrix} dv$$

$$R_{al}(t) = \sum_{k=0}^{n-1} R_{al}(0)^k \frac{t^k}{k!} + \frac{1}{T\Theta}\int_0^t (t-v)^{\Theta-1} \left[\tau\sigma I_{al} - (\psi+\mu_h) R_{al}(t)\right] dv$$

$$C_p(t) = \sum_{k=0}^{n-1} C_p(0)^k \frac{t^k}{k!} + \frac{1}{T\Theta}\int_0^t (t-v)^{\Theta-1} \left[\rho I_l(t) + \theta I_{al}(t) - \mu_b C_p(t)\right] dv$$

$$S_v(t) = \sum_{k=0}^{n-1} S_v(0)^k \frac{t^k}{k!} + \frac{1}{T\Theta}\int_0^t (t-v)^{\Theta-1} \left[\Omega_v - \beta_v (I_a(t) + I_{al}(t)) S_v(t) - \mu_v S_v(t)\right] dv$$

$$I_v(t) = \sum_{k=0}^{n-1} I_v(0)^k \frac{t^k}{k!} + \frac{1}{T\Theta}\int_0^t (t-v)^{\Theta-1} \left[\beta_v (I_a(t) + I_{al}(t)) S_v(t) - \mu_v I_v(t)\right] dv$$

4. Existence and Uniqueness of Solutions

Let Γ be a convex, bounded, and closed subset of a Banach space Υ and $\chi:\Gamma\rightarrow\Gamma$ a condensing map, where Υ has a fixed point in Γ . Consider the initial value problem on the cylinder $\Delta=\{(t,n)\in\mathbb{R}\times\Upsilon:t\in[0,T],y\in\Gamma(0,\kappa)\}$ fixed for some $T>0$, $\kappa>0$, and suppose $\exists\Delta\in(0,\zeta)$, and S_h, I_a, I_l, I_{al}, V_h, R_a, R_l, R_{al}, C_p, S_v, $I_v\in L_{1/\Delta}(0,T,\mathbb{R}^+)$. It follows that $|(\mathbb{R},n)-\mathbb{R}(t,x)|\leq L_1(t)\|n-x\|$, $\forall\ (t,n)$, $(t,x)\in\mathbb{R}$. so that,

$$\begin{aligned}
S_h &= S_{0h}+S_{1h},\\
I_a &= I_{0a}+I_{1a},\\
I_l &= I_{0l}+I_{1l},\\
I_{al} &= I_{0al}+I_{1al},\\
V_h &= V_{0h}+V_{1h},\\
R_a &= R_{0a}+R_{1a},\\
R_l &= R_{0l}+R_{1l},\\
R_{al} &= R_{0al}+R_{1al},\\
C_p &= C_{0p}+C_{1p},\\
S_v &= S_{0v}+S_{1v},\\
I_v &= I_{0v}+I_{1v},
\end{aligned}$$

and the suppositions below hold:

(1) S_{0h}, I_{0a}, I_{0l}, I_{0al}, V_{0h}, R_{0a}, R_{0l}, R_{0al}, C_{0p}, S_{0v}, and I_{0v} are bounded and lipszchitz.

(2) S_{1h}, I_{1l}, I_{1al}, V_{1h}, R_{1a}, R_{1l}, C_{1p}, S_{1v}, and I_{1v} are compact and bounded.

(3) $|(\mathbb{R},n)-\mathbb{R}(t,x)|\leq L_1(t)\|n-x\|$, $\forall\ (t,n)$, $(t,x)\in\mathbb{R}$.

Employing the Riemann-Liouville integral to both sides of equation (3) we obtain the system of integral equations below:

$$\begin{aligned}
S_h(t) &= S_h(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(t-\eta)^{\varpi-1}S_{0h}(\eta,S_h(\eta))\,d\eta+\int_0^t(t-\eta)^{\varpi-1}S_{1h}(\eta)\,d\eta,\\
I_a(t) &= I_a(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(t-\eta)^{\varpi-1}I_{0a}(\eta,I_a(\eta))\,d\eta+\int_0^t(t-\eta)^{\varpi-1}I_{1a}(\eta)\,d\eta,
\end{aligned} \tag{25}$$

$$\begin{aligned}
I_l(t) &= I_l(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1} I_{0l}(\eta,I_l(\eta))\,d\eta+\int_0^t (t-\eta)^{\varpi-1} I_{1l}(\eta)\,d\eta,\\
I_{al}(t) &= I_{al}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1} I_{0al}(\eta,I_{al}(\eta))\,d\eta+\int_0^t (t-\eta)^{\varpi-1} I_{1al}(\eta)\,d\eta,\\
V_h(t) &= V_h(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1} V_{0h}(\eta,V_h(\eta))\,d\eta+\int_0^t (t-\eta)^{\varpi-1} V_{1h}(\eta)\,d\eta,\\
R_a(t) &= R_a(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1} R_{0a}(\eta,R_a(\eta))\,d\eta+\int_0^t (t-\eta)^{\varpi-1} R_{1a}(\eta)\,d\eta,\\
R_l(t) &= R_l(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1} R_{0l}(\eta,R_l(\eta))\,d\eta+\int_0^t (t-\eta)^{\varpi-1} R_{1l}(\eta)\,d\eta,\\
R_{al}(t) &= R_{al}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1} R_{0al}(\eta,R_{al}(\eta))\,d\eta+\int_0^t (t-\eta)^{\varpi-1} R_{1al}(\eta)\,d\eta,\\
C_p(t) &= C_p(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1} C_{0p}(\eta,C_p(\eta))\,d\eta+\int_0^t (t-\eta)^{\varpi-1} C_{1p}(\eta)\,d\eta,\\
S_v(t) &= S_v(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1} S_{0v}(\eta,S_v(\eta))\,d\eta+\int_0^t (t-\eta)^{\varpi-1} S_{1v}(\eta)\,d\eta,\\
I_v(t) &= I_v(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1} I_{0v}(\eta,I_v(\eta))\,d\eta+\int_0^t (t-\eta)^{\varpi-1} I_{1v}(\eta)\,d\eta.
\end{aligned}$$

Theorem 2. *Based on the suppositions 1 and 2 above, the initial value problem has at least one solution in the interval* $[0,T]$ *dependent on the condition*

$$\xi=\frac{\nu\,\|L\|_{1/\nabla}\,T^{\mathrm{K}}}{\Lambda(\varpi)}<1, \tag{26}$$

where $\mathrm{K}=\zeta-\nabla$ *and* $\Gamma=\left(\frac{1-\nabla}{\zeta-\nabla}\right)^{1-\nabla}$.

Proof. Let υ be such that $\psi(0)+\frac{1}{\Lambda(\varpi)}\nu\left(\|\mathrm{H}_1\|_{1/\nabla}+\|\mathrm{H}_2\|_{1/\nabla}\right)T^{\mathrm{K}}\le\upsilon$ and suppose that the closed ball in a Banach space $([0,T],\Upsilon)$, with $\sup\|\cdot\|$.

Considering a Banach space $([0,T],\Upsilon), n\mapsto S_{0h}+S_{1h}n$ with

$$
\begin{aligned}
S_{0h}(t) &= S_{1h}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}S_{0h}(\eta,n(\eta))\,d\eta,\\
S_{1h}(t) &= S_{1h}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}S_{1h}(\eta,n(\eta))\,d\eta,\\
I_{0a}(t) &= I_{1a}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}I_{0a}(\eta,n(\eta))\,d\eta,\\
I_{1a}(t) &= I_{1a}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}I_{1a}(\eta,n(\eta))\,d\eta,\\
I_{0l}(t) &= I_{1l}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}I_{0l}(\eta,n(\eta))\,d\eta,\\
I_{1l}(t) &= I_{1l}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}I_{1l}(\eta,n(\eta))\,d\eta,\\
I_{0al}(t) &= I_{1al}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}I_{0al}(\eta,n(\eta))\,d\eta,\\
I_{1al}(t) &= I_{1al}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}I_{1al}(\eta,n(\eta))\,d\eta,\\
V_{0h}(t) &= V_{1h}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}V_{0h}(\eta,n(\eta))\,d\eta,\\
V_{1h}(t) &= V_{1h}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}V_{1h}(\eta,n(\eta))\,d\eta,\\
R_{0a}(t) &= R_{1a}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}R_{0a}(\eta,n(\eta))\,d\eta,\\
R_{1a}(t) &= R_{1a}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}R_{1a}(\eta,n(\eta))\,d\eta,\\
R_{0l}(t) &= R_{1l}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}R_{0l}(\eta,n(\eta))\,d\eta,\\
R_{1l}(t) &= R_{1l}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}R_{1l}(\eta,n(\eta))\,d\eta,
\end{aligned}
\tag{27}
$$

$$\begin{aligned}
R_{0al}(t) &= R_{1al}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}R_{0al}(\eta,n(\eta))\,d\eta,\\
R_{1al}(t) &= R_{1al}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}R_{1al}(\eta,n(\eta))\,d\eta,\\
C_{0p}(t) &= C_{1p}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}C_{0p}(\eta,n(\eta))\,d\eta,\\
C_{1p}(t) &= C_{1p}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}C_{1p}(\eta,n(\eta))\,d\eta,\\
S_{0v}(t) &= S_{1v}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}S_{0v}(\eta,n(\eta))\,d\eta,\\
S_{1v}(t) &= S_{1v}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}S_{1v}(\eta,n(\eta))\,d\eta,\\
I_{0v}(t) &= I_{1v}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}I_{0v}(\eta,n(\eta))\,d\eta,\\
I_{1v}(t) &= I_{1v}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^{tr}(1-\eta)^{\varpi-1}I_{1v}(\eta,n(\eta))\,d\eta,
\end{aligned}$$

We need to prove that $S_h(\Gamma_\upsilon)\subset\Gamma_\upsilon$, for $n\in\Gamma_\upsilon$.

$$\begin{aligned}
\|S_h(t)\| &\leq |S_h(0)|+\tfrac{1}{\Lambda(\varpi)}\int_0^{t}(t-\eta)^{\varpi-1}S_{0h}(\eta,n(\eta))\,d\eta\\
&\leq |S_h(0)|+\tfrac{1}{\Lambda(\varpi)}\int_0^{t}(t-\eta)^{\varpi-1}S_{oh}(\eta,n(\eta))\,d\eta\\
&\quad+\tfrac{1}{\Lambda(\varpi)}\int_0^{t}(t-\eta)^{\varpi-1}S_{1h}(\eta,n(\eta))\,d\eta \qquad (28)\\
&\leq |S_h(0)|+\tfrac{1}{\Lambda(\varpi)}\left(\int_0^{t}(t-\eta)^{\frac{\varpi-1}{1-\nabla}}d\eta\right)^{1-\nabla}\left(\int_0^{t}\mathrm{M}_1^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}\\
&\quad+\tfrac{1}{\Lambda(\varpi)}\left(\int_0^{t}(t-\eta)^{\frac{\varpi-1}{1-\nabla}}d\eta\right)^{1-\nabla}\left(\int_0^{t}\mathrm{M}_2^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}
\end{aligned}$$

$$\leq \quad |S_h(0)| + \frac{\nu_1\left(\|M_1\|_{1/\nabla}+\|M_2\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_1} \leq \upsilon_1.$$

Using the same strategy, we get

$$\begin{aligned}
\|I_a(t)\| &\leq |I_a(0)| + \frac{\nu_2\left(\|M_3\|_{1/\nabla}+\|M_4\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_2} \leq \upsilon_2, \\
\|I_l(t)\| &\leq |I_l(0)| + \frac{\nu_3\left(\|M_5\|_{1/\nabla}+\|M_6\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_3} \leq \upsilon_3, \\
\|I_{al}(t)\| &\leq |I_{al}(0)| + \frac{\nu_4\left(\|M_7\|_{1/\nabla}+\|M_8\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_4} \leq \upsilon_4, \\
\|V_h(t)\| &\leq |V_h(0)| + \frac{\nu_5\left(\|M_9\|_{1/\nabla}+\|M_{10}\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_5} \leq \upsilon_5, \\
\|R_a(t)\| &\leq |R_a(0)| + \frac{\nu_6\left(\|M_{11}\|_{1/\nabla}+\|M_{12}\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_6} \leq \upsilon_6, \\
\|R_l(t)\| &\leq |R_l(0)| + \frac{\nu_7\left(\|M_{13}\|_{1/\nabla}+\|M_{14}\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_7} \leq \upsilon_7, \\
\|R_{al}(t)\| &\leq |R_{al}(0)| + \frac{\nu_8\left(\|M_{15}\|_{1/\nabla}+\|M_{16}\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_8} \leq \upsilon_8, \\
\left\|C_p(t)\right\| &\leq \left|C_p(0)\right| + \frac{\nu_9\left(\|M_{17}\|_{1/\nabla}+\|M_{18}\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_9} \leq \upsilon_9, \\
\|S_v(t)\| &\leq |S_v(0)| + \frac{\nu_{10}\left(\|M_{19}\|_{1/\nabla}+\|M_{20}\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_{10}} \leq \upsilon_{10}, \\
\|I_v(t)\| &\leq |I_v(0)| + \frac{\nu_{11}\left(\|M_{21}\|_{1/\nabla}+\|M_{22}\|_{1/\nabla}\right)}{\Lambda(\varpi)}T^{\mathrm{K}_{11}} \leq \upsilon_{11},
\end{aligned} \tag{29}$$

and therefore, $\Big(S_h(\Gamma_\upsilon), I_a(\Gamma_\upsilon), I_l(\Gamma_\upsilon), I_{al}(\Gamma_\upsilon), V_h(\Gamma_\upsilon), R_a(\Gamma_\upsilon), R_l(\Gamma_\upsilon), R_{al}(\Gamma_\upsilon),$
$C_p(\Gamma_\upsilon), S_v(\Gamma_\upsilon), I_v(\Gamma_\upsilon)\Big) \subset \Gamma_\upsilon$

We now prove that S_{0h}, I_{0a}, I_{0l}, I_{0al}, V_{0h}, R_{0a}, R_{0l}, R_{0al}, C_{0p}, S_{0v}, and I_{0v} are contractions. For $n, x \in \Gamma_\upsilon$, we have

$$\begin{aligned}
\|S_{0h}n(t)-S_{1h}x(t)\| &\leq \tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1}L(\eta)\,|n(\eta)-x(\eta)|\,d\eta \\
&\leq \tfrac{1}{\Lambda(\varpi)}\left(\int_0^t (t-\eta)^{\frac{\varpi-1}{1-\nabla}}\,d\eta\right)^{1-\nabla}\left(L^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}\|n-x\| \\
&\leq \Psi_i\,\|n-x\|, \\
\|I_{0a}n(t)-I_{1a}x(t)\| &\leq \tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1}L(\eta)\,|n(\eta)-x(\eta)|\,d\eta \\
&\leq \tfrac{1}{\Lambda(\varpi)}\left(\int_0^t (t-\eta)^{\frac{\varpi-1}{1-\nabla}}\,d\eta\right)^{1-\nabla}\left(L^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}\|n-x\| \\
&\leq \Psi_i\,\|n-x\|, \\
\|I_{0l}n(t)-I_{1l}x(t)\| &\leq \tfrac{1}{\Lambda(\varpi)}\textstyle\int_0^t (t-\eta)^{\varpi-1}L(\eta)\,|n(\eta)-x(\eta)|\,d\eta \qquad (30)\\
&\leq \tfrac{1}{\Lambda(\varpi)}\left(\int_0^t (t-\eta)^{\frac{\varpi-1}{1-\nabla}}\,d\eta\right)^{1-\nabla}\left(L^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}\|n-x\| \\
&\leq \Psi_i\,\|n-x\|, \\
\|I_{0l}n(t)-I_{1l}x(t)\| &\leq \tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1}L(\eta)\,|n(\eta)-x(\eta)|\,d\eta \\
&\leq \tfrac{1}{\Lambda(\varpi)}\left(\int_0^t (t-\eta)^{\frac{\varpi-1}{1-\nabla}}\,d\eta\right)^{1-\nabla}\left(L^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}\|n-x\| \\
&\leq \Psi_i\,\|n-x\|, \\
\|I_{0al}n(t)-I_{1al}x(t)\| &\leq \tfrac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1}L(\eta)\,|n(\eta)-x(\eta)|\,d\eta
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Lambda(\varpi)}\left(\int_0^t (t-\eta)^{\frac{\varpi-1}{1-\nabla}}\,d\eta\right)^{1-\nabla}\left(L^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}\|n-x\| \\
&\leq \Psi_i\|n-x\|,
\end{aligned}$$

$$\begin{aligned}
\|V_{0h}n(t)-V_{1h}x(t)\| &\leq \frac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1}L(\eta)\,|n(\eta)-x(\eta)|\,d\eta \\
&\leq \frac{1}{\Lambda(\varpi)}\left(\int_0^t (t-\eta)^{\frac{\varpi-1}{1-\nabla}}\,d\eta\right)^{1-\nabla}\left(L^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}\|n-x\| \\
&\leq \Psi_i\|n-x\|,
\end{aligned}$$

$$\begin{aligned}
\left\|C_{0p}n(t)-C_{1p}x(t)\right\| &\leq \frac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1}L(\eta)\,|n(\eta)-x(\eta)|\,d\eta \\
&\leq \frac{1}{\Lambda(\varpi)}\left(\int_0^t (t-\eta)^{\frac{\varpi-1}{1-\nabla}}\,d\eta\right)^{1-\nabla}\left(L^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}\|n-x\| \\
&\leq \Psi_i\|n-x\|,
\end{aligned}$$

$$\begin{aligned}
\|S_{0v}n(t)-S_{1v}x(t)\| &\leq \frac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1}L(\eta)\,|n(\eta)-x(\eta)|\,d\eta \\
&\leq \frac{1}{\Lambda(\varpi)}\left(\int_0^t (t-\eta)^{\frac{\varpi-1}{1-\nabla}}\,d\eta\right)^{1-\nabla}\left(L^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}\|n-x\| \\
&\leq \Psi_i\|n-x\|,
\end{aligned}$$

$$\begin{aligned}
\|I_{0v}n(t)-I_{1v}x(t)\| &\leq \frac{1}{\Lambda(\varpi)}\int_0^t (t-\eta)^{\varpi-1}L(\eta)\,|n(\eta)-x(\eta)|\,d\eta \\
&\leq \frac{1}{\Lambda(\varpi)}\left(\int_0^t (t-\eta)^{\frac{\varpi-1}{1-\nabla}}\,d\eta\right)^{1-\nabla}\left(L^{1/\nabla}(\eta)\,d\eta\right)^{\nabla}\|n-x\| \\
&\leq \Psi_i\|n-x\|,
\end{aligned}$$

Where

$$\Psi_i = \frac{\nu_i \|L\|_{1/\nabla} T^{\lambda_i}}{\Lambda(\varpi)} < 1,\ i \in \{1,2,3,4,5,6,7,8,9,10,11\} \tag{31}$$

These equations prove that $S_{0h}, I_{0a}, I_{0l}, I_{0al}, V_{0h}, R_{0a}, R_{0l}, R_{0al}, C_{0p}, S_{0v}$, and I_{0v} are contractions and satisfy

$$\begin{aligned}
\|S_{oh}(n) - S_{1h}(x)\| &\leq \Psi_i \|n-x\|, \\
\|I_{oa}(n) - I_{1a}(x)\| &\leq \Psi_i \|n-x\|, \\
\|I_{oa}(n) - I_{1a}(x)\| &\leq \Psi_i \|n-x\|, \\
\|I_{ol}(n) - I_{1l}(x)\| &\leq \Psi_i \|n-x\|, \\
\|I_{oal}(n) - I_{1al}(x)\| &\leq \Psi_i \|n-x\|, \\
\|V_{oh}(n) - V_{1h}(x)\| &\leq \Psi_i \|n-x\|, \\
\|R_{oa}(n) - R_{1a}(x)\| &\leq \Psi_i \|n-x\|, \\
\|R_{ol}(n) - R_{1l}(x)\| &\leq \Psi_i \|n-x\|, \\
\|R_{oal}(n) - R_{1al}(x)\| &\leq \Psi_i \|n-x\|, \\
\left\|C_{op}(n) - C_{1p}(x)\right\| &\leq \Psi_i \|n-x\|, \\
\|S_{ov}(n) - S_{1v}(x)\| &\leq \Psi_i \|n-x\|, \\
\|I_{ov}(n) - I_{1v}(x)\| &\leq \Psi_i \|n-x\|,
\end{aligned} \tag{32}$$

for $i \in \{1,2,3,4,5,6,7,8,9,10,11\}$

Next we prove that S_{1h}, I_{1a}, I_{1l}, I_{1al}, V_{1h}, R_{1a}, R_{1l}, R_{1al}, C_{1p}, S_{1v}, and I_{1v} are compact. For $0 \leq k_1 \leq k_2 \leq T$ we have

(33)

$$\begin{aligned}
\|S_{1h}k(k_1) - S_{1h}x(k_2)\| &\leq \frac{1}{\Lambda(\varpi)}\left| \int_0^{k_2} (k_2-\eta)^{\varpi-1} S_{1h}(\eta, n(\eta))\, d\eta \right. \\
&\quad \left. - \int_0^{k_1} (k_1-\eta)^{\varpi-1} S_{1h}(\eta, n(\eta))\, d\eta \right| \\
&\leq \frac{1}{\Lambda(\varpi)} \int_0^{k_1} \left((k_1-\eta)^{\varpi-1} - (k_2-\eta)^{\varpi-1} \right) M_{11}(\eta)\, d\eta \\
&\quad + \int_0^{k_2} (k_2-\eta)^{\varpi-1} M_{11}(\eta)\, d\eta \\
&\leq \frac{1}{\Lambda(\varpi)} \left[\int_0^{k_1} \left((k_1-\eta)^{\varpi-1} - (k_2-\eta)^{\varpi-1} \right)^{\frac{1}{1-\nabla}} d\eta \right]^{1-\nabla} (M_{11}(\eta)\, d\eta)^{\nabla} \\
&\quad + \frac{1}{\Lambda(\varpi)} \left(\int_0^{k_2} (k_2-\eta)^{\frac{\varpi-\nabla}{1-\nabla}} d\eta \right) \left(M_{11}^{1/\nabla}(\eta)\, d\eta \right)^{\nabla} \\
&\leq \frac{\nu_i}{\Lambda(\varpi)} \left[k_1^{\frac{\varpi-\nabla}{1-\nabla}} - k_2^{\frac{\varpi-\nabla}{1-\nabla}} + (k_2-k_1)^{\frac{\varpi-\nabla}{1-\nabla}} \right]^{1-\nabla} \|M_{11}\|_{1/\nabla} \\
&\quad + \frac{\nu_i}{\Lambda(\varpi)} (k_2-k_1)^{\varpi-\nabla} \|M_{11}\|_{1/\nabla} \\
&\leq \frac{\nu_i}{\Lambda(\varpi)} \left[(k_2-k_1)^{\frac{\varpi-\nabla}{1-\nabla}} \right] \|M_{11}\|_{1/\nabla} + \frac{\nu_i}{\Lambda(\varpi)} (k_2-k_1)^{\varpi-\nabla} \|M_{11}\|_{1/\nabla} \\
&\leq \frac{2\nu_i \|M_{11}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2-k_1)^{\varpi-\nabla}
\end{aligned}$$

Using the same strategy, we obtain

(34)
$$
\begin{aligned}
\|I_{1a}k(k_1) - I_{1a}x(k_2)\| &\leq \tfrac{2\nu_i \|M_{21}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2 - k_1)^{\varpi-\nabla}, \\
\|I_{1l}k(k_1) - I_{1l}x(k_2)\| &\leq \tfrac{2\nu_i \|M_{31}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2 - k_1)^{\varpi-\nabla}, \\
\|I_{1al}k(k_1) - I_{1al}x(k_2)\| &\leq \tfrac{2\nu_i \|M_{41}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2 - k_1)^{\varpi-\nabla}, \\
\|V_{1h}k(k_1) - V_{1h}x(k_2)\| &\leq \tfrac{2\nu_i \|M_{51}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2 - k_1)^{\varpi-\nabla}, \\
\|R_{1a}k(k_1) - R_{1a}x(k_2)\| &\leq \tfrac{2\nu_i \|M_{61}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2 - k_1)^{\varpi-\nabla}, \\
\|R_{1l}k(k_1) - R_{1l}x(k_2)\| &\leq \tfrac{2\nu_i \|M_{71}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2 - k_1)^{\varpi-\nabla}, \\
\|R_{1al}k(k_1) - R_{1al}x(k_2)\| &\leq \tfrac{2\nu_i \|M_{81}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2 - k_1)^{\varpi-\nabla}, \\
\left\|C_{1p}k(k_1) - C_{1p}x(k_2)\right\| &\leq \tfrac{2\nu_i \|M_{91}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2 - k_1)^{\varpi-\nabla}, \\
\|S_{1v}k(k_1) - S_{1v}x(k_2)\| &\leq \tfrac{2\nu_i \|M_{101}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2 - k_1)^{\varpi-\nabla}, \\
\|I_{1v}k(k_1) - I_{1v}x(k_2)\| &\leq \tfrac{2\nu_i \|M_{111}\|_{1/\nabla}}{\Lambda(\varpi)} (k_2 - k_1)^{\varpi-\nabla},
\end{aligned}
$$

for $\nu_i \in \{1, 2, 3, ..., 11\}$.

From the Arzela-Ascoli principle, it can be concluded that $S_{1h}(\Gamma_\zeta)$, $I_{1a}(\Gamma_\zeta)$, $I_{1l}(\Gamma_\zeta)$, $I_{1al}(\Gamma_\zeta)$, $V_{1h}(\Gamma_\zeta)$, $R_{1a}(\Gamma_\zeta)$, $R_{1l}(\Gamma_\zeta)$, $R_{1al}(\Gamma_\zeta)$, $C_{1p}(\Gamma_\zeta)$, $S_{1v}(\Gamma_\zeta)$, $I_{1v}(\Gamma_\zeta)$ are relatively compact, which means that S_{1h}, I_{1a}, I_{1l}, I_{1al}, V_{1h}, R_{1a}, R_{1l}, R_{1al}, C_{1p}, S_{1v}, and I_{1v} are compact. Since S_{0h}, I_{0a}, I_{0l}, I_{0al}, V_{0h}, R_{0a}, R_{0l}, R_{0al}, C_{0p}, S_{0v}, and I_{0v} are contractions and S_{1h}, I_{1a}, I_{1l}, I_{1al}, V_{1h}, R_{1a}, R_{1l}, R_{1al}, C_{1p}, S_{1v}, and I_{1v} are compact and therefore completely continuous. The maps $S_h = S_{0h} + S_{1h}$, $I_a = I_{0a} + I_{1a}$, $I_l = I_{0l} + I_{1l}$, $I_{al} = I_{0al} + I_{1al}$, $V_h = V_{0h} + V_{1h}$, $R_a = R_{0a} + R_{1a}$, $R_l = R_{0l} + R_{1l}$, $R_{al} = R_{0al} + R_{1al}$, $C_p = C_{0p} + C_{1p}$, $S_v = S_{0v} + S_{1v}$, and $I_v = I_{0v} + I_{1v}$ are condensing on Γ_υ, and we get the existence of fixed points of S_h, I_a, I_l, I_{al}, V_h, R_a, R_l, R_{al}, C_p, S_v, and I_v respectively. Finally we prove that the given initial value problem has solution in the real interval $[0,T]$. To prove this, we consider supposition 3 Condition (31), and the map H given by

$$\begin{aligned}
\mathrm{H}[S_h(t)] &= S_h(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}S_h(\eta,S_h(\eta))\,d\eta,\\
\mathrm{H}[I_a(t)] &= I_a(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}I_a(\eta,I_a(\eta))\,d\eta,\\
\mathrm{H}[I_l(t)] &= I_l(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}I_l(\eta,I_l(\eta))\,d\eta,\\
\mathrm{H}[I_{al}(t)] &= I_{al}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}I_{al}(\eta,I_{al}(\eta))\,d\eta,\\
\mathrm{H}[V_h(t)] &= V_h(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}V_h(\eta,V_h(\eta))\,d\eta,\\
\mathrm{H}[R_a(t)] &= R_a(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}R_a(\eta,R_a(\eta))\,d\eta, \qquad (35)\\
\mathrm{H}[R_l(t)] &= R_l(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}R_l(\eta,R_l(\eta))\,d\eta,\\
\mathrm{H}[R_{al}(t)] &= R_{al}(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}R_{al}(\eta,R_{al}(\eta))\,d\eta,\\
\mathrm{H}[C_p(t)] &= C_p(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}C_p(\eta,C_p(\eta))\,d\eta,\\
\mathrm{H}[S_v(t)] &= S_v(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}S_v(\eta,S_v(\eta))\,d\eta,\\
\mathrm{H}[I_v(t)] &= I_v(0)+\tfrac{1}{\Lambda(\varpi)}\int_0^t(1-\eta)^{\varpi-1}I_v(\eta,I_v(\eta))\,d\eta.
\end{aligned}$$

For $S_{0h}(t), I_{0a}(t), I_{0l}(t), I_{0al}(t), V_{0h}(t), R_{0a}(t), R_{0l}(t), R_{0al}(t), C_{0p}(t), S_{0v}(t), I_{0v}(t)\in\Gamma_\zeta$, we get

$$
\begin{aligned}
\left|\mathrm{H}\left[S_h(t)\right]-F\left[S_h(t)\right]\right| &\leq \frac{1}{\Lambda(\varpi)}\int_0^t (1-\eta)^{\varpi-1} L_1(\eta)\left|S_{0h}(\eta)-S_{1h}(\eta)\right| d\eta \\
&\leq \frac{1}{\Lambda(\varpi)}\left[\int_0^t (t-\eta)^{\frac{\varpi-1}{1-\nabla}} d\eta\right]^{1-\nabla}\left[\int_0^t L_1^{1/\nabla}(\eta)\, d\eta\right]^{\nabla} \\
&\leq \frac{v_1\|L_1\|_{1/\nabla} T^{\mathrm{K}_1}}{\Lambda(\varpi)}\left\|S_{0h}-S_{1h}\right\|;
\end{aligned}
\tag{36}
$$

Using the same strategy, we obtain

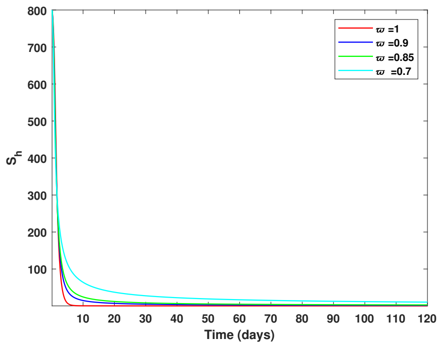
$$
\begin{aligned}
\left|\mathrm{H}\left[I_a(t)\right]-F\left[I_a(t)\right]\right| &\leq \frac{v_2\|L_2\|_{1/\nabla} T^{\mathrm{K}_2}}{\Lambda(\varpi)}\left\|I_{0a}-I_{1a}\right\|, \\
\left|\mathrm{H}\left[I_l(t)\right]-F\left[I_l(t)\right]\right| &\leq \frac{v_3\|L_3\|_{1/\nabla} T^{\mathrm{K}_3}}{\Lambda(\varpi)}\left\|I_{0l}-I_{1l}\right\|, \\
\left|\mathrm{H}\left[I_{al}(t)\right]-F\left[I_{al}(t)\right]\right| &\leq \frac{v_4\|L_4\|_{1/\nabla} T^{\mathrm{K}_4}}{\Lambda(\varpi)}\left\|I_{0al}-I_{1al}\right\|, \\
\left|\mathrm{H}\left[V_h(t)\right]-F\left[V_h(t)\right]\right| &\leq \frac{v_5\|L_5\|_{1/\nabla} T^{\mathrm{K}_5}}{\Lambda(\varpi)}\left\|V_{0h}-V_{1h}\right\|, \\
\left|\mathrm{H}\left[R_a(t)\right]-F\left[R_a(t)\right]\right| &\leq \frac{v_6\|L_6\|_{1/\nabla} T^{\mathrm{K}_6}}{\Lambda(\varpi)}\left\|R_{0a}-R_{1a}\right\|, \\
\left|\mathrm{H}\left[R_l(t)\right]-F\left[R_l(t)\right]\right| &\leq \frac{v_7\|L_7\|_{1/\nabla} T^{\mathrm{K}_7}}{\Lambda(\varpi)}\left\|R_{0l}-R_{1l}\right\|, \\
\left|\mathrm{H}\left[R_{al}(t)\right]-F\left[R_{al}(t)\right]\right| &\leq \frac{v_8\|L_8\|_{1/\nabla} T^{\mathrm{K}_8}}{\Lambda(\varpi)}\left\|R_{0al}-R_{1al}\right\|, \\
\left|\mathrm{H}\left[C_p(t)\right]-F\left[C_p(t)\right]\right| &\leq \frac{v_9\|L_9\|_{1/\nabla} T^{\mathrm{K}_9}}{\Lambda(\varpi)}\left\|C_{0p}-C_{1p}\right\|, \\
\left|\mathrm{H}\left[S_v(t)\right]-F\left[S_v(t)\right]\right| &\leq \frac{v_{10}\|L_{10}\|_{1/\nabla} T^{\mathrm{K}_{10}}}{\Lambda(\varpi)}\left\|S_{0v}-S_{1v}\right\|, \\
\left|\mathrm{H}\left[I_v(t)\right]-F\left[I_v(t)\right]\right| &\leq \frac{v_{11}\|L_{11}\|_{1/\nabla} T^{\mathrm{K}_{11}}}{\Lambda(\varpi)}\left\|I_{0v}-I_{1v}\right\|.
\end{aligned}
\tag{37}
$$

For the cases above, condition(31) is assured, and the existence of the unique solution for the model is thus proved.

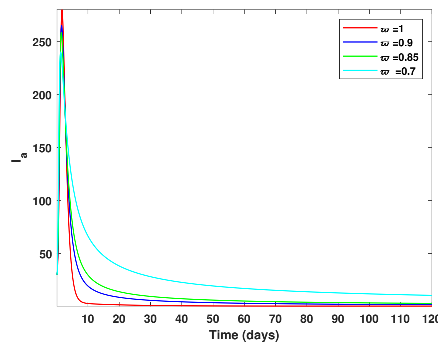
□

In this section, the numerical simulation results are presented with a step size of $h = 0.001$. The numerical scheme employed was based on Adam-type predictor-corrector as extensively examined in [32]. The parameter values utilised some were obtained in [1] and others were estimated: $b = 0.01$, $\Omega_h = 0.001$, $k = 0.01$, $w = 0.01$, $\psi = 0.07$, $\beta_h = 0.005$, $\pi = 0.08$, $d = 0.001$, $\mu_h = 0.05$, $\alpha = 0.33$, $\phi = 0.2$, $\beta_l = 0.005$, $\sigma = 0.05$, $m = 0.2$, $\rho = 0.02$, $\eta = 0.08$, $\theta = 0.04$, $\Omega_h = 0.4$, $q = 0.02$, $\tau = 0.01$, $\mu_b = 0.0025$, $\gamma = 0.006$, $\Omega_v = 0.005$, $\beta_v = 0.05$, $\mu_v = 0.008$, $\kappa = 10000$, $v = 0.5$, $g = 0.001$

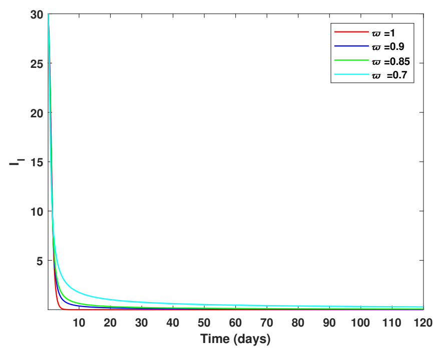
Figure 2(a) represents the number of Susceptible individuals (S_h) and as the fractional order derivative increases from 0.7 towards 1 the number of Susceptible individuals reduce. This is reasonable since more people get infected and move out of the (S_h) class. In Figure 2(b), the number of individuals infected with Anthrax (I_a) reduce as the fractional order derivatives increase. Therefore, increasing the fractional order can help curb the disease spread. Figure 2(c) represents the individuals infected with Listeriosis (I_l). Increasing the fractional order from 0.7 towards 1 leads to a decrease in the number of individuals in this class. Similar to the case of the Anthrax disease, the spread of Listeriosis can also be reduced by increasing the fractional order. Figure 2(d) shows the dynamics of the coinfected individuals. We observe that the number of individuals infected with both Anthrax and Listeriosis (I_{al}) increases as the fractional order derivative decreases from 1 towards 0.7. From the results obtained in 2(b) and 2(c), it is plausible therefore to say that if we increase the fractional order instead, the number of coinfected individuals will decrease. Figure 2(e) shows how the number of individuals Vaccinated V_h changes with the fractional order. The observed decrease in the number of individuals getting vaccinated as the fractional order increases is justifiable since an increase in the fractional order causes a decrease in the number of individuals infected with both or any of the diseases.



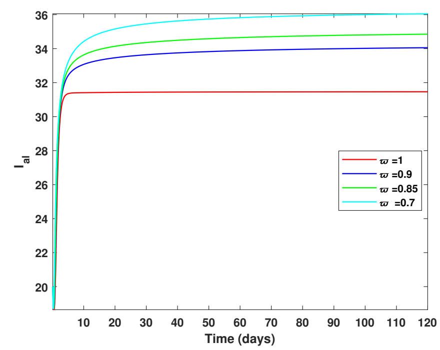
(a)



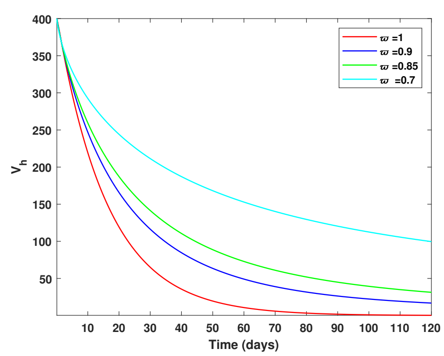
(b)



(c)



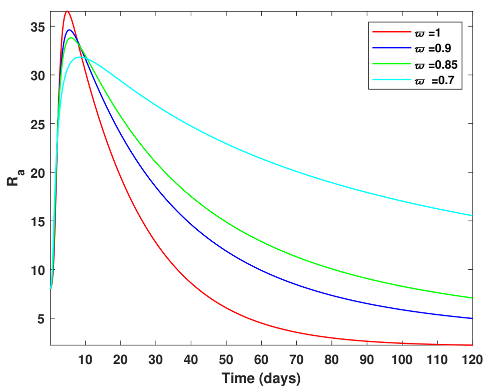
(d)



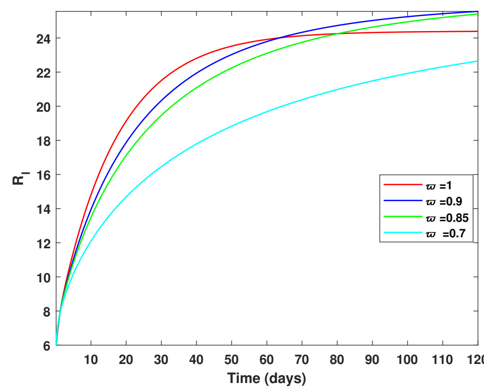
(e)

FIGURE 2. Numerical simulation of model 4 via Power law for different values of ϖ

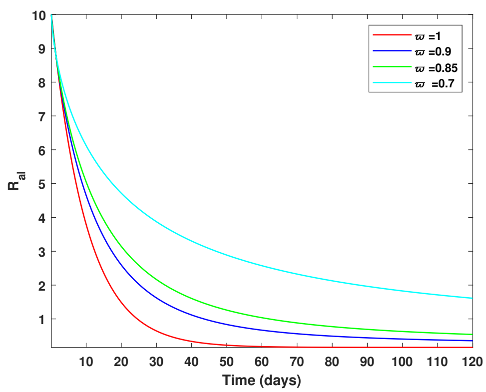
Figure 3(a) is for the number of individuals recovered from Anthrax (R_a). It shows that the number of individuals reduce as the fractional order derivatives increase towards 1. Thus a decrease in the fractional order is required for more people to recover from Anthrax. In Figure 3(b), Unlike the case with the individuals recovering from Anthrax, we observe an increase in the number of individuals who have recovered from Listeriosis (R_l) as the fractional order derivative increases from 0.7 towards 1. In this case therefore, an increase in the fractional order derivative is needed for more listeriosis infected persons to recover. Figure 3(c) represents individuals recovered from both Anthrax and Listeriosis (R_{al}). Here, the number of individuals reduce as the fractional order derivative increase. It is clear that these dynamics are influenced by the changes that occur in the number of Anthrax recovered individuals as the fractional order derivative changes. Figure 3(d) shows the changes in the population of pathogen infested animal carcasses (C_p) in the soil. we observe a decrease in the concentration of the carcasses as the fractional order derivative increases towards 1. Hence, we need to increase the fractional order derivative in order to decrease the concentration of the pathogens in the environment and intend reduce the spread of Anthrax. In Figure 3(e) the number of susceptible animals (S_v) increases as the fractional order derivative increases. This is justifiable because more animals get infected and move out of the susceptible class as the fractional order derivative increases. In Figure 3(f) the number of animals infected with Anthrax (I_v) reduces as the fractional order derivative increases from 0.7 towards 1.



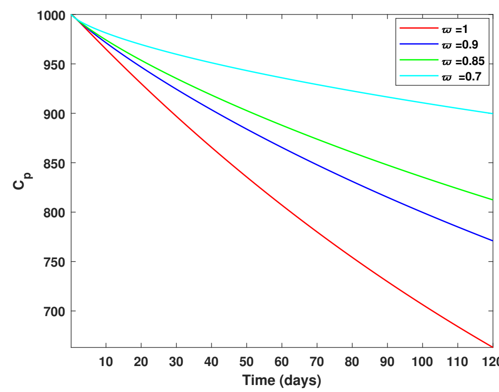
(a)



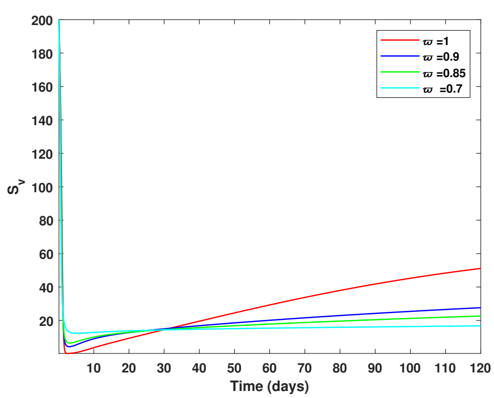
(b)



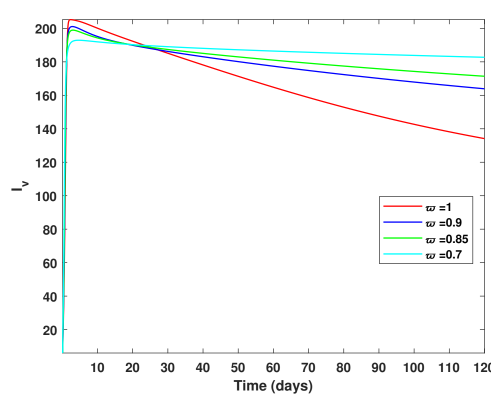
(c)



(d)



(e)



(f)

FIGURE 3. Numerical simulation of model 4 via Power law for different values of ϖ continues.

5. Conclusion

We analyse a coinfection model of anthrax and listeriosis using frcational order derivative, in particular, the Liouville- Caputo operator. Our model is the fractional order version of the integer order model of Osman and Makinde [1]. The basic properties of the model such as existence and uniqueness and positivity of solutions were investigated. The stability analysis of Liouville-Caputo operator with Power Law was also studied. we use Adams-type predictor-corrector method to qualitatively analyse the model trajectories subject to certain initial conditions. We simulate the model by varying fractional orders ($\varpi = 0.7,\ 0.85,\ 0.9,\ 0.1$) and the numerical simulation results suggest that the coinfection dynamics depend notably on the fractional order derivative and some specific model parameters. Decreasing the prevalence of the coinfection entails decreasing the prevalence and spread of the individual diseases. The figures obtained in the simulations indicate that the prevalence and spread of the individual diseases is decreased by an increase in the fractional order derivative, thus, the fractional order derivative practically drives the dynamics of the coinfection. The figures obtained show similar numerical simulation results to those obtained in the Anthrax Listeriosis integer order coinfection model [1]. In the classical case $\varpi = 1$, the results obtained are in agreement with those obtained by Osman and Makinde and suggests that our fractional model is well posed. The use of the Liouville-Caputo operator, permits a better description in the history of the biological process. We thus suggest that complex models are much better examined using fractional order derivatives.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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