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A STOCHASTIC EPIDEMIC MODEL WITH GENERAL INCIDENCE RATE CONTROL APPROACH

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Abstract. We develop an SIR epidemic model with a nonlinear incidence rate and treatment function from a deterministic frame to a stochastic one. The optimal control problem for the deterministic model is discussed. We investigate an optimal control problem for the stochastic epidemic model.

Keywords: stochastic epidemic model; Pontryagin's maximum principle; stochastic optimal control.

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1. Introduction

Many researchers have established and studied different types of stochastic epidemic models (See, for example, [1, 2, 3, 4]) . Stochastic models represent an essential means to describe the spread of infectious diseases and can show supplemental realism compared to their corresponding deterministic models [1, 5]. Therefore, many mathematicians have proposed and studied the dynamics of a stochastic generalized SIR epidemic model by using different approaches

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(See, for example, [6, 7, 8, 9, 10]) and with disease modeled by a different incidence rate [11]. We see that the incidence rate is the number of new cases per population at risk in a given period of time at which people transfer from the compartment of susceptible individuals to the compartment of infective individuals. For example, Zhao and Jiang [12] investigated a stochastic SIRS epidemic models with saturated incidence rate. Also, they studied the persistence in the means, extinction of the disease, and the existence of stationary distribution. In [13] the authors studied a generalized stochastic SIR epidemic models with a non-linear incidence rate. Dieu in [14] proposed a stochastic SIR epidemic model with Beddington–DeAngelis incidence rate. They investigated sufficient and almost necessary condition for the permanence of the stochastic epidemic model and the asymptotic global stability of disease free equilibiruim.

On the other hand, the application of optimal control theory in epidemiology represents a significant means for proposing effective strategies that assist governments to get efficient arrangements to fight against infectious diseases while optimizing the cost of the treatment of infection and vaccination. Recently, many scholars have investigated optimal control models and given strategies that minimize the infected population [15, 16, 17]. Laaroussi and Rachik in [18] developed and investigated an optimal control problem for a spatiotemporal SIR epidemic model to minimize the number of infected individuals balanced against the vaccination and treatment costs. In the work [19], the authors investigate the treatment and vaccination employing an optimal control approach to reduce the number of infected individuals. In the work, the authors introduced a stochastic SIVR epidemic and they proposed a control strategy [20].

The rest of this paper is organized as follows. Section 3, presents the deterministic, stochastic system. The deterministic optimal control problem is discussed in Section 3. Section 4, investigate the stochastic optimal control problem for our stochastic model. Finally, in Section 4, we close the paper with a conclusion and future recherche direction.

2. THE MODELS

Rajasekar et al. In [21], introduce and investigate a stochastic SIR model with saturated incidence rate and saturated treatment function where the populations are subdivided into three classes, namely, Susceptible $S(t)$, infected $I(t)$, Recovered $R(t)$.

$$(1)\qquad \begin{cases} \frac{dS(t)}{dt} = A - \rho S(t) - \frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)}, \\ \frac{dI(t)}{dt} = \frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)} - (\rho+\theta+\mu)I(t) - \frac{rI(t)}{1+\eta I(t)}, \\ \frac{dR(t)}{dt} = \mu I(t) + \frac{rI(t)}{1+\eta I(t)} - \rho R(t), \end{cases}$$

where A is the recruitment rate of the population, ρ is the natural death rate of the population, μ is the recovery rate of the infective individuals, θ represents the death rate due to the epidemic. $\frac{\beta SI}{1+kI}$ is the saturated incidence rate of a disease [22], where β is the disease transmission coefficient. $\frac{rI}{1+\eta I}$ is saturated treatment function; r is the cure rate and η measures the effect of delayed treatment for infectious individuals (For more detail see, [23]).

The stochastic differential equations represent a generalization of the ordinary differential equations and integrate the random effects of the systems. Therefore, it has been used in modeling for several fields such as physics, finance, medicine, and biology. Then, the stochastic differential equations have the following form:

$$(2)\qquad dx(t) = f(x(t),t)dt + g(x(t),t)dB(t),\ t \geq 0,\ with\ x(0) = x_0 \in \mathbb{R}^n$$

where $f : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{R}^{n\times m}$. According to [24], under the following condition there exists a unique positive solution $x(t),\ t \geq 0$ to Eqs. 2.

(C): both f and g satisfy the local Lipschitz condition and the linear growth condition. Namely, their exist a positive constant θ_k such that

$$|f(x,t) - f(y,t)| \vee |g(x,t) - g(y,t)| \leq \theta_k |x-y|,$$

for any $t \geq 0$ and those $x, y \in \mathbb{R}^n$ with $x \vee y \leq k$, and there is moreover an $\theta > 0$ such that

$$|f(x,t)| \vee |g(x,t)| \leq \theta(1+x),\ for\ any\ x \in \mathbb{R}^n.$$

In the applied case, the stochastic systems show more precisely the reality by including the environmental effects, which are an essential aspect in biological environments. So epidemic models are often subject to random noise [11]. For this reason, Tornatore et al. in [6] studied the effect of white noise on the *SIR* epidemic model. They investigated a condition for the extinction of disease by showing that the disease-free equilibrium of deterministic system (case where: $\sigma = 0$) is globally asymptotically stable if the infection coefficient β is less than to $\min\left\{\lambda+\mu-\frac{\sigma^2}{2}, 2\mu\right\}$. Note that the proposed condition coincides with the basic reproduction number of the deterministic system (case where: $\sigma = 0$). In [26], Ji et al. have generalized the stochastic model proposed by Tornatore et al. in [25] to a population composed of multiple groups of individuals. The authors showed the existence and uniqueness of a global positive solution to the model on the following set,

$$\Gamma \;=\; \left\{(S_1, I_1, ..., S_n, I_n):\; S_k > 0,\; I_k > 0, S_k + I_k \leq \frac{\Lambda_k}{d_k},\; k = 1, 2, ..., n,\; a.s.\right\}$$

In addition, they proved that the disease-free equilibrium P_0 of system is stochastically asymptotically stable in the large if $\mathscr{R}_0 < 1$ and β_{kj} is irreducible, also they showed that if $\mathscr{R}_0 > 1$ and β_{kj} is irreducible the solution of system fluctuates around the endemic equilibrium of deterministic system.

Then, we consider the following stochastic epidemic model with nonlinear incidence rate

$$(3)\qquad \begin{cases} \frac{dS(t)}{dt} = \left[A - \rho S(t) - \frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)}\right]dt - \frac{\sigma S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)}dW(t), \\ \frac{dI(t)}{dt} = \left[\frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)} - (\rho+\theta+\mu)I(t) - \frac{rI(t)}{1+\eta I(t)}\right]dt \\ \qquad + \frac{\sigma S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)}dW(t), \\ \frac{dR(t)}{dt} = \left[(\mu+u_2(t))I(t) + \frac{rI(t)}{1+\eta I(t)} + u_1(t)S(t) - \rho R(t)\right]dt, \end{cases}$$

where $W(t)$ is independent standard Brownian motions defined on a complete probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}\right)$ with the filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions, and σ represents the intensity of $W(t)$.

3. THE OPTIMAL CONTROL PROBLEM

The optimum control represents a valuable and important tool used to reduce the transmission of epidemic diseases in human populations. In this section, we are interested in a deterministic optimal control problem. For this, we introduce the following deterministic system incorporating control :

$$(4) \qquad \begin{cases} \frac{dS(t)}{dt} = A - \rho S(t) - \frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)} - u_1(t)S(t), \\ \frac{dI(t)}{dt} = \frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)} - (\rho+\theta+\mu+u_2(t))I(t) - \frac{rI(t)}{1+\eta I(t)}, \\ \frac{dR(t)}{dt} = (\mu+u_2(t))I(t) + \frac{rI(t)}{1+\eta I(t)} + u_1(t)S(t) - \rho R(t), \end{cases}$$

where $u(t)$ represent the treatment control of infectious individuals in a time interval $[0,T]$. The optimal control problem is to minimize the following objective functional

$$(5) \qquad J(u_1,u_2) = \int_0^T \left(I(t) + \frac{B_1}{2}u_1^2(t) + \frac{B_2}{2}u_2^2(t) \right) dt,$$

where B_1 and B_2 are the cost coefficients, and $S(0),\ I(0),\ R(0) \geq 0$. The terms $\frac{B_1}{2}u_1^2$ and $\frac{B_2}{2}u_2^2$ are associated with the vaccination control u_1 and infection treatment control u_2, respectively. Minimizing equation (5) gives an optimal controls u_1^* and u_2^* such that

$$J(u_1^*,u_2^*) = \min_{u\in U} J(u_1(t),u_2(t)),$$

with admissible controls set are given by

$$U = \{(u_1,u_2) :\ u_i \text{ are Lebesgue measurable},\ 0 \leq u_i(t) \leq 1,\ 0 \leq t \leq T,\ i = 1,2\},$$

and subjected to the constraint presented by the system (4). To determine the exact formulation of our optimal control u_1 and u_2. We use Pontryagin's Maximum Principle (see,[28]), which converts equations (4) and (5) in a problem of minimizing. The Hamiltonian is defined by

$$H(S,I,R,u) = L(I,u_1,u_2) + \sum_{i=1}^{3} l_i(t) f_i(S,I,R).$$

With L is the Lagrangian of the problem given by

$$L(I,u_1,u_2) = I(t) + \frac{B_1}{2}u_1^2(t) + + \frac{B_2}{2}u_2^2(t),$$

and f_i is the right side of the difference equation of the i^{th} state variable, $l_i, (i = 1,2,3)$ are the adjoint variables.

Concerning the existence of an optimal control we use the result in (Lukes, [27]), and we obtain the following theorem.

Theorem 3.1. *Consider the control problem with system (4). There exists an optimal control* $(u_1^*, u_2^*) \in U$ *such that*

$$J(u_1^*, u_2^*) = \min_{u \in U} J(u_1, u_2)$$

Proof. To establish the existence of an optimal control it is simple to check that

- **I.:** The set of controls and corresponding state variables is nonempty,
- **II.:** The admissible set U is convex and closed,
- **III.:** The right hand side of the state system is bounded by a linear function in the state and control variables,
- **IV.:** The integrand of the objective functional is convex on U,
- **V.:** There exist constants $\varpi_1 > 0$, $\varpi_2 > 0$, and $q > 1$ such that the integrand $L(S, I, R, u)$ of the objective functional verifies,

$$L(S, I, R, u) \geq \varpi_2 + \varpi_1 (|u|^2)^{q/2}$$

indeed, we mention that the set of admissible controls U is closed and bounded and also convex by definition □

By using Pontryagin's Maximum Principle, we obtain the following theorem

Theorem 3.2. *Given the optimal controls* u_1^* *and* u_2^* *the solutions* S^*, I^* *and* R^* *of the corresponding state system (4), there exists adjoint variables* l_1, l_2 *and* l_3 *satisfying :*

$$\begin{aligned}
\frac{dl_1}{dt} &= \frac{(l_1(t) - l_2(t))\beta I^*(t)(1 + k_1 I^*(t))}{(1 + k_1 S^*(t) + k_2 I^*(t) + k_3 S^*(t) I^*(t))^2} + l_1(t)\rho + (l_3(t) - l_1(t))u_1^*(t), \\
\frac{dl_2}{dt} &= \frac{(l_1(t) - l_2(t))\beta S^*(t)(1 + k_1 S^*(t))}{(1 + k_1 S^*(t) + k_2 I^*(t) + k_3 S^*(t) I^*(t))^2} + l_2(t)(\rho + \theta + \mu + u_2^*(t)) - 1 \\
&\quad + (l_2(t) - l_3(t))\frac{r}{(1 + \eta I^*(t))^2} - l_3(t)(\mu + u_2^*(t)), \\
\frac{dl_3}{dt} &= l_3(t)\rho, \qquad (6)
\end{aligned}$$

with the transversality conditions at time T

$$l_i(t) = 0, \;\; i = 1, 2, 3.$$

Moreover, the optimal controls u_1^* and u_2^* are given by

$$u_1^*(t) = \min\left(1, \max\left(0, \frac{(l_1(t)-l_3(t))S^*}{B_1}\right)\right), \tag{7}$$

$$u_2^*(t) = \min\left(1, \max\left(0, \frac{(l_2(t)-l_3(t))I^*}{B_2}\right)\right), \tag{8}$$

for $t \in [0,T]$

Proof. The adjoint equations and transversality conditions can be obtained by using Pontryagin's maximum principle [1] such that

$$\frac{dl_1}{dt} = -\frac{\partial H}{\partial S}, \; \frac{dl_2}{dt} = -\frac{\partial H}{\partial I}, \frac{dl_3}{dt} = -\frac{\partial H}{\partial R}, \text{ with } l_i(T)=0, \text{ for } i=1,2,3. \tag{9}$$

The optimal controls u_1^* and u_2^* can be solved from the optimality condition:

$$\begin{aligned}\frac{\partial H}{\partial u_1} &= B_1u_1(t) - (l_3(t)-l_1(t))S(t) = 0,\\ \frac{\partial H}{\partial u_2} &= B_2u_2(t) - (l_2(t)-l_3(t))I(t) = 0.\end{aligned}$$

That are

$$\begin{aligned}u_1(t) &= \frac{(l_3(t)-l_1(t))S(t)}{B_1},\\ u_2(t) &= \frac{(l_2(t)-l_3(t))I(t)}{B_2}.\end{aligned}$$

By the bounds in U of the controls, it is easy to obtain u_1^* and u_2^* are given in (7) and (16) the form of system (4) . □

4. The Stochastic Optimal Control

In this section, we give the stochastic optimization problem and demonstrate its solution. We consider the following system

(10)
$$\begin{cases}\frac{dS(t)}{dt} = \left[A - \rho S(t) - \frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)} - u_1(t)S(t)\right]dt - \frac{\sigma S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)}dW(t),\\ \frac{dI(t)}{dt} = \left[\frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)} - (\rho+\theta+\mu+u_2(t))I(t) - \frac{rI(t)}{1+\eta I(t)}\right]dt\\ \qquad + \frac{\sigma S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)}dW(t),\\ \frac{dR(t)}{dt} = \left[(\mu+u_2(t))I(t) + \frac{rI(t)}{1+\eta I(t)} + u_1(t)S(t) - \rho R(t)\right]dt,\end{cases}$$

Our aim is to obtain optimal controls for infection treatment and vaccination that minimizes the objective functional which for an initial state x_0 is defined as follows

$$\mathbb{E}_{0,x_0}\left[\int_0^T \left(I(t)+\frac{B_1}{2}u_1^2(t)+\frac{B_2}{2}u_2^2(t)\right)dt\right] \tag{11}$$

Such that the earlier deterministic optimal problem, we assume that there is a nonnegative constants $u_{i,max} \leq 1$, (i=1,2) such that $u_i(t) \leq u_{i,max}$, (i=1,2) a.s. The class of admissible control law is

$$\mathscr{M} = \{u = (u_1, u_2) : u_i \text{ are adapted, and } 0 \leq u_i(t) \leq u_{i,max} \text{ a.s., } (i = 1,2)\} \tag{12}$$

We define the following performance criterion to solve the stochastic control problem

$$\mathscr{X}(t,x,u_1,u_2) = \mathbb{E}_{0,x}\left[\int_0^T \left(I(t)+\frac{B_1}{2}u_1^2(t)+\frac{B_2}{2}u_2^2(t)\right)dt\right], \tag{13}$$

note that the expectation is conditional on the state of the system being a fixed value x at time t. The optimal controls u and the value function represented as follows

$$\Psi(t,x) = \inf_{u\in\mathscr{M}} \mathscr{X}(t,x,u) = \mathscr{X}(t,x,u^*) \tag{14}$$

We set a control law that minimizes the value $\mathscr{X} : \mathscr{M} \longrightarrow \mathbb{R}_+$ presented by (13). In order to show the solution of our stochastic optimal control problem for the system (10) given $\mathscr{M}$ in (12) and $\mathscr{X}$ in (13). We aim to find the value function Ψ defined in (15) and an optimal controls function

$$u^*(t) = \arg\inf_{u\in\mathscr{M}} \mathscr{X}(t,x,u) \in \mathscr{M}. \tag{15}$$

Now we give in the following theorem the expression for the optimal controls u_1^* and u_2^*.

Theorem 4.1. *A solution to the optimal vaccination and treatment problem stated in Problem (12) is of the form*

$$\begin{aligned} u_1^*(t) &= \min\left\{1, \max\left\{0, \frac{(\Psi_S-\Psi_R)S(t)}{B_1}\right\}, u_{1,m}\right\}, \\ u_2^*(t) &= \min\left\{1, \max\left\{0, \frac{(\Psi_I-\Psi_R)S(t)}{B_2}\right\}, u_{2,m}\right\}. \end{aligned} \tag{16}$$

Proof. Note that, using the dynamic programming approach we can determine (16). Then, Applying the Itô's formula on Ψ, we get

$$\begin{aligned}
\mathscr{L}\Psi &= \left[A - \rho S(t) - \frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)} - u_1(t)S(t)\right]\Psi_S(t) \\
&+ \left[\frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)} - (\rho+\theta+\mu+u_2(t))I(t) - \frac{rI(t)}{1+\eta I(t)}\right]\Psi_I(t) \\
&+ \left[(\mu+u_2(t))I(t) + \frac{rI(t)}{1+\eta I(t)} + u_1(t)S(t) - \rho R(t)\right]\Psi_R(t) \\
&\frac{1}{2}\left(\frac{\sigma S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)}\right)^2\Psi_{SS}(t) \\
&+\frac{1}{2}\left(\frac{\sigma S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)}\right)^2\Psi_{II}(t) \\
&-\left(\frac{\sigma S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)}\right)^2\Psi_{SI}(t).
\end{aligned}$$

Applying the Hamilton-Jacobi-Bellman theory, we must determine the infinitum as follows

$$\inf_{u\in\mathscr{M}}\left[I(t) + \frac{B_1}{2}u_1^2(t) + \frac{B_2}{2}u_2^2(t)\right]. \tag{17}$$

Then, we establish the necessary condition for the optimal control. Via (), we get

$$B_1u_1(t) + (\Psi_R - \Psi_S)S(t) = 0,$$

$$B_2u_2(t) + (\Psi_R - \Psi_I)I(t) = 0.$$

We consider the bounds u_1 and u_2 and with the identical reason in the deterministic characterization; the asserted expressions emerge. □

5. Conclusion

In this paper, we studied an optimal control problem for the proposed stochastic SIR epidemic model with a nonlinear incidence rate and treatment function. Two different control are presented to characterize the effectiveness of treatment, and the effectiveness of vaccination for the susceptible individuals. The objectify of the presented optimal controls problem is to minimizing the number of infected individuals and the cost(s) of controls. we shown approximate solutions for optimal stochastic control by exploiting the similarity between the two controls forms: deterministic and Stochastic.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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