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OPTIMAL CONTROL FOR A DISCRETE TIME EPIDEMIC MODEL WITH ZONES EVOLUTION

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Abstract. In this paper, we present a new mathematical model to describe the evolution of an infectious disease in regions and between individuals. For this purpose we considered two systems, the first one for humans $S_iI_iR_i$, where S_i represents the number of susceptible, I_i of infected and R_i of cured. The second system $Z_i^S Z_i^I Z_i^R$ represents the different types of regions, where Z_i^s is the number of susceptible regions, where there are only susceptible people, after visiting an infected person, a susceptible region is likely to be infected, which we will note Z_i^I, the last compartment Z_i^R denotes the infected regions, which are restored after the recovery of all infected people. In addition, we considered three control strategies u, v and w to control the spread of the virus within regions and between individuals. Numerical examples are provided to illustrate the effectiveness of our proposed control strategy.

Keywords: mathematical model; discrete-time systems; optimal control; Covid-19; contagious virus; Pontryagin maximum.

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1. INTRODUCTION

The world has witnessed various large epidemics and pandemics that have the ability to progress in time and space and exceed the usual level of morbidity in a given territory. Nowadays, pandemics spread over the territory of an entire country, several countries, or sometimes even beyond the borders of one continent. These contagious diseases have become widespread and affect a significant proportion of the population [1, 2]. Thus, serious challenges should be faced when a disease outbreaks, especially in international capacity for prevention measures and response strategies [3]. Epidemics have been a threat to human lives and have caused numerous losses in every field of society including economics. Therefore, relevant policies, counter measures, and control approaches should be taken into consideration in order to prevent the continuous spread of an epidemic [4]. Policy makers need to take a wise response to the outbreak with limited resources, along with balancing investments in public health and curative services, beside enhancing adequate measures related to the pandemic prevention [5, 6]. The study of epidemic spread mechanisms helps to combat and cease the impact of infectious diseases, along with the search for new drugs, vaccination and preventive measures [7].Therefore, mathematics have been intertwined with every major discipline in the biological and biomedical sciences including epidemiology that aimed for describing the transmission process of an epidemic and providing further understanding in disease transmission and spread mechanisms [8, 9]. Epidemiological modeling has been widely used especially after its greater result in controlling the propagation of infectious diseases within defined population and for various geographical regions. Moreover, it provides an estimate for the potential scale of an epidemic, and helps to pinpoint key factors in the disease transmission process, then recommend effective control and preventive measures [10, 11]. The application of mathematical models submits numerous fundamental perceptions into epidemic outbreaks and their control, it is nearly very hard to conclude these assumptions only from infection data [12]. The most and simplest model used in this regard is the Susceptible-Infectives-Removed (SIR) model initially proposed by Kermack and McKendrick stands for the idea of passing from “susceptible” to “infected” when getting contact to an infectious material. The model is called SIR model when it concerns recovery with permanent immunity and the entire population that is susceptible to the epidemic

is divided into three disjoint groups: S (susceptible), I (infected), R (recovered or removed) [13, 14]. S- Susceptible means healthy, not contagious, but can get infected, because they do not have immunity, I- Infected people who are sick and have a contagious symptoms and have the ability to spread the infection, R- Recovered healthy, not contagious and can not get infected, because they have immunity. The group of susceptibles includes part of the population confirmed infection, but not yet sick. Therefore, the number of susceptible decreases over the account of those who became infected, and group of infected decreases at the expense of those who recovered or died. The rate of infection is proportional to the number of infected individuals, and the number of susceptible. Each infected person, in turn, has a constant chance of recovery per unit of time [15, 16]. The submission of quarantine measures on the epidemic transmission process using epidemic models allows to estimate the parameters of the model, such as the rates of morbidity and recovery, which can be used in more complex models spatio-temporal epidemiological models [17, 18]. Such models make it possible to take into account the uneven distribution of the population, changes in population mobility and the frequency of susceptible-infected contacts due to quarantine measures. Using the obtained analytical and numerical solutions, various stages of the epidemic, as well as its undulating nature helps to determine characteristics, dynamics and impact of pandemics, as well as performance assessments events in various settings [19, 20]. Practically, many epidemiological models are made to control the development of infectious diseases such as SARS [21], HIV [22], Ebola [23], Influenza [24], Tuberculosis [25], Cholera [26], Measles [27] and others [28, 29]. However, with the effect of spatio-temporal spread of epidemics, mathematical modeling should take into account the geographical criterion to show the spatial spread of an infectious disease within different geographical zones. The aspects of time and space shape any pandemic that could be transmitted from one region to another [30, 31]. The history is full of such examples, the black death is the most famous epidemic that comes in mind when talking about epidemics in history, since it wiped out a large part of the European population and spread across North Africa and the island of Greenland. The disease had been raging in the East since about 1320, it destroyed up 90% of the population of the Chinese province of Hobei, affected India and Central Asia,

and gradually spread throughout Europe through trade [32, 33]. The Ebola epidemic in 2013-2015 West Africa started in Guinea and spreading between countries across land borders, it moved from Sierra Leone to Liberia, and by air to Nigeria and by land to Senegal, cases keep increasing and moving to Spain, the United Kingdom, and the United States [34, 35]. After that the authorities of some neighboring states closed their borders with countries affected by the epidemic. Among them are Kenya, Cameroon, Senegal and South Africa. Mauritania closed its land border with Mali in October after detecting the first case of the virus there [36]. In [37] a study has been made to show that travel restriction especially air flights in the United States in 2001 was effective to delay influenza dynamics by 2 weeks. In the recent decades, humans are moving from one place to another, thereby becoming exposed to a variety of infectious diseases [38]. The recent virus Covid-19 known as severe acute respiratory syndrome coronavirus is an obvious illustration of pandemics that have the ability to spread through continents, it starts from its origin in Wuhan, Hubei Province, China then expand to Asia. In Europe, France was reported as the first official case of the virus, followed by Germany and Finland, in a short period of time all 27 countries of the European Union were affected. and cases keep moving across the world [39, 40]. The beginning of January 2020 travel restrictions were introduced in China, this fact reduces epidemic progression in China, thus the huge effect was obvious internationally, because it starts to slow the speed of infection transmission, travel restriction has no doubt a remarkable impact on the spread of epidemics especially in the first phase. Modeling results have shown that the travel restriction to and from China has an effect on case importation that were reduced approximately to 80% [41]. When the delta variant was Shown in UK, European countries were rushing to close their borders with the United kingdom to prevent the entry of a new and potentially more transmissible variant of SARS-CoV-2, the most models show that travel and flight bans and border closures reduced the arrival of people with COVID-19 to many countries at the start of the outbreak [42]. A model is made in [43] using a data from China to explain how travel policies affect the dynamics of the national and international spread of this pandemic. The multi-regional SIR model in [45, 46] as a discrete time model and an accurate illustration of mathematical modeling of temporal and spatial expand of pandemics, the infection moves from one area to another due to air travel, the mathematical model is used in a variety

of geographical areas in order to control the way in which an epidemic moves within different geographical zones. Travel restriction policies may provide a significant delay in transmission of infectious diseases, especially along with application of other preventive measures, thus such preventive measures may not always reduce the speed of a pandemic spread, delays could affect the regional epidemics and enhance the epidemic to a long pandemic season. Their effectiveness requires actually the absence of any cases in the isolated territory, beside the total travel bans to stop cross-territory movements. However, not only restrictive measures played a role, but also active educational and explanatory work with the population. Some control systems can be found in the following references [47, 48, 49].

The rest of the paper is organized as follows. In section 2, we present the discrete time $SIRZ^SZ^IZ^R$ mathematical model describing the evolution of a contagious virus that takes into account its between individuals and regions. In addition, we provide a numerical simulation without control to our model.The optimal control problem of the considered model is investigated in section 3.Results and discussion are provided to ensure the effectiveness of our strategies of controls in section 4. To conclude our article, a conclusion is given in section 5.

2. THE MODEL WITH VACCINATION AND TRAVEL-BLOCKING

2.1. Presentation of the model. The spread of an infectious disease makes us think several times before choosing and deciding the travel destinations. Our model is divided into two systems, the first $S_iI_iR_i$ of human, where S represents the number of susceptible, I infected and R the recovered. the compartment of the second system $Z_i^sZ_i^IZ_i^R$ represents the different types of regions, $Z_i^s$is the number of susceptible regions, where there are only people susceptible, after visiting an infected person, a susceptible area is at risk of becoming infected , that we will note Z_i^I, the last compartment Z_i^Rdesignates the infected areas, which are restored after the recovery of all infected. The model resulting from these assumptions is governed by the following system

$$\begin{aligned} S_{i+1} &= \Lambda + S_i - \alpha S_i I_i - \mu S_i \\ I_{i+1} &= I_i + \alpha S_i I_i - \mu I_i - rI_i - \rho I_i \\ R_{i+1} &= R_i + \rho I_i - \mu R_i \end{aligned} \tag{1}$$

$$Z_{i+1}^{S} = Z_i^s - \beta Z_i^s I_i + \theta Z_i^R \tag{2}$$

$$Z_{i+1}^{I} = Z_i^I + \beta Z_i^s I_i - \gamma Z_i^I$$

$$Z_{i+1}^{R} = Z_i^R + \gamma Z_i^I - \theta Z_i^R \tag{3}$$

with initial conditions $S_0 \geq 0, I_0 \geq 0, R_0 \geq 0, Z_0^S \geq 0, Z_0^I \geq 0$ and $Z_0^R \geq 0$ and where $i \in \{0,1,...,N-1\}$.

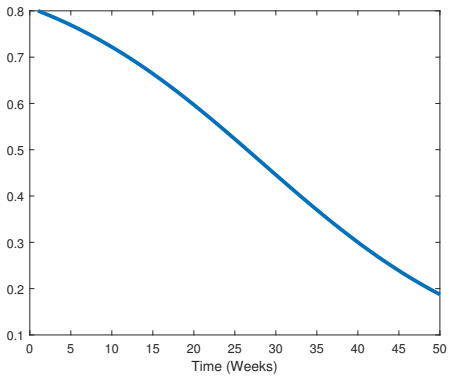
After one unit of time, susceptible individuals (S) may remain in the susceptible compartment, or become infected with a contact with infectious individuals $\alpha S_i I_i$ and move to the infectious compartment (I), or die at a per capita μS_i. The infected individuals (I) may stay in the infected compartment or get recovered at per capita ρI and move to the recovered compartment (R), or die due to the infection at a per capita rI_i, or die naturally. The recovered individuals may remain in its compartment or die. On the other words, the susceptible regions (Z^S) may remain in this compartment or become infected regions (Z^I) since visiting infected person $\beta Z_i^S I_i$ and then move to the compartment Z^I. The infected regions stay infected or get recovered at per capita γZ_i^I. The last compartment is Z^R, θ is the losing removal individuals' immunity rate.

Parameters description can be found in Table 1.

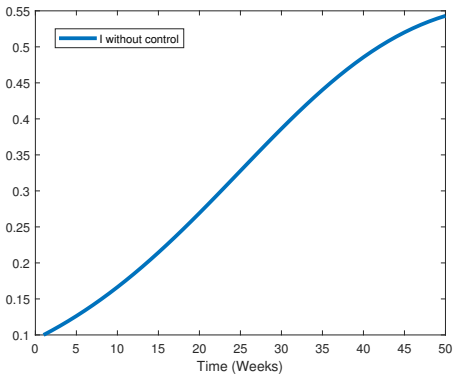
Parameter	Physical interpretation
Λ	The incidence of susceptible
α	The rate of people who were infected by contact with infected.
β	The rate of regions which become infected since infected people.
θ	The rate of convertion from recovered regions to susceptible regions.
μ	The natural death rate.
r	Mortality since the virus.
ρ	The rate of individuals who were recovered from the disease.
γ	The rate of regions who were recovered from the disease.

TABLE 1. List of all parameters of system (1)

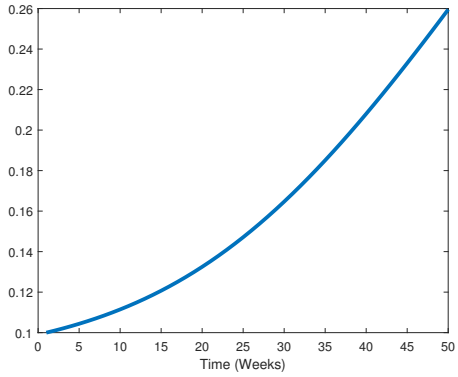
2.2. Simulation without control. In this part we are interessted of plotting the graphs of susceptible, infected and recovered people in 50 weeks in order to visualize their behavior. In addition, we will plot the graphs of susceptible, infected and recovered.



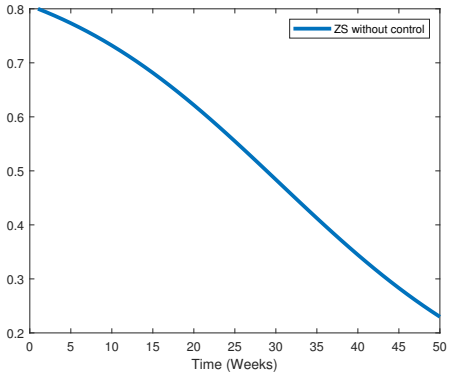
(A) Susceptible people



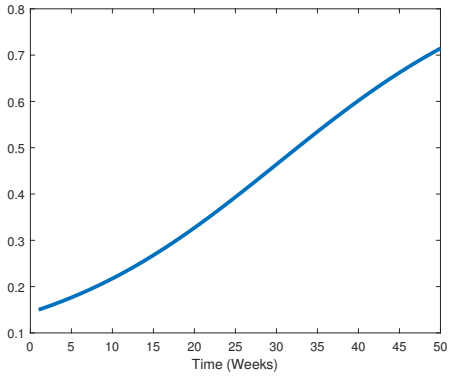
(B) Infected people



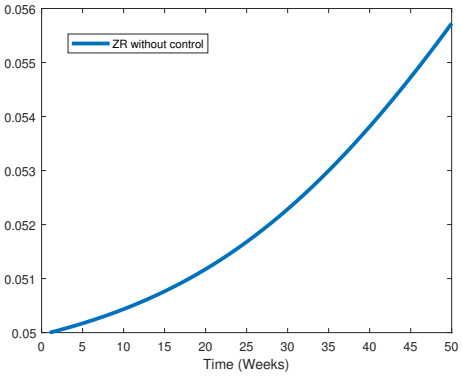
(C) Recovered people



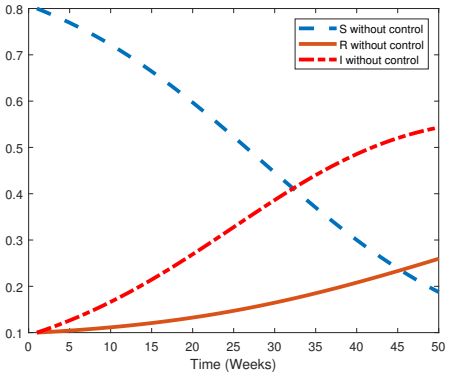
(D) Susceptible regions



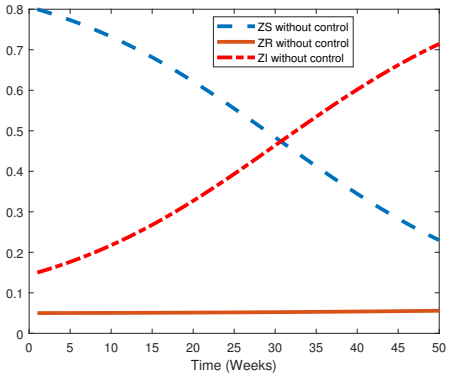
(E) Infected regions



(F) Recovered regions



(G) The susceptible, infected and recovered people.



(H) The susceptible, infected and recovered regions.

Figure 1

In figure 1a, we remark that the number of susceptible people decrease with time and this happened because these people get the virus and consequently become infected, which obviously schown in figure 1b. We constate again in figure 1d that the susceptible regions decrease with time, this due to these regions become infected after visiting of infected individuals and this is shown in figure 1e. In figure 1c, we remark that the recovered people weakly increasing and that clearly appears in figure 1g.

3. THE OPTIMAL CONTROL PROBLEM

3.1. Presentation of the controls. The discrete-time controlled system associated with (1) is given as follows

$$\begin{aligned} S_{i+1} &= \Lambda + S_i - \alpha u_i S_i I_i - \mu S_i - v_i S_i \\ I_{i+1} &= I_i + \alpha u_i S_i I_i - \mu I_i - r I_i - \rho I_i \qquad (4) \\ R_{i+1} &= R_i + \rho I_i - \mu R_i + v_i S_i \end{aligned}$$

$$Z^s_{i+1} = Z^s_i - \beta Z^s_i I_i + \theta Z^R_i \qquad (5)$$

$$Z^i_{i+1} = Z^I_i + \beta w_i Z^s_i I_i - \gamma Z^I_i$$

$$Z^R_{i+1} = Z^R_i + \gamma Z^I_i - \theta Z^R_i \qquad (6)$$

with initial conditions $S_0 \geq 0, I_0 \geq 0, R_0 \geq 0, Z^S_0 \geq 0, Z^I_0 \geq 0$ and $Z^R_0 \geq 0$ and where $i \in \{0,1,...,N-1\}$.

The strategie of control v represent the vaccination of susceptible individuals and thus protect them of the virus, while the strategies of controls u and w represent the travel-blocking where we block the meeting between susceptible and infected people and the visiting of infected poeple to the susceptible regions.

3.2. Objective functional. The problem is to minimize the objective functional $J(u,v,w)$ given by

$$J(u,v,w) = (\alpha^I I_N - \alpha^R R_N + \alpha^Z Z^I_N) + \sum_{i=0}^{N-1} \left(\alpha^I I_i - \alpha^R R_i + \alpha^Z Z^I_i + \frac{A}{2}(u_i)^2 + \frac{B}{2}(v_i)^2 + \frac{C}{2}(w_i)^2 \right)$$

where $A > 0, B > 0, C > 0, \alpha^I > 0, \alpha^R > 0, \alpha^Z > 0$ are the weight constants of controls, $u = (u_0,...,u_{N-1})$,$v = (v_0,...,v_{N-1})$ and $w = (w_0,...,w_{N-1})$, and N is the final time of our

strategy of control. Our goal is to minimize the infectd individuals and the infected regions, minimize the cost of applying controls and increase the number of removed. In other words, we are seeking optimal controls u^*, v^* and w^* such that

$$J(u^*,v^*,w^*) = min\{J(u,v,w)/u \in \mathscr{U}, v \in \mathscr{V}, w \in \mathscr{W}\} \tag{7}$$

where $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$ are the control sets defined by

$$\mathscr{U} = \{u/u_{min} \leq u_i \leq u_{max}, i = 0,...,N-1\} \tag{8}$$

$$\mathscr{V} = \{v/v_{min} \leq v_i \leq v_{max}, i = 0,...,N-1\} \tag{9}$$

$$\mathscr{W} = \{w/w_{min} \leq w_i \leq w_{max}, i = 0,...,N-1\} \tag{10}$$

such that $0 < u_{min} < u_{max} < 1$, $0 < v_{min} < v_{max} < 1$ and $0 < w_{min} < w_{max} < 1$.

3.3. Sufficient conditions.

Theorem 3.1. *There exists an optimal control* $(u^*,v^*,w^*) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}$ *such that*

$$J(u^*,v^*,w^*) = min\{J(u,v,w)/u \in \mathscr{U}, v \in \mathscr{V}, w \in \mathscr{W}\}$$

subject to the control system (1) and initial conditions.

Proof. Since the parameters of the system are bounded and there are a finite number of time steps, that is I, S, R, Z^S, Z^I and Z^R are uniformly bounded for all (u,v,w) in the control set $\mathscr{U} \times \mathscr{V} \times \mathscr{W}$, thus $J(u,v,w)$ is also bounded for all $(u,v,w) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}$. Which implies that $\inf_{(u,v,w)\in \mathscr{U}\times\mathscr{V}\times\mathscr{W}} J(u,v,w)$ is finite, and there exists a sequence $(u^n,v^n,w^n) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}$ such that

$$\lim_{n\to+\infty} J(u^n,v^n,w^n) = \inf_{(u,v,w)\in\mathscr{U}\times\mathscr{V}\times\mathscr{W}} J(u,v,w)$$

and corresponding sequences of states I^n, S^n, R^n and Z^{Sn}, Z^{In}, Z^{Rn}. Since there is a finite number of uniformly bounded sequences, there exists $(u^*,v^*,w^*) \in \mathscr{U} \times \mathscr{V} \times \mathscr{W}$ and I^*, S^*, R^* and Z^{S*}, Z^{I*}, Z^{R*} such that, on a sequence,

$$(u^n, v^n, w^n) \to (u^*, v^*, w^*)$$

$$I^n \to I^*$$

$$S^n \to S^*$$

$$R^n \to R^*$$

$$Z^{Sn} \to Z^{s*}$$

$$Z^{In} \to Z^{I*}$$

$$Z^{Rn} \to Z^{R*}$$

Finally, due to the finite dimensional structure of the system (1) and the objective function $J(u,v,w)$, (u^*, v^*, w^*) is an optimal control with corresponding states I^*, S^*, R^*,Z^{S*},Z^{I*} and Z^{R*}. This completes the proof. □

3.4. Necessary conditions. By using a discrete version of the Pontryagin's maximum principle [44], we derive necessary conditions for our optimal controls. For this purpose, we define the Hamiltonian as

$$\begin{aligned}\mathscr{H}_i &= \left(\alpha^I I_i - \alpha^R R_i + \alpha^Z Z_i^I + \frac{A}{2}(u_i)^2 + \frac{B}{2}(v_i)^2 + \frac{C_k}{2}(w_i)^2\right) + (\zeta_{1,i+1}\left[S_i - \alpha u_i S_i I_i - \mu S_i - v_i S_i\right] \\ &\quad \zeta_{2,i+1}\left[I_i + \alpha u_i S_i I_i - \mu I_i - r I_i - \rho I_i\right] + \zeta_{3,i+1}\left[R_i + \rho I_i - \mu R_i + v_i S_i\right] \\ &+ \quad \zeta_{4,i+1}\left[Z_i^s - \beta w_i Z_i^s I_i + \theta Z_i^R\right] + \zeta_{5,i+1}\left[Z_i^I + \beta w_i Z_i^s I_i - \gamma Z_i^I\right] + \zeta_{6,i+1}\left[Z_i^R + \gamma Z_i^I - \theta Z_i^R\right])\end{aligned}$$

Theorem 3.2. *Given optimal controls u^*, v^*, w^* and solutions I_i^*, S_i^*, R_i^*,Z_i^{S*},Z_i^{I*} and Z_i^{R*}, there exists $\zeta_{k,i}$, $i = 1...N$, $k = 1,2,...,6$, the adjoint variables satisfying the following equations*

$$\begin{aligned}\Delta\zeta_{1,i} &= -\zeta_{1,i+1}\left[1 - \alpha I_i u_i - \mu - v_i\right] - \zeta_{2,i+1}\left[\alpha u_i I_i\right] - \zeta_{3,i+1} v_i \\ \Delta\zeta_{2,i} &= -\alpha^I - \zeta_{1,i+1}\left[-\alpha S_i u_i\right] - \zeta_{2,i+1}\left[1 + \alpha u_i S_i - \mu - r - \rho\right] \\ &\quad -\zeta_{3,i+1}\left[\rho\right] - \zeta_{4,i+1}\left[\beta w_i Z_i^s\right] - \zeta_{5,i+1}\left[\beta w_i Z_i^s\right] \\ \Delta\zeta_{3,i} &= -\left[-\alpha^R + \zeta_{3,i+1}\left[1 - \mu\right]\right] \\ \Delta\zeta_{4,i} &= -\left[\zeta_{4,i+1}\left[1 - \beta w_i I_i\right] + \zeta_{5,i+1}\left[\beta w_i I_i\right]\right]\end{aligned}$$

$$\begin{aligned}\Delta\zeta_{5,i} &= -\left[\alpha^Z + \zeta_{5,i+1}\left[1-\gamma\right] + \zeta_{6,i+1}\left[\gamma\right]\right]\\ \Delta\zeta_{6,i} &= -\left[\zeta_{6,i+1}(1-\theta) + \zeta_{4,i+1}\theta\right]\end{aligned}$$

where $\zeta_{1,N}=0, \zeta_{2,N}=\alpha^I, \zeta_{3,N}=-\alpha^R, \zeta_{4,N}=0, \zeta_{5,N}=\alpha^Z, \zeta^j_{6,N}=0$, *are the transversality conditions. In addition* $u^*=\left(u^*_0,...,u^*_{N-1}\right)$, $v^*=\left(v^*_0,...,v^*_{N-1}\right)$ *and* $w^*=\left(w^*_0,...,w^*_{N-1}\right)$ *are given by*

$$\begin{aligned}u^*_i &= min\left\{max\left\{u_{min}, \frac{\left(\zeta_{1,i+1}-\zeta_{2,i+1}\right)\left[\alpha S_i I_i\right]}{A}\right\}, u_{max}\right\}, i=1,...,N-1,\\ v^*_i &= min\left\{max\left\{v_{min}, \frac{\left(\zeta_{1,i+1}-\zeta_{3,i+1}\right)S_i}{B}\right\}, v_{max}\right\}, i=0,...,N-1,\\ w^*_i &= min\left\{max\left\{w_{min}, \frac{\left(\zeta_{4,i+1}-\zeta_{5i+1}\right)\beta Z^s_i I_i}{C}\right\}, w_{max}\right\}, i=0,...,N-1,\end{aligned} \tag{11}$$

Proof. Using the discrete version of the Pontryagin's maximum principle [44], we obtain the following adjoint equations:

$$\begin{aligned}\Delta\zeta_{1,i} &= -\frac{\partial\mathscr{H}_i}{\partial S_i}\\ &= -\left[\zeta_{1,i+1}\left[1-\alpha u_i I_i-\mu-v_i\right]+\zeta_{2,i+1}\left[\alpha u_i I_i\right]+\zeta_{3,i+1}v_i\right]\\ \Delta\zeta_{2,i} &= -\frac{\partial\mathscr{H}_i}{\partial I_i}\\ &= -\left[\alpha^I+\zeta_{1,i+1}\left[-\alpha u_i S_i\right]+\zeta_{2,i+1}\left[1+\alpha u_i S_i-\mu-r-\rho\right]\right.\\ &\quad \left.+\zeta_{3,i+1}\left[\rho\right]+\zeta_{4,i+1}\left[\beta w_i Z^s_i\right]+\zeta_{5,i+1}\left[\beta w_i Z^s_i\right]\right]\\ \Delta\zeta_{3,i} &= -\frac{\partial\mathscr{H}_i}{\partial R_i}\\ &= -\left[-\alpha^R+\zeta_{3,i+1}\left[1-\mu\right]\right]\\ \Delta\zeta_{4,i} &= -\frac{\partial\mathscr{H}_i}{\partial Z^S_i}\\ &= -\left[\zeta_{4,i+1}\left[1-\beta w_i I_i\right]+\zeta_{5,i+1}\left[\beta w_i I_i\right]\right]\\ \Delta\zeta_{5,i} &= -\frac{\partial\mathscr{H}_i}{\partial Z^I_i}\\ &= -\left[\alpha^Z+\zeta_{5,i+1}\left[1-\gamma\right]+\zeta_{6,i+1}\left[\gamma\right]\right]\\ \Delta\zeta_{6,i} &= -\frac{\partial\mathscr{H}_i}{\partial Z^R_i}\\ &= -\left[\zeta_{6,i+1}(1-\theta)+\zeta_{4,i+1}\theta\right]\end{aligned}$$

with $\zeta_{1,N} = 0, \zeta_{2,N} = \alpha^I, \zeta_{3,N} = -\alpha^R$. To obtain the optimality conditions we take the variation with respect to controls (u_i, v_i, w_i) and set it equal to zero

$$\frac{\partial \mathscr{H}_i}{\partial u_i} = A\,u_i - (\zeta_{1,i+1} - \zeta_{2,i+1})\,[\alpha S_i I_i] = 0$$

$$\frac{\partial \mathscr{H}_i}{\partial v_i} = Bv_i - (\zeta_{1,i+1} - \zeta_{3,i+1})\,S_i = 0$$

$$\frac{\partial \mathscr{H}_i}{\partial w_i} = Cw_i - (\zeta_{4,i+1} - \zeta_{5i+1})\,\beta Z_i^s I_i = 0$$

Then we obtain the optimal control

$$u_i = \frac{(\zeta_{1,i+1} - \zeta_{2,i+1})\,[\alpha S_i I_i]}{A}$$

$$v_i = \frac{(\zeta_{1,i+1} - \zeta_{3,i+1})\,S_i}{B}$$

$$w_i = \frac{(\zeta_{4,i+1} - \zeta_{5i+1})\,\beta Z_i^s I_i}{C}$$

By the bounds in $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$ of the controls in the definitions (8),(9) and (10), it is easy to obtain u_i^*, v_i^* and w_i^* in the following form

$$u_i^* = min\left\{max\left\{u_{min}, \frac{(\zeta_{1,i+1} - \zeta_{2,i+1})\,[\alpha S_i I_i]}{A}\right\}, u_{max}\right\}, i = 0,...,N-1$$

$$v_i^* = min\left\{max\left\{v_{min}, \frac{(\zeta_{1,i+1} - \zeta_{3,i+1})\,I_i}{B}\right\}, v_{max}\right\}, i = 0,...,N-1$$

$$w_i^* = min\left\{max\left\{w_{min}, \frac{(\zeta_{4,i+1} - \zeta_{5i+1})\,\beta Z_i^s I_i}{C}\right\}, w_{max}\right\}, i = 0,...,N-1$$

□

4. Results and Discussion

4.1. Numerical simulation. In the following, the numerical simulations associated to the previously stated optimization problem (7) are given. Specifically, the code is written in MATLAB (see Algorithm (1)) and we perform simulations of our results. A discrete iterative method is applied in order to solve the systems of optimality, and it converges following an adequate test such as the one for FBSM. The state system (12) is then solved forward in time under the initial assumption, followed by the adjoint system (13) being solved backward in time because of the transversality conditions. After that, we will make sure to actualize the optimal control values

through the state and co-state values (14) acquired in the preceding steps. Lastly, the preceding steps are implemented when the required tolerance standard is achieved.

Algorithm 1 Determination of states of the controlled system and controls u, v and w.

Require: $S_0, I_0, R_0, Z_0^s, Z_0^I, Z_0^R, N, u(0) = v(0) = w(0) = 0, \zeta_{1,N} = 0, \zeta_{2,N} = \alpha^I, \zeta_{3,N} = -\alpha^R, \zeta_{4,N} = 0, \zeta_{5,N} = \alpha^Z, \zeta_{6,N} = 0.$

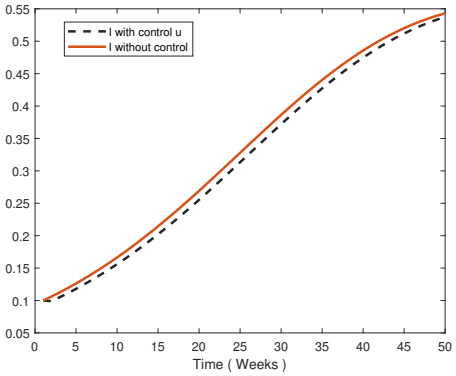
for i=1,...,N-1 **do**

$$
\begin{cases}
S_{i+1} = S_i - \alpha u_i S_i I_i - \mu S_i - v_i S_i \\
I_{i+1} = I_i + \alpha u_i S_i I_i - (\mu + r + \rho) I_i \\
R_{i+1} = R_i + \rho I_i - \mu R_i + v_i S_i \\
Z_{i+1}^s = Z_i^s - \beta w_i Z_i^s I_i + \theta Z_i^R \\
Z_{i+1}^I = Z_i^I + \beta w_i Z_i^s I_i - \gamma Z_i^I \\
Z_{i+1}^R = Z_i^R + \gamma Z_i^I - \theta Z_i^R
\end{cases} \tag{12}
$$

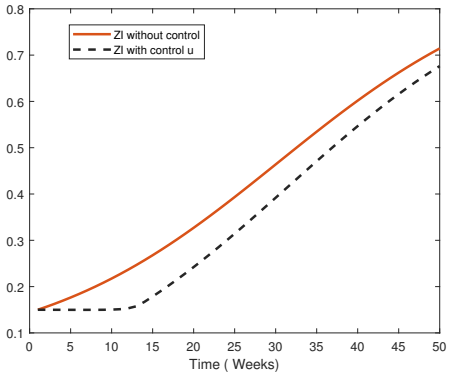
$$
\begin{cases}
\zeta_{1,i} = \zeta_{1,i+1}(2 - \alpha u_i I_i - \mu - v_i) + \zeta_{2,i+1} \alpha u_i I_i \\
+ \zeta_{3,i+1} v_i \\
\zeta_{2,i} = \zeta_{2,i+1}(2 + \alpha u_i S_i - (\mu + r + \rho)) + \alpha^I \\
- \zeta_{1,i+1} \alpha u_i S_i + \zeta_{3,i+1}(\rho) - \zeta_{4,i+1} \beta w_i Z_i^s \\
+ \zeta_{5,i+1} \beta w_i Z_i^s \\
\zeta_{3,i} = \zeta_{3,i+1}(2 - \mu) - \alpha^R \\
\zeta_{4,i} = \zeta_{4,i+1}(2 - \beta w_i I_i) + \zeta_{5,i+1} \beta w_i I_i \\
\zeta_{5,i} = \zeta_{5,i+1}(2 - \gamma) + \alpha^Z + \zeta_{6,i+1} \gamma \\
\zeta_{6,i} = \zeta_{6,i+1}(2 - \theta) + \zeta_{4,i+1} \theta
\end{cases} \tag{13}
$$

$$
\begin{cases}
u_i = \frac{(\zeta_{1,i+1} - \zeta_{2,i+1}) \alpha S_i I_i}{A} \\
v_i = \frac{(\zeta_{1,i+1} - \zeta_{3,i+1}) S_i}{B} \\
w_i = \frac{(\zeta_{4,i+1} - \zeta_{5,i+1}) \beta Z_i^s I_i}{B}
\end{cases} \tag{14}
$$

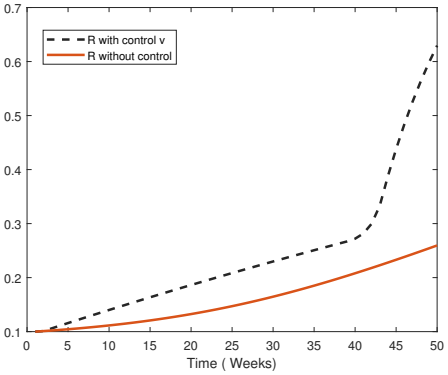
end for



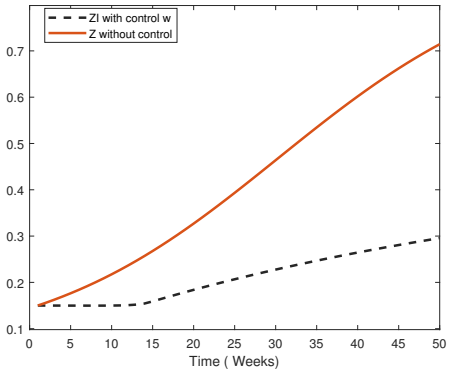
(A) Effect of control u on Infected Individuals



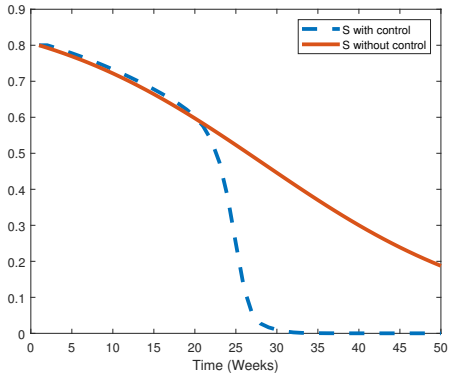
(B) Effect of control u on Infected Regions



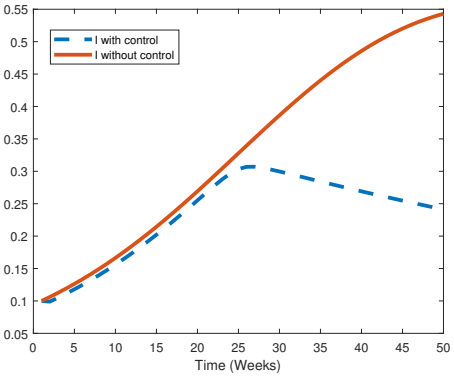
(C) Effect of control v on Recovered Individuals



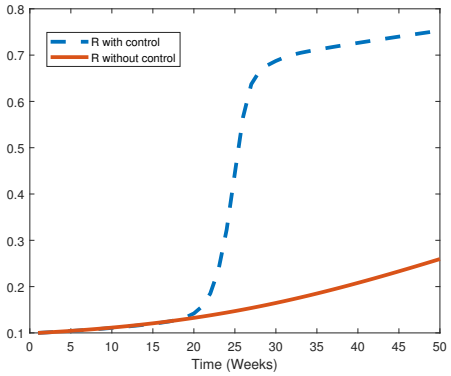
(D) Effect of control w on Infected Regions



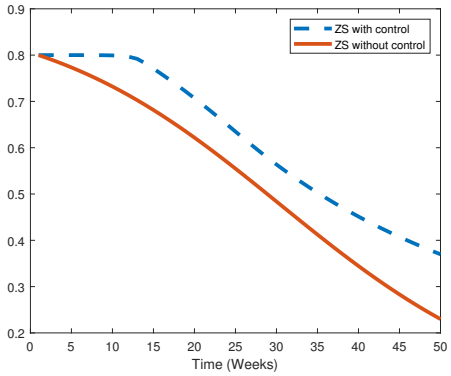
(E) Susceptible Individuals applying all controls u, v and w



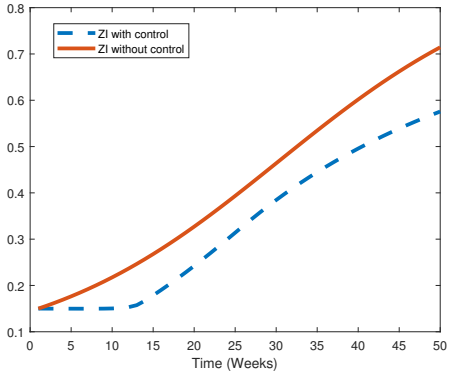
(F) Infected Individuals applying all controls u, v and w



(G) Recovered Individuals applying all controls u, v and w



(H) Susceptible Regions applying all controls u, v and w



(I) Infected Regions applying all controls u, v and w

5. Discussion

5.1. Strategy one : Applying only control u. This strategy aims to reduce the number of infected individuals and the number of infected zones by reducing contact between infected and susceptible individuals, either through quarantine, travel restrictions, or closure of public places at a given time. The effect of this strategy on the infected and the infected areas are presented in the Figures 1a, 1b.

5.2. Strategy two : Applying only control v. The objective of this strategy is to increase the number of persons who recover, by sensitizing those at risk to the severity of the disease or by

vaccinating them, and through Figure 1c we see the effectiveness of this strategy when after forty weeks the number of recovered persons has increased from 30% to 60%.

5.3. Strategy three : Applying only control w. This control strategy focuses on reducing the number of infected zones by restricting travel to and from these areas and by placing barriers around the affected areas and preventing the infected individual from moving into the uninfected areas. Figure 1d shows the impact of this program in reducing the number of infected areas by 40% in 50 weeks.

5.4. Strategy four : Applying all controls u, v and w. The goal of this strategy is to combine simultaneously the previous strategies. In Figures 1f, 1g and 1i, this strategy proved successful in reducing the number of infected people and the number of infected zones and in increasing sufficiently the recovered individuals to control the spread of the epidemic.

In the following section, we will present a conclusion of our work.

6. Conclusion

In this paper, we have presented a new mathematical model describing the evolution of the virus between people and regions. We have divided this model into two systems, the first $S_iI_iR_i$ which describe the virus evolution between individuals and the second $Z_i^S Z_i^I Z_i^R$ which describe the virus evolution between regions. Numerical simulation to prove the efficiency of our strategies of controls are given.

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Data Availability

The disciplinary data used to support the findings of this study have been deposited in the Network Repository (http://www.networkrepository.com).

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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