Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2022, 2022:38
https://doi.org/10.28919/cmbn/7261
ISSN: 2052-2541

A STOCHASTIC VIRAL MODEL WITH CELL-TO-CELL TRANSMISSION

AMINE EL KOUFI*

Laboratory of Analysis, Modeling and Simulation (LAMS), Faculty of Sciences Ben M'sik, Hassan II University, P.O Box 7955 Sidi Othman, Casablanca, Morocco

Abstract. The purpose of this paper is to investigate the dynamics of a stochastic virus dynamical model with both cell-to-virus infection and cell-to-cell transmission and cure rate. We show the existence of a unique global positive solution of our system, we give a sufficient conditions for persistence of the infected cells and virus, we prove that the virus-free equilibrium is almost sure exponentially stable. Numerical simulations are achieved to illustrate theoretical results.

Keywords: Stochastic model; virus dynamical model; exponential stability; persistence.

2010 AMS Subject Classification: 92D30, 47H05.

1. INTRODUCTION AND PRELIMINARIES

In nature, there are many dangerous viruses, which cause various infectious diseases such as the coronavirus (COVID-19), the human immunodeficiency virus (HIV), the hepatitis B virus (HBV). Usually, there are two modes of virus transmission: by viral to cell contamination into the extracellular area, or by cell-to-cell transmission involving direct cell to cell contact. Therefore, many mathematical models have been proposed and developed To modeling the viral infection dynamics. One among them the following viral model with both cell-to-virus infection

*Corresponding author

E-mail address: elkoufiamine1@gmail.com

Received February 12, 2022

and cell-to-cell transmission and cure rate proposed by Zhang et al. [2]

$$\begin{aligned} \dot{x} &= \Lambda - dx - f(y,v)x + \rho y, \\ \dot{y} &= f(y,v)x - (a+\rho)\, y, \\ \dot{v} &= ky - uv. \end{aligned} \tag{1}$$

Where x, y and v represent the number of host cells, infected cells, and free virus at time t, respectively. Λ is the regeneration rate of host cells. Free virus is produced by infected cells at a rate ky. Also, d, a and u are, respectively, the death rates of host cells, infected cells, and free virus. ρ is the cure rate. Here, $f(y,v)x = (\beta y + \alpha v)\, x$ represents the total infection rate of host cells, αvx represents the infection rate of host cells by free viruses, and βyx is the infection rate of host cells by direct contact with an infected cell, β, and α are the infection coefficients.
The basic reproduction number [2] of (1) model is as follows

$$\mathscr{R}_0 = \frac{\Lambda(\alpha k + \beta u)}{du(a+\rho)}.$$

Moreover, the system (1) has a unique virus-free equilibrium $E_0 = \left(\frac{\Lambda}{d}, 0, 0\right)$ If $\mathscr{R}_0 \leq 1$. When $\mathscr{R}_0 > 1$, E_0 is still present and there exists an equilibrium namely, endemic equilibrium $E_1 = (x^*, y^*, v^*)$, where

$$\begin{aligned} x^* &= \frac{\Lambda}{d\mathscr{R}_0}, \\ y^* &= \frac{\Lambda}{a}\left(1 - \frac{1}{\mathscr{R}_0}\right), \\ v^* &= \frac{k}{u} y^*. \end{aligned}$$

The dynamical behavior of model (1) is as follows:

Theorem 1.1. [2] *1. If $\mathscr{R}_0 \leq 1$, the virus-free equilibrium E_0 is globally asymptotically stable. 2. If $1 < \mathscr{R}_0 < 1 + \delta$, the epidemic equilibrium E_1 is globally asymptotically stable, where $\delta = \frac{(\beta\Lambda + (a-\rho)d + \sqrt{(\beta\Lambda + (a-\rho d))^2 + 4a\rho d^2}}{2\rho d}$.*

However, it's important to incorporate the effect of environmental noises such that the white noise because biological systems are often subject to environmental noise and deterministic models do not integrate the effect of the fluctuating environment. Consequently, many

mathematicians have developed epidemic models with stochastic differential equations(see, [5, 6, 7, 8, 9]). Then, assuming that the environmental noise is proportional to the variables, we obtain the following stochastic virus dynamical model with both cell-to-virus infection and cell-to-cell transmission and cure rate as follows:

$$\begin{aligned} \dot{x} &= [\Lambda - dx - f(y,v)x + \rho y]\,dt + \sigma_1 x dB_1(t), \\ \dot{y} &= [f(y,v)x - (a+\rho)\,y]\,dt + \sigma_2 y dB_2(t), \\ \dot{v} &= [ky - uv]\,dt + \sigma_3 v dB_3(t). \end{aligned} \tag{2}$$

Where $B = (B_1(t), B_2(t), B_3(t))$ are independent Brownian motions defined in a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. The components of B are assumed to be mutually independent. The non-negative constants σ_1, σ_2 and σ_3 denote the intensities of the stochastic perturbations.

Let us denote by $\mathbb{R}^n_+$ the set of points in $\mathbb{R}^n$ having only nonnegative coordinates. Throughout this paper we assume to have a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration, $\{\mathscr{F}_t\}_{t\geq 0}$, that is right continuous and with $\mathscr{F}_0$ contains all P-null sets. We consider the following stochastic differential system (3), for an d-dimensional Brownian motion $B(t)$ on Ω.

$$dx(t) = h(t,x)dt + g(t,x)dB(t) \quad t \geq 0. \tag{3}$$

A solution with initial value $x(0) = x_0$ is denoted by $x(t,x_0)$. Assume that $h(t,0) = g(t,0) = 0$ for all $t \geq 0$.

Concerning a stochastic process $x(t)$ which is a function of $B(t)$, when we wish to distinguish a specific Brownian path ω, we can write $x(t,\omega)$.

The differential operator $\mathscr{L}$ with the function displayed in equation (3), is defined for a function $V(t,x) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^d)$ by the formula

$$\mathscr{L}V(t,x) = V_t(t,x) + V_x(t,x)h(t,x) + \frac{1}{2}Tr\left[g^t(t,x)V_{xx}g(t,x)\right],$$

where Tr means trace and t denotes the transpose of a matrix. And

$$V_t(t,x) = \frac{\partial V}{\partial t},\ V_x(t,x) = \left(\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_n}\right),\ V_{xx}(t,x) = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right).$$

By using Itô's formula, we have

$$dV(t,x) = \mathscr{L}V(t,x)dt + V_x(t,x)g(t,x)dB(t).$$

There exist diverse definitions of stability for the equilibrium points of (SDE). We must concentrate on one of them.

Definition 1.1. [11]*The equilibrium $x=0$ of the system (3) is said to be almost surely exponentially stable if for all $x_0 \in \mathbb{R}^n$,*

$$\limsup_{t\to+\infty} \frac{1}{t} \ln|x(t,x_0)| < 0 \quad \textit{almost surely } (a.s.).$$

The following lemma are quite useful in the proof of almost sure exponential stability main theorem.

Lemma 1.1. [3]*For $k \in \mathbb{N}$, let $X(t) = (X_1(t), X_2(t), ..., X_d(t))$ be a bounded $\mathbb{R}^d$ valued function. Let $(t_{0,n})$ be any increasing unbounded sequence of positive real numbers. Then there is a family of sequences $\left(t_{l,n}\right)$ such that for each $l \in \{1,2,....,d\}$, $\left(t_{l,n}\right)$ is a subsequence of $\left(t_{l-1,n}\right)$ and the sequence $X_l(t_{l,n})$ converges to the largest limit point of the sequence $X_l(t_{l-1,n})$.*

Lemma 1.2. [4]*For any initial value $(x(0), y(0), v(0)) \in \mathbb{R}^3_+$, the solution $(x(t), y(t), v(t))$ of model (2) has the following properties:*

$$\begin{aligned}
\lim_{t\to\infty} \frac{x(t)}{t} &= 0, \ \lim_{t\to\infty} \frac{y(t)}{t} = 0, \ \lim_{t\to\infty} \frac{v(t)}{t} = 0, \ \ a.s. \\
\limsup_{t\to\infty} \frac{\ln x(t)}{t} &\leq 0, \ \limsup_{t\to\infty} \frac{\ln y(t)}{t} \leq 0, \ \limsup_{t\to\infty} \frac{\ln v(t)}{t} \leq 0, \ \ a.s. \\
\lim_{t\to\infty} \frac{\int_0^t x(s)dB_1(s)}{t} &\leq 0, \ \lim_{t\to\infty} \frac{\int_0^t y(s)dB_2(s)}{t} \leq 0, \ \lim_{t\to\infty} \frac{\int_0^t v(s)dB_3(s)}{t} \leq 0, \ \ a.s.
\end{aligned}$$

The rest of this paper is organized as follows. In Section 2, we show the existence of unique positive global solution to the given stochastic system. exponential stability and persistence in mean results are investigated in Section 3 and Section 4, respectively. In Section 5, the analytical results are illustrated with the support of numerical examples. Finally, we terminate the paper with conclusion and future directions.

2. Existence and Uniqueness of the Global Positive Solution

In this section, we will show the existence, positivity of solutions of model (2).

Theorem 2.1. *For any initial value $X(0) = (x(0), y(0), v(0)) \in \mathbb{R}^3_+$, there is a unique solution $X(t) = (x(t), y(t), v(t))$ on $t \geq 0$ which remains in $\mathbb{R}^3_+$ with probability one.*

Proof 2.1. *Since the coefficients of system (2) are locally Lipshitz continuous, for any initial value* $X(0) \in \mathbb{R}^3_+$, *there exists a unique local solution* $X(t)$ *on* $t \in [0, \tau_e)$, *where* τ_e *is the explosion time*[10]. *We need to show that this solution is global almost surely that is,* $\tau_e = \infty$ *a.s. Let* n_0 *be sufficiently large such that every component of* $X(0)$ *lies within the interval* $\left[\frac{1}{n_0}, n_0\right]$. *For each integer* $n \geq n_0$, *define the stopping times:*

$$\tau_n = \inf\left\{t \in [0,\tau_e) \ : \ x(t) \notin \left(\frac{1}{n}, n\right) or\, y(t) \notin \left(\frac{1}{n}, n\right) or v(t) \notin \left(\frac{1}{n}, n\right)\right\},$$

where we set $\inf\{\emptyset\} = \infty$ *(as usual* $\emptyset$ *denotes the empty set). Obviously,* τ_n *is increasing as* $n \to \infty$. *Set* $\tau_\infty = \lim_{n\to\infty} \tau_n$, *and* $\tau_\infty \leq \tau_e$ $(a.s)$. *Now we need to show* $\tau_\infty = \infty$ *a.s. If this statement is violated, then there exist* $T > 0$ *and* $\varepsilon \in (0,1)$ *such that*

$$\mathbb{P}(\tau_\infty \leq T) > \varepsilon.$$

Hence, there is an integer $n_1 \geq n_0$ *such that*

$$\mathbb{P}(\tau_\infty \leq T) \geq \varepsilon, \quad for\ all\ n \geq n_1. \tag{4}$$

Define a C^2-*function* $V_1 : \mathbb{R}^3_+ \longrightarrow \mathbb{R}_+$ *by*

$$V_1(x,y,v) = \left(x - c - c\ln\frac{x}{c}\right) + (y - 1 - \ln y) + (v - 1 - \ln v),$$

where $c > 0$ *is sufficiently small constant in order to have* $c\alpha < u$ *and* $c\beta + k < a$. *Using Itô's formula for all* $t \in [0, \tau_n)$, *we have*

$$dV_1(x,y,v) = \mathscr{L}V_1 dt + (x-c)dB_1(t) + (y-1)\,dB_2(t) + (v-1)\,dB_3(t),$$

where

$$\begin{aligned}
\mathscr{L}V_1 &= \left(1 - \frac{c}{x}\right)[\Lambda - dx - f(y,v)x + \rho y] + \left(1 - \frac{1}{y}\right)[f(y,v)x - (a+\rho)\,y] \\
&\quad + \left(1 - \frac{1}{v}\right)[ky - uv] \\
&\leq c(\beta y + \alpha v) + (k-a)\,y - uv + \Lambda + cd + a + \rho + u + \frac{1}{2}\left(\sigma_1^2 c + \sigma_2^2 + \sigma_3^2\right).
\end{aligned}$$

Note that

$$(c\beta + k - a)\,y < 0 \quad and \quad (c\alpha - u)\,v < 0.$$

Hence,

$$dV_1(x,y,v) \leq \mathscr{K}\,dt + (x-c)dB_1(t) + (y-1)\,dB_2(t) + (v-1)\,dB_3(t),$$

where $\mathscr{K} = \Lambda + cd + a + \rho + u + \frac{1}{2}\left(\sigma_1^2 c + \sigma_2^2 + \sigma_3^2\right)$. *Integrating both sides of the above inequality from* 0 *to* $\tau_n \wedge T$ *yields*

$$\int_0^{\tau_n \wedge T} dV_1(x,y,v) \leq \int_0^{\tau_n \wedge T} \mathscr{K}\,ds + M_1(\tau_n \wedge T),$$

where $\tau_n \wedge T = \min\{\tau_n, T\}$ *and*

$$M_1(s) = \int_0^s (x-c)dB_1(r) + \int_0^s (y-1)\,dB_2(r) + \int_0^s (v-1)\,dB_3(r).$$

Note that $M_1(s)$ *is a mean zero martingale process. Then taking the expectations leads to*

$$\mathbb{E}V_1\left(x(\tau_n \wedge T), y(\tau_n \wedge T), v(\tau_n \wedge T)\right) \leq V_1\left(x(0), y(0), v(0)\right) + \mathscr{K}T.$$

Set $\Omega_n = \{\tau_n \leq T\}$ *for* $m \geq m_1$ *and by (4), we have* $\mathbb{P}(\tau_\infty \leq T) \geq \varepsilon$ *for each* $m \geq m_1$. *we have*

$$\begin{aligned} V_1\left(x(\tau_n \wedge T), y(\tau_n \wedge T), v(\tau_n \wedge T)\right) &\geq \min_{i \in \{1,c\}} \left\{ \left(n - i - i\ln\frac{n}{i}\right), \left(\frac{1}{n} - i - i\ln\frac{1}{in}\right) \right\} \\ &:= L_n. \end{aligned}$$

Then we obtain

$$V_1\left(X(0)\right) + \mathscr{K}T \geq \mathbb{E}\left[1_{\Omega_n} V_1\left(x(\tau_n \wedge T), y(\tau_n \wedge T), v(\tau_n \wedge T)\right)\right] \geq \varepsilon L_n,$$

where 1_{Ω_n} *is the indicator function of* Ω_n. *Letting* $n \to +\infty$ *leads to the contradiction* $\infty = V_1\left(x(0), y(0), v(0)\right) + \mathscr{K}T < \infty$. *So we must therefore have* $\tau_\infty = \infty$ *a.s. This completes the proof.* □

3. PERSISTENCE IN MEAN

The goal of this section is to give a sufficient condition for the persistence in mean of virus and infected cells. For simplicity, we introduce the following notation:

$$\langle x \rangle_t = \frac{1}{t}\int_0^t x(s)ds.$$

The definition of persistence in mean is given by

Definition 3.1. *Model (2) is said to be persistence in the mean, if*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t y(s)ds>0,\quad \liminf_{t\to\infty}\frac{1}{t}\int_0^t v(s)ds>0\ \ a.s.$$

Let

$$\mathscr{R}_s^*=\frac{\Lambda(\alpha k+\beta u)}{\left(u+\frac{\sigma_3^2}{2}\right)\left(a+\rho+\frac{\sigma_2^2}{2}\right)\left(d+\frac{\sigma_1^2}{2}\right)}.$$

Theorem 3.1. *Let $(x(t),y(t),v(t))$ be the solution of system (2) with any initial value $(x(0),y(0),v(0))\in\mathbb{R}_+^3$. Assume that $\mathscr{R}_s^*>1$. Then*

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t v(s)ds &\geq \frac{k}{\alpha k+\beta u}\left(d+\frac{\sigma_1^2}{2}\right)[\mathscr{R}_s^*-1]>0,\ a.s.\\ \liminf_{t\to\infty}\frac{1}{t}\int_0^t y(s)ds &\geq \frac{u}{\alpha k+\beta u}\left(d+\frac{\sigma_1^2}{2}\right)[\mathscr{R}_s^*-1]>0,\ a.s.\end{aligned}$$

Proof 3.1. *We define the function Y as follows:*

$$Y(x,y,v)=-\ln x-q_1\ln y-q_2\ln v-\frac{\beta}{k}v,$$

where q_1 and q_2 are constants to be determined later. By Itô's formula, we get

$$dY(x,y,v)=\mathscr{L}Ydt-\sigma_1dB_1(t)-q_1\sigma_2dB_2(t)-q_2\sigma_3dB_3(t). \tag{5}$$

Where

$$\begin{aligned}\mathscr{L}Y &= -\frac{\Lambda}{x}-\frac{q_1\alpha vx}{y}-\frac{q_2ky}{v}+d+\beta y+\alpha v-\frac{\rho y}{x}+\frac{\sigma_1^2}{2}-q_1\beta x+q_1(a+\rho)\\ &\quad+\frac{q_1\sigma_2^2}{2}+q_2u+\frac{q_2\sigma_3^2}{2}-\beta y+\frac{\beta u}{k}v\\ &\leq -3\sqrt[3]{\Lambda q_1\alpha q_2k}+d+\frac{\sigma_1^2}{2}+q_1\left[a+\rho+\frac{\sigma_2^2}{2}\right]+q_2\left[u+\frac{\sigma_3^2}{2}\right]+\frac{\alpha k+\beta u}{k}v.\end{aligned}$$

Hence, we can choose q_1 and q_2 :

$$q_1=\frac{\Lambda(\alpha k+\beta u)}{\left(u+\frac{\sigma_3^2}{2}\right)\left(a+\rho+\frac{\sigma_2^2}{2}\right)^2},\quad q_2=\frac{\Lambda(\alpha k+\beta u)}{\left(u+\frac{\sigma_3^2}{2}\right)^2\left(a+\rho+\frac{\sigma_2^2}{2}\right)},$$

such that

$$q_1\left(a+\rho+\frac{\sigma_2^2}{2}\right)=q_2\left(u+\frac{\sigma_3^2}{2}\right)=\frac{\Lambda(\alpha k+\beta u)}{\left(u+\frac{\sigma_3^2}{2}\right)\left(a+\rho+\frac{\sigma_2^2}{2}\right)}.$$

Therefore,

$$\begin{aligned}\mathscr{L}Y &\leq -\frac{\Lambda(\alpha k+\beta u)}{\left(u+\frac{\sigma_3^2}{2}\right)\left(a+\rho+\frac{\sigma_2^2}{2}\right)}+d+\frac{\sigma_1^2}{2}+\frac{\alpha k+\beta u}{k}v\\ &= -\left(d+\frac{\sigma_1^2}{2}\right)\left[\frac{\Lambda(\alpha k+\beta u)}{\left(u+\frac{\sigma_3^2}{2}\right)\left(a+\rho+\frac{\sigma_2^2}{2}\right)\left(d+\frac{\sigma_1^2}{2}\right)}-1\right]+\frac{\alpha k+\beta u}{k}v\\ &:= -\left(d+\frac{\sigma_1^2}{2}\right)[\mathscr{R}_s^*-1]+\frac{\alpha k+\beta u}{k}v.\end{aligned}$$

Integrating both sides of (5) *from 0 to t and dividing by t yields that*

$$\begin{aligned}&\frac{\ln x(0)-\ln x(t)}{t}+\frac{q_1(\ln y(0)-\ln y(t))}{t}+\frac{q_2(\ln v(0)-\ln v(t))}{t}+\frac{\beta}{k}\frac{v(0)-v(t)}{t}\\ &\leq -\left(d+\frac{\sigma_1^2}{2}\right)[\mathscr{R}_s^*-1]+\frac{\alpha k+\beta u}{k}\langle v\rangle_t-\frac{\sigma_1}{t}\int_0^t dB_1(s)-\frac{q_1\sigma_2}{t}\int_0^t dB_2(s)\\ &\qquad -\frac{q_2\sigma_3}{t}\int_0^t dB_3(s).\end{aligned}$$

By strong law of large numbers [1]*, we obtain*

$$\lim_{t\to\infty}\frac{\sigma_i}{t}\int_0^t dB_i(s)=0\ a.s.,\ for\ (i=1,2,3).$$

Then by Lemma 1.2, we get

$$\liminf_{t\to\infty}\langle v\rangle_t\geq\frac{k}{\alpha k+\beta u}\left(d+\frac{\sigma_1^2}{2}\right)[\mathscr{R}_s^*-1]>0\ a.s.$$

From system (2), we have

$$\frac{v(t)-v(0)}{t}=k\langle y\rangle_t-u\langle v\rangle_t+\frac{\sigma_3}{t}\int_0^t v(s)dB_3(s).$$

Making use of Lemma 1.2, we get

$$\liminf_{t\to\infty}\langle y\rangle_t\geq\frac{u}{\alpha k+\beta u}\left(d+\frac{\sigma_1^2}{2}\right)[\mathscr{R}_s^*-1]>0\ a.s.$$

The theorem is proved. □

4. EXPONENTIAL STABILITY

Let define the following subset Δ of sample paths:

$$\Delta = \left\{ \omega \in \Omega : \ x(t,\omega), y(t,\omega), v(t,\omega) \in \mathbb{R}^3_+ \text{ for all } t \geq 0 \right\}.$$

Next, we introduce some important notation and property for our theorem on almost sure exponential stability.

$$H(t) = \theta_1 \left(\frac{\Lambda}{d} - x \right) + \theta_2 y + \theta_3 v, \tag{6}$$

and

$$V_2(t) = \ln H(t).$$

Where

$$\begin{aligned} \theta_1 &= \alpha k + \beta u, \\ \theta_2 &= u(a+\rho), \\ \theta_3 &= \alpha(a+\rho). \end{aligned} \tag{7}$$

Proposition 4.1. *If*

$$\limsup_{t\to\infty} \langle \mathscr{L} V_2(X(t)) \rangle < 0 \quad a.s.$$

Then $H(t)$ converges exponentially to 0 a.s.

Proof 4.1. *By the Itô's formula, we have*

$$V_2(X(t)) = V_2(X(0)) + \int_0^t \mathscr{L} V_2(X(s))\, ds + M_2(t)$$

where

$$\begin{aligned} M_2(t) &= \int_0^t \frac{\sigma_1 x(s)}{H(X(s))} dB_1(s) + \int_0^t \frac{\sigma_2 y(s)}{H(X(s))} dB_1(s) + \int_0^t \frac{\sigma_3 v(s)}{H(X(s))} dB_1(s) \\ &= \int_0^t \left(\frac{\sigma_1 x(s)}{H(X(s))} + \frac{\sigma_2 y(s)}{H(X(s))} + \frac{\sigma_3 v(s)}{H(X(s))} \right) dB(s) \end{aligned}$$

The strong law of large numbers for local martingales see [1]*, implies that*

$$\lim_{t\to\infty} \frac{M_2(t)}{t} = 0 \quad a.s.$$

And we have

$$\lim_{t\to\infty}\frac{V_2(X(0))}{t}.$$

Then

$$\limsup_{t\to\infty} V_2(X(t)) = \limsup_{t\to\infty}\langle \mathscr{L}V_2(X(t))\rangle \ a.s.$$

The proof of proposition is completed. □

We calculate $\mathscr{L}V_2$,

$$\begin{aligned}\mathscr{L}V_2 &= -\frac{\theta_1}{H}\left[\Lambda - dx - f(y,v)x + \rho y\right] + \frac{\theta_2}{H}\left[f(y,v)x - (a+\rho)\,y\right] \\ &\quad + \frac{\theta_3}{H}\left[ky - uv\right] - \frac{1}{2}\left[\left(\frac{\theta_1\sigma_1 x}{H}\right)^2 + \left(\frac{\theta_2\sigma_2 y}{H}\right)^2 + \left(\frac{\theta_3\sigma_3 v}{H}\right)^2\right].\end{aligned}$$

Through Lemma 1.1 we can see, for every sample path $\omega\in\Delta$, a sequence t_n which is increasing and unbounded, such that

$$\limsup_{t\longrightarrow\infty}\langle \mathscr{L}V_2(\omega)\rangle_t = \lim_{n\longrightarrow\infty}\mathscr{L}\langle V_2(\omega)\rangle_{t_n},$$

also for which we can define the following limits :

$$\bar{x} = \lim_{n\longrightarrow\infty}\langle x\rangle_{t_n}, \quad \bar{y} = \lim_{n\longrightarrow\infty}\left\langle\frac{y}{H}\right\rangle_{t_n}, \quad \bar{v} = \lim_{n\longrightarrow\infty}\left\langle\frac{v}{H}\right\rangle_{t_n}, \quad r = \lim_{n\longrightarrow\infty}\left\langle\frac{\frac{\Lambda}{d}-x}{H}\right\rangle_{t_n}.$$

The conditions $\theta_1 r + \theta_2\bar{y} + \theta_3\bar{v} = 1$ can be seen to hold.

Let

$$\Phi(\theta) = \Phi(\theta_1,\theta_2,\theta_3) = \limsup_{t\longrightarrow\infty}\langle \mathscr{L}V_2\rangle_t.$$

Then

$$\begin{aligned}\Phi(\theta) &= \theta_1\left[-dr + (\beta\bar{y}+\alpha\bar{v})\,\bar{x} - \rho\bar{y}\right] + \theta_2\left[(\beta\bar{y}+\alpha\bar{v})\,\bar{x} - (a+\rho)\,\bar{y}\right] \\ &\quad + \frac{\theta_3}{H}\left[k\bar{y} - u\bar{v}\right] - \frac{1}{2}\left[(\theta_1\sigma_1\bar{x})^2 + (\theta_2\sigma_2\bar{y})^2 + (\theta_3\sigma_3\bar{v})^2\right].\end{aligned} \tag{8}$$

Define a parameter as follows :

$$\mathscr{R}_s = \frac{\Lambda(\alpha k+\beta u)}{du(a+\rho)+\hat{g}},$$

whrer $\hat{g}$ is a minimum of the continuous and positive function $g:[0,1]\longrightarrow\mathbb{R}$ defined as

$$g(z) = \frac{\theta_2\theta_3\left[(\sigma_2 z)^2 + \sigma_3^2(1-z)^2\right]}{2\left[\beta\theta_3 z + \alpha\theta_2(1-z)\right]}$$

Theorem 4.1. *If $\mathscr{R}_s < 1$,Then y and v are almost surely converge exponentially to* 0.

Proof 4.2. *For $\theta_2$and θ_3 define in (7) and for $\theta_1 = 0$, we define*

$$H_* = \theta_2 y + \theta_3 v, \quad \textit{and} \quad V_* = \ln H_*.$$

We complet the proof of theorem by prove that

$$\limsup_{t\longrightarrow\infty} \langle \mathscr{L} V_* \rangle_t < 0.$$

Let $\Phi_* = \Phi(0,\theta_2,\theta_3)$. We need to prove that $\Phi_* < 0$. From (8), we have

$$\begin{aligned} \Phi_* &= \theta_2 \left[(\beta\bar{y} + \alpha\bar{v})\bar{x} - (a+\rho)\bar{y} \right] + \frac{\theta_3}{H} \left[ky - u\bar{v} \right] - \frac{1}{2} \left[(\theta_2\sigma_2\bar{y})^2 + (\theta_3\sigma_3\bar{v})^2 \right] \\ &< \theta_2 \frac{\Lambda}{d} (\beta\bar{y} + \alpha\bar{v}) - \theta_4 (\beta\bar{y} + \alpha\bar{v}) - \frac{1}{2} \left[(\theta_2\sigma_2\bar{y})^2 + (\theta_3\sigma_3\bar{v})^2 \right]. \end{aligned}$$

Notice that

$$(\theta_2\sigma_2\bar{y})^2 + (\theta_3\sigma_3\bar{v})^2 = \frac{(\theta_2\sigma_2\bar{y})^2 + (\theta_3\sigma_3\bar{v})^2}{\beta\bar{y} + \alpha\bar{v}} (\beta\bar{y} + \alpha\bar{v}),$$

and $\theta_3\bar{v} = 1 - \theta_2\bar{y}$. Thus

$$\Phi_* < \theta_2 \frac{\Lambda}{d} (\beta\bar{y} + \alpha\bar{v}) - \theta_4 (\beta\bar{y} + \alpha\bar{v}) - g(\theta_2\bar{y}) (\beta\bar{y} + \alpha\bar{v})$$

Therefore we obtain

$$\begin{aligned} \Phi_* &< \theta_2 \frac{\Lambda}{d} (\beta\bar{y} + \alpha\bar{v}) - \theta_4 (\beta\bar{y} + \alpha\bar{v}) - g_* (\beta\bar{y} + \alpha\bar{v}) \\ &= \theta_4 (\mathscr{R}_s - 1) (\beta\bar{y} + \alpha\bar{v}) < 0. \end{aligned}$$

□

Theorem 4.2. *If $\mathscr{R}_s < 1$, then the virus-free equilibrium is almost surely exponentially stable.*

Proof 4.3. *From Theorem 4.1, we have $\lim_{t\to+\infty} y(t) = 0$ a.s. and $\lim_{t\to+\infty} v(t) = 0$ a.s.. suppose that for some subset $\tilde{\Delta}$ of Δ with $\mathbb{P}(\tilde{\Delta}) > 0$, on $\tilde{\Delta}$ we have.*

$$\lim_{t\to+\infty} \left(\frac{\Lambda}{d} - x(t) \right) \neq 0. \tag{9}$$

Therefore, from (6) *and* (8). *In particular we choose $\theta_1 = \theta_2 = \theta_3 = 1$. From (9) and by the definition of $\bar{y}$ and $\bar{v}$, on $\tilde{\Delta}$ we have*

$$\bar{y} = 0 \ \ a.s \quad \textit{and} \quad \bar{v} = 0 \ a.s.$$

So, from (8) *it follows that*

$$\Phi(\theta) \leq -dr - \frac{1}{2}\sigma_1^2\bar{x}^2.$$

Therefore,

$$\Phi(\theta) < 0 \quad a.s.$$

From Proposition 4.1 it follows that on $\tilde{\Delta}$*, we have that*

$$\lim_{t\to+\infty}\left(\frac{\Lambda}{d} - x(t)\right) = 0 \quad a.s.$$

This is a contradiction, the proof is completed. □

5. Numerical Simulation

To illustrate our analytical results, we give numerical simulations of model (2) using Euler scheme[12].

Example 5.1. *We consider Stochastic Virus system with parameters* $\Lambda = 20, d = 0.2, \beta = 0.0008, \alpha = 0.0005, \rho = 0.1, a = 0.02, k = 2, u = 1, \sigma_1 = 0.03, \sigma_2 = 0.04, \sigma_3 = 0.06$*. By calculation, we have* $\mathscr{R}_s^* = 1.4733 > 1$*, in this case, The infected cells and virus persistence in mean Figure 1 illustrates this result. So, if the white noise is large enough such that* $\mathscr{R}_s^* < 1$ *The infected cells and virus go extinction.*

Example 5.2. *In this example, we save the same parameter values as those in example 5.1, besides, we choose* $\sigma_1 = 0.3$*,* $\sigma_2 = 0.4$*,* $\sigma_3 = 0.6$*,* $\rho = 0.17$*. A simple calculation yields that* $\mathscr{R}_s = 0.4986 < 1$*, therefore according to Theorem 4.2, the virus-free equilibrium almost sure exponential stability. as illustrated in Figure 3.*

6. Conclusion

In this paper, we have introduced and analyzed a stochastic virus dynamical model with both cell-to-virus infection and cell-to-cell transmission by including random perturbations of white noise into variables. Firstly, We have shown that the unique global solution of the stochastic system (2) is positive for any initial value. Employing some methods of stochastic analysis we proved that the infected cells and virus persistence in mean if $\mathscr{R}_s^* > 1$. And If $\mathscr{R}_s < 1$ then the virus-free equilibrium is almost sure exponentially stable. We can introduce and investigate a

new version of the stochastic Virus model (2) by introducing a different type of noise which is the Lévy noise. We will investigate this case in our future works.

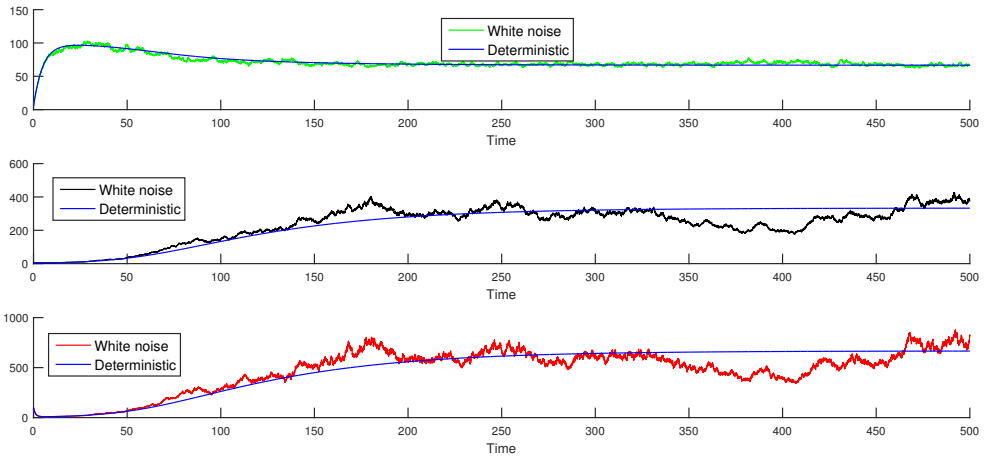

FIGURE 1. Numerical simulation of the path $x(t)$, $y(t)$ and $v(t)$ for the stochastic and deterministic systems as given in Example 5.1.

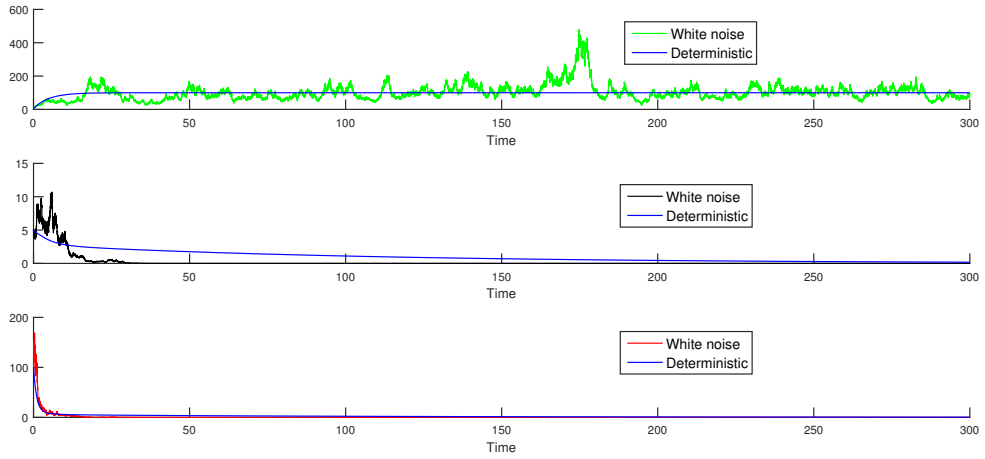

FIGURE 2. Numerical simulation of the path $x(t)$, $y(t)$ and $v(t)$ for the stochastic and deterministic systems as given in Example 5.2.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] R.S. Liptser, A strong law of large numbers for local martingales, Stochastics. 3 (1980), 217–228. https://doi.org/10.1080/17442508008833146.

[2] T. Zhang, X. Meng, T. Zhang, Global dynamics of a virus dynamical model with cell-to-cell transmission and cure rate, Comput. Math. Meth. Med. 2015 (2015), 758362. https://doi.org/10.1155/2015/758362.

[3] P.J. Witbooi, Stability of a stochastic model of an SIR epidemic with vaccination, Acta Biotheor. 65 (2017), 151–165. https://doi.org/10.1007/s10441-017-9308-5.

[4] Y. Zhou, W. Zhang, Threshold of a stochastic SIR epidemic model with Lévy jumps, Physica A: Stat. Mech. Appl. 446 (2016), 204–216. https://doi.org/10.1016/j.physa.2015.11.023.

[5] Q. Liu, D. Jiang, Threshold behavior in a stochastic SIR epidemic model with Logistic birth, Physica A: Stat. Mech. Appl. 540 (2020), 123488. https://doi.org/10.1016/j.physa.2019.123488.

[6] Y. Cai, Y. Kang, M. Banerjee, W. Wang, A stochastic SIRS epidemic model with infectious force under intervention strategies, J. Differ. Equ. 259 (2015), 7463–7502. https://doi.org/10.1016/j.jde.2015.08.024.

[7] Y. Liu, Y. Zhang, Q. Wang, A stochastic SIR epidemic model with Lévy jump and media coverage, Adv. Differ. Equ. 2020 (2020), 70. https://doi.org/10.1186/s13662-020-2521-6.

[8] A. El Koufi, J. Adnani, A. Bennar, N. Yousfi, Analysis of a stochastic SIR model with vaccination and nonlinear incidence rate, Int. J. Differ. Equ. 2019 (2019), 9275051. https://doi.org/10.1155/2019/9275051.

[9] A.E. Koufi, A. Bennar, N. Yousfi, Dynamics of a stochastic SIRS epidemic model with regime switching and specific functional response, Discr. Dyn. Nat. Soc. 2020 (2020), 5898456. https://doi.org/10.1155/2020/5898456.

[10] R. Situ, Theory of stochastic differential equations with jumps and applications, mathematical and analytical techniques with applications to engineering, New York, Springer, (2005).

[11] X. Mao. Stochastic differential equations and applications, Horwood, Chichester, (1997).

[12] P.E. Kloeden, E. Platen, Numerical solution of stochastic differential equations, Springer-Verlag Berlin Heidelberg, (1992).