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LOCAL POLYNOMIAL BI-RESPONSES MULTI-PREDICTORS NONPARAMETRIC REGRESSION FOR PREDICTING THE MATURITY OF MANGO (*GADUNG KLONAL 21*): A THEORETICAL DISCUSSION AND SIMULATION

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Abstract: Determination of mango maturity level can be solved by non-destructive analysis to support one of the SDGs, sustainable production. It is a single example with two responses and several predictors. They created a regression model to handle problems with bi-responses multi-predictors, particularly for local polynomial estimators. This research aims to theoretically build a nonparametric regression model estimate using a local polynomial bi-response multi-predictor. The model can be used to predict the parameters of mango maturity, including the sweetness and acidity of mango. We create the algorithm and R code to show the performance of a bi-response

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multi-predictor local polynomial estimator based on simulation of three functions data, including trigonometric, exponential, and polynomial functions. The simulation proved that determining the optimal bandwidth based on the generalized cross-validation criterion is the most critical stage in the estimation process. If the bandwidth is too large, the estimation plot will too smooth, and vice versa. The optimal bandwidth gives the best estimation with the mean square error (MSE) and mean absolute percentage error (MAPE) values less than MSE and MAPE values of non-optimal bandwidth.
Keywords: local polynomial; mango; nonparametric regression; prediction; sustainable production.

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1. INTRODUCTION

Regression analysis is a statistical technique for examining the functional relationship between response and predictor variables. In regression analysis, there are two approaches namely parametric regression approach and nonparametric regression approach. Nonparametric regression approach offers an estimation method of model parameters. It only requires weak identification assumptions, which reduces the risk of model misspecification [1]. Its only purpose is to be smoother and more flexible in determining the shape of its regression function [2].

Nonparametric regression analysis methods can be used to develop predictive models. Nonparametric regression uses several estimators, including the Kernel [3,4], local linear [5–7], local polynomial [8–10], spline [3,4,11–18], and Fourier series [19]. The local polynomial is a well-known estimator. In nonparametric regression modeling, one estimator with a distinct form is the local polynomial estimator. The function is calculated locally at the point to be calculated in local polynomial regression. This local estimation allows for the capture of nonlinearities in every estimation phase without the effect of data set outliers [20]. The method is data-driven and easy to configure [21].

Local polynomial regression can be implemented in multiple responses and multiple predictors cases. Implementations of local polynomial regression in multiple response cases can be found in various areas, including estimating the growth curve of children for two years using a local linear estimator bi-responses [22], modeling children's weight in East Java [23], and modeling the number of maternal and infant deaths [24]. However, there is no study which has developed an estimation model for prediction the cases using a local polynomial estimator in bi-responses and multi-predictors variables.

Responsible consumption and production are global goals for sustainable development [25]. Agricultural product producers must also produce their products sustainably and environmentally friendly. One method is to reduce all waste in the manufacturing process, including evaluating the final product's quality characteristics. In general, the sensory qualities of food products are tested, and their chemical content is tested in the laboratory. This process is wasteful and time-consuming because sample preparation takes a long time and requires skilled workers.

Mango, also known as the "King of Fruit" of East Asia [26], is agricultural. Several quality characteristics, including firmness, Total Soluble Solids (TSS), Total Titratable Acid (TTA), dry matter, and pH of mangoes, should be tested on harvested mangoes. However, the two most commonly used variables are TSS and TTA, indicating mango sweetness and acidity. These two variables are widely used as indicators of mango ripeness. In the laboratory, TSS and total acidity analyses were performed destructively. However, many non-destructive tests have been developed using the Near Infra-red Spectroscopy tool that generates absorbance spectra data. A regression model of bi-responses multi-predictors was created based on the absorbance spectra data to predict TSS and total acidity value.

The research majority of fruit quality attribute prediction used parametric regression analysis of uni-response, such as the Partial Least Square Regression (PLSR) [27–30], Principal Component Regression (PCR) [27,28], Multiple Linear Regression (MLR) [27–29] and Simple Linear Regression (SLR) [31,32]. A few studies predict the fruit quality attributes using a nonparametric regression approach, including Nicolaï et al. [33] used the Kernel Partial Least Square Regression (KPLSR) method and [34] used Support Vector Machine (SVM) Regression method. Predictions of the acidity and sweetness of mangoes have been investigated using local polynomial regression uni-response multi-predictor [10,35,36]. Under these conditions, there has been less research on local polynomial nonparametric regression bi-response multi-predictor to predict fruit maturity based on the NIR spectroscopy test results.

This research aims to develop a model of local polynomial nonparametric regression with bi-responses and multiple predictor variables applied to predict the maturity level of mango. This research theoretically discusses how we estimate the model using a local polynomial estimator. Besides, we simulated this model in the data that have a trigonometric function. The results of this research are expected to be applied in cases of prediction of mango maturity.

2. PRELIMINARIES

Previous research on estimating local linear regression models for bi-response uni-predictor cases has been carried out [22]. This model is then used to estimate median growth curves for children up to two years old. Given pairs of observations $(x, y^{(1)}, y^{(2)})$ that follow bi-response nonparametric regression model:

(1) $$\underset{\sim}{y} = \underset{\sim}{f}(x) + \underset{\sim}{\varepsilon}$$

where $\underset{\sim}{f}(x) = \left(f_1(x), f_2(x)\right)^T$ is a vector of the unknown smooth function, $\underset{\sim}{y} = (y^{(1)}, y^{(2)})^T$ and $\underset{\sim}{\varepsilon} = (\varepsilon^{(1)}, \varepsilon^{(2)})^T$ is a random error with mean $\underset{\sim}{0}$ and variance Σ where $E(\underset{\sim}{\varepsilon}) = \underset{\sim}{0}$, $Cov(\varepsilon_{ri}, \varepsilon_{si}) = \begin{cases} \rho\sigma_{ri}\sigma_{si}, i = j \\ 0, \quad i \neq j \end{cases}, r, s = 1, 2.$

Function $\underset{\sim}{f}(x)$ in (1) is a smooth function assumed continuous and has $(d+1)$ continuous derivatives on an interval $x = x_0$. We can approach the function $f_1(x)$ and $f_2(x)$ by Taylor series about x_0 as follows:

(2) $$f_1(x) = \sum_{j=0}^{d^{(1)}} \frac{f_1^{(j)}(x_0)}{j!}(x - x_0)^j = \sum_{j=0}^{d^{(1)}} \beta_j^{(1)}(x_0)(x - x_0)^j$$

where $\beta_j^{(1)}(x_0) = \dfrac{\left(f_1^{(j)}(x_0)\right)}{j!}$, and

(3) $$f_2(x) = \sum_{j=0}^{d^{(2)}} \frac{f_2^{(j)}(x_0)}{j!}(x-x_0)^j = \sum_{j=0}^{d^{(2)}} \beta_j^{(2)}(x_0)(x-x_0)^j$$

where $\beta_j^{(2)}(x_0) = \frac{\left(f_2^{(j)}(x_0)\right)}{j!}$.

Based on equations (2) and (3), $\underset{\sim}{f}(x)$ can be expressed as follows:

(4) $$\underset{\sim}{f}(x) = X_{x_0}\underset{\sim}{\beta}(x_0),\ x \in (x_0 - h, x_0 + h)$$

where

$$X_{x_0} = \begin{bmatrix} 1 & (x-x_0) & \cdots & (x-x_0)^{d^{(1)}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & (x-x_0) & \cdots & (x-x_0)^{d^{(2)}} \end{bmatrix}$$

If the degree of polynomial, d = 1, we get a local linear estimator. We estimate $\underset{\sim}{f}(x)$ based on kernel local linear estimator around point x_0 by taking n-pairs data sample $\left\{x_i, y_i^{(1)}, y_i^{(2)}\right\}_{i=1}^{n}$. So, Equation (4) can be written as:

(5) $$\underset{\sim}{f}(\underset{\sim}{x}) = \mathbf{X}_{x_0}\underset{\sim}{\beta}_0(x_0)$$

where $\underset{\sim}{f}(\underset{\sim}{x}) = \left(f_1(\underset{\sim}{x}), f_2(\underset{\sim}{x})\right)^T$, $f_r(\underset{\sim}{x}) = \left(f_r(x_1) \quad f_r(x_2) \quad \cdots \quad f_r(x_n)\right)^T$, $\underset{\sim}{\beta}_0(x_0) = \left(\underset{\sim}{\beta}_0^{(1)}(x_0), \underset{\sim}{\beta}_0^{(2)}(x_0)\right)^T$

and

$$X_{x_0} = \begin{pmatrix} X_{x_0}^{(1)} & 0 \\ 0 & X_{x_0}^{(2)} \end{pmatrix}, X_{x_0}^{(r)} = \begin{bmatrix} 1 & x_1 - x_0 \\ 1 & x_2 - x_0 \\ \vdots & \vdots \\ 1 & x_n - x_0 \end{bmatrix}$$

Based on Equation (5), Equation (1) can be expressed as follows:

(6) $$\underset{\sim}{Y} = X_{x_0}\underset{\sim}{\beta}(x_0) + \underset{\sim}{\varepsilon}$$

where $\underset{\sim}{Y} = \left(\underset{\sim}{y}^{(1)}, \underset{\sim}{y}^{(2)}\right)^T$ and $\underset{\sim}{y}^{(r)} = \left(y_i^{(1)}, y_i^{(2)}\right)^T$.

We obtain estimator $\underset{\sim}{\beta}(x_0)$ in Equation (6) by using weighted least square (WLS) method:

(7) $$Q(x_0) = \left(\underset{\sim}{Y} - \mathbf{X}_{x_0}\underset{\sim}{\beta}(x_0)\right)^T \mathbf{\Sigma}^{-1}\mathbf{K}_h(x_0)\left(\underset{\sim}{Y} - \mathbf{X}_{x_0}\underset{\sim}{\beta}(x_0)\right)$$

where weighted matrix $\mathbf{\Sigma}^{-1}$ is the invert of the covariance matrix of $\underset{\sim}{\varepsilon}$ and $\mathbf{K}_h(x_0)$ be the nr x nr diagonal matrix of weights: $\mathbf{K}_h(x_0) = diag\left(\underset{\sim}{K}_h\left(x_i - x_0\right), \underset{\sim}{K}_h\left(x_i - x_0\right)\right)$ and $K_h(.)$ is the kernel function, that is:

$$\underset{\sim}{K}_h\left(x_i - x_0\right) = \left(K_h\left(x_1 - x_0\right), K_h\left(x_2 - x_0\right), ..., K_h\left(x_n - x_0\right)\right)^T$$

The estimation of $\underset{\sim}{\beta}\left(x_0\right)$ is obtained by minimizing $Q(x_0)$, and we get:

(8) $$\hat{\underset{\sim}{\beta}}\left(x_0\right) = \left[\mathbf{X}_{x_0}^T \mathbf{\Sigma}^{-1} \mathbf{K}_h\left(x_0\right) \mathbf{X}_{x_0}\right]^{-1} \mathbf{X}_{x_0}^T \mathbf{\Sigma}^{-1} \mathbf{K}_h\left(x_0\right) \underset{\sim}{Y}$$

So, estimation of $\underset{\sim}{f}(x_0)$ based on kernel estimator can be written as:

(9) $$\hat{\underset{\sim}{f}}(x) = \hat{\underset{\sim}{\beta}}(x_0)$$

By substituting equation (8) into (9), we have:

$$\hat{\underset{\sim}{f}}(x) = \left[\mathbf{X}_{x_0}^T \mathbf{\Sigma}^{-1} \mathbf{K}_h\left(x_0\right) \mathbf{X}_{x_0}\right]^{-1} \mathbf{X}_{x_0}^T \mathbf{\Sigma}^{-1} \mathbf{K}_h\left(x_0\right) \underset{\sim}{Y}$$

In the local polynomial regression, a parameter, i.e., the bandwidth (h), controls the fit smoothness and affects the bias-variance trade-off. We use the generalized-cross-validation criterion for determining optimal h.

Estimation of the function $\hat{f}_j(x_i)$ for each response is:

$$\hat{f}_r(x_i) = A(h_j)\underset{\sim}{y}^{(r)},\ r = 1,2$$

where $A(h_j)$ represents matrix as follows:

$$A\left(h_j\right) = e_1^T\left[X_{x_i}^{(d_j)T} K_{hj}(x_i) X_{x_i}^{(d_j)}\right]^{-1} X_{x_i}^{(d_j)T} K_{hj}(x_i)$$

To obtain optimal h_j based on the GCV criterion, we minimize [37,38]:

$$GCV(h_j) = \frac{n^{-1}\sum_{i=1}^{n}(y_i^{(j)} - \hat{y}_i^{(j)})^2}{\left(1 - tr\left[\mathrm{I} - A(h_j)\right]/n\right)^2}$$

3. MAIN RESULTS

In this section, we theoretically describe the estimation of local polynomial nonparametric regression bi-responses and multi-predictors if applied to predict the sweetness (TSS value) and acidity (pH value) of mango *Gadung Klonal 21* based on absorbance spectra data of NIR spectroscopy.

3.1. Estimating Model

The results of the estimating local polynomial bi-responses and multi-predictors nonparametric regression model are presented in the following Theorem and Lemma.

Theorem. *Given the data in pairs* $\left(x_{i1}, x_{i2}, ..., x_{ip}, y_i^{(r)}\right), i = 1, 2, ..., n$ *where i is the number of observations*, *p is the number of predictor variables, and r is the number of response variables. They meet the bi-responses multi-predictors nonparametric regression model as follows:*

(10) $$y_i^{(r)} = \sum_{j=1}^{p} \underset{\sim}{f}\left(x_{ij}\right) + \varepsilon_i^{(r)}$$

where $\underset{\sim}{y}_i = \left(\underset{\sim}{y}_i^{(1)}, \underset{\sim}{y}_i^{(2)}\right)^T$, $\underset{\sim}{f}\left(x_{ij}\right) = \left(f_1\left(x_{ij}\right), f_2\left(x_{ij}\right)\right)^T$ *is a function of unknown shape and* $\varepsilon_i = \left(\varepsilon_i^{(1)}, \varepsilon_i^{(2)}\right)^T$ is *the measurement error with mean is* $\mathbf{0}$, *and variance is* $\mathbf{\Sigma}_i$, *so local polynomial estimator for estimating nonparametric regression function of bi-responses multi-predictors at* x_j *around* x_{0j} *is*:

(11) $$\hat{\underset{\sim}{f}}(x_{ij}) = x^*(\underset{\sim}{x}_0)\left((\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\mathbf{X}(\underset{\sim}{x}_0)\right)^{-1}\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y}$$

where $x^*(\underset{\sim}{x}_0) = \begin{pmatrix} \underset{\sim}{x}_1^*(\underset{\sim}{x}_0) & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{x}_2^*(\underset{\sim}{x}_0) \end{pmatrix}$ *;* $x^*(\underset{\sim}{x}_0)$ *is a vector of size* $2n$ x $(2+d_1+d_2)$; $\underset{\sim}{\beta}(\underset{\sim}{x}_0)$ *is a vector of size* $(2+d_1+d_2)$ x 1; $d_1 = \sum_{j=1}^{p} d_j^{(1)}$; $d_2 = \sum_{j=1}^{p} d_j^{(2)}$; $\underset{\sim}{\varepsilon}$ *is a vector of size* $2n$x1; d_1 *and* d_2 *are the polynomial degree of first and second response, and* d_j *is the polynomial degree of the* j^{th} *predictor for* $j = 1, 2, \ldots, p$.

Proof of Theorem. We use a local polynomial estimator to estimate the regression function in estimating bi-response values. Equation (4) is a smooth function assumed to have an unknown shape and is estimated using a nonparametric approach based on a local polynomial estimator. The regression function $\underset{\sim}{f}\left(x_{ij}\right)$ is a smooth function with continuous and differentiable properties. The differentiable function can be approximated by the Taylor series expansion. The Taylor series for $f_1(x_j)$ and $f_2(x_j)$ at x_j around x_{0j} can be expressed as follows:

$$\left.\begin{aligned} f_1(x_1) &= f_1(x_{01}) + (x_1 - x_{01})\frac{\left(f_1^1(x_{01})\right)}{1!} + (x_1 - x_{01})^2 \frac{\left(f_1^2(x_{01})\right)}{2!} + \ldots + (x_1 - x_{01})^{d_1^{(1)}} \frac{\left(f_1^{d_1^{(1)}}(x_{01})\right)}{d_1^{(1)}!} \\ f_1(x_2) &= f_1(x_{02}) + (x_2 - x_{02})\frac{\left(f_1^1(x_{02})\right)}{1!} + (x_2 - x_{02})^2 \frac{\left(f_1^2(x_{02})\right)}{2!} + \ldots + (x_2 - x_{02})^{d_2^{(1)}} \frac{\left(f_1^{d_2^{(1)}}(x_{02})\right)}{d_2^{(1)}!} \\ &\vdots \\ f_1(x_p) &= f_1(x_{0p}) + (x_p - x_{0p})\frac{\left(f_1^1(x_{0p})\right)}{1!} + (x_p - x_{0p})^2 \frac{\left(f_1^2(x_{0p})\right)}{2!} + \ldots + (x_p - x_{0p})^{d_p^{(1)}} \frac{\left(f_1^{d_p^{(1)}}(x_{0p})\right)}{d_p^{(1)}!} \end{aligned}\right\} \tag{12}$$

Equation (12) can be written in the notation:

$$f_1(x_p) = \sum_{j=0}^{p} \frac{\left(f_1^{d_j^{(1)}}(x_{0j})\right)}{d_j^{(1)}!}(x - x_0)^{d_j^{(1)}} = \sum_{j=0}^{p} \beta_j^{(1)}(x_{0j})(x_j - x_{0j})^{d_j^{(1)}} \tag{13}$$

$$\left.\begin{aligned} f_2(x_1) &= f_2(x_{01}) + (x_1 - x_{01})\frac{\left(f_2^1(x_{01})\right)}{1!} + (x_1 - x_{01})^2 \frac{\left(f_2^2(x_{01})\right)}{2!} + \ldots + (x_1 - x_{01})^{d_1^{(2)}} \frac{\left(f_2^{d_1^{(2)}}(x_{01})\right)}{d_1^{(2)}!} \\ f_2(x_2) &= f_2(x_{02}) + (x_2 - x_{02})\frac{\left(f_2^1(x_{02})\right)}{1!} + (x_2 - x_{02})^2 \frac{\left(f_2^2(x_{02})\right)}{2!} + \ldots + (x_2 - x_{02})^{d_2^{(2)}} \frac{\left(f_2^{d_2^{(2)}}(x_{02})\right)}{d_2^{(2)}!} \\ &\vdots \\ f_2(x_p) &= f_2(x_{0p}) + (x_p - x_{0p})\frac{\left(f_2^1(x_{0p})\right)}{1!} + (x_p - x_{0p})^2 \frac{\left(f_2^2(x_{0p})\right)}{2!} + \ldots + (x_p - x_{0p})^{d_p^{(2)}} \frac{\left(f_2^{d_p^{(2)}}(x_{0p})\right)}{d_p^{(2)}!} \end{aligned}\right\} \tag{14}$$

Equation (14) can be written in the notation:

$$f_2(x_p) = \sum_{j=0}^{p} \frac{\left(f_2^{d_j^{(2)}}(x_{0j})\right)}{d_j^{(2)}!}(x - x_0)^{d_j^{(2)}} = \sum_{j=0}^{p} \beta_j^{2}(x_{0j})(x_j - x_{0j})^{d_j^{(2)}} \tag{15}$$

Suppose $\beta_m^{(1)}(x_{0j}) = \dfrac{\left(f_1^{m_{1j}}(x_{0j})\right)}{m_{1j}!}; m = 0,1,2,...,d;\ j = 1,2,3,...,p$ and

$\beta_c^{(2)}(x_{0j}) = \dfrac{\left(f_2^{c_{2j}}(x_{0j})\right)}{c_{2j}!}; c = 0,1,2,...,d;\ j = 1,2,3,...,p$, then equations (12) and (14) can be written as follows:

$$\left.\begin{array}{c} f_1(x_1) = \beta_0^{(1)}(x_{01}) + (x_1 - x_{01})\beta_1^{(1)}(x_{01}) + (x_1 - x_{01})^2 \beta_2^{(1)}(x_{01}) + ... + (x_1 - x_{01})^{d_1^{(1)}} \beta_{d_1^{(1)}}^{(1)}(x_{01}) \\ f_1(x_2) = \beta_0^{(1)}(x_{02}) + (x_2 - x_{02})\beta_1^{(1)}(x_{02}) + (x_2 - x_{02})^2 \beta_2^{(1)}(x_{02}) + ... + (x_2 - x_{02})^{d_2^{(1)}} \beta_{d_2^{(1)}}^{(1)}(x_{02}) \\ \vdots \\ f_1(x_p) = \beta_0^{(1)}(x_{0p}) + (x_p - x_{0p})\beta_1^{(1)}(x_{0p}) + (x_p - x_{0p})^2 \beta_2^{(1)}(x_{0p}) + ... + (x_p - x_{0p})^{d_p^{(1)}} \beta_{d_p^{(1)}}^{(1)}(x_{0p}) \end{array}\right\} \quad (16)$$

$$\left.\begin{array}{c} f_2(x_1) = \beta_0^{(2)}(x_{01}) + (x_1 - x_{01})\beta_1^{(2)}(x_{01}) + (x_1 - x_{01})^2 \beta_2^{(2)}(x_{01}) + ... + (x_1 - x_{01})^{d_1^{(2)}} \beta_{d_1^{(2)}}^{(2)}(x_{01}) \\ f_2(x_2) = \beta_0^{(2)}(x_{02}) + (x_2 - x_{02})\beta_1^{(2)}(x_{02}) + (x_2 - x_{02})^2 \beta_2^{(2)}(x_{02}) + ... + (x_2 - x_{02})^{d_2^{(2)}} \beta_{d_2^{(2)}}^{(2)}(x_{02}) \\ \vdots \\ f_2(x_p) = \beta_0^{(2)}(x_{0p}) + (x_p - x_{0p})\beta_1^{(2)}(x_{0p}) + (x_p - x_{0p})^2 \beta_2^{(2)}(x_{0p}) + ... + (x_p - x_{0p})^{d_p^{(2)}} \beta_{d_p^{(2)}}^{(2)}(x_{0p}) \end{array}\right\} \quad (17)$$

where $x_j \in \left(x_{0j} - h_j, x_{0j} + h_j\right)$; $d_p^{(1)}$ is the polynomial degree in the regression function of the first response in the p^{th} predictor, $d_p^{(2)}$ is the polynomial degree in the regression function of the second response in the p^{th} predictor. Based on equations (16) and (17), it can be formed into:

$$f_1(\underset{\sim}{x}) = \sum_{j=1}^{p} f_1(x_j) \quad ; \quad f_2(\underset{\sim}{x}) = \sum_{j=1}^{p} f_2(x_j)$$

$$f_1(\underset{\sim}{x}) = f_1(x_1) + f_1(x_2) + ... + f_1(x_p); \quad f_2(\underset{\sim}{x}) = f_2(x_1) + f_2(x_2) + ... + f_2(x_p)$$

$$\begin{array}{c} f_1(\underset{\sim}{x}) = \left(\beta_0^{(1)}(x_{01}) + (x_1 - x_{01})\beta_1^{(1)}(x_{01}) + (x_1 - x_{01})^2 \beta_2^{(1)}(x_{01}) + ... + (x_1 - x_{01})^{d_1^{(1)}} \beta_{d_1^{(1)}}^{(1)}(x_{01})\right) + \\ \left(\beta_0^{(1)}(x_{02}) + (x_2 - x_{02})\beta_1^{(1)}(x_{02}) + (x_2 - x_{02})^2 \beta_2^{(1)}(x_{02}) + ... + (x_2 - x_{02})^{d_2^{(1)}} \beta_{d_2^{(1)}}^{(1)}(x_{02})\right) \\ + ...\left(\beta_0^{(1)}(x_{0p}) + (x_p - x_{0p})\beta_1^{(1)}(x_{0p}) + (x_p - x_{0p})^2 \beta_2^{(1)}(x_{0p}) + ... + (x_p - x_{0p})^{d_p^{(1)}} \beta_{d_p^{(1)}}^{(1)}(x_{0p})\right) \end{array} \quad (18)$$

$$\begin{array}{c} f_2(\underset{\sim}{x}) = \left(\beta_0^{(2)}(x_{01}) + (x_1 - x_{01})\beta_1^{(2)}(x_{01}) + (x_1 - x_{01})^2 \beta_2^{(2)}(x_{01}) + ... + (x_1 - x_{01})^{d_1^{(2)}} \beta_{d_1^{(2)}}^{(2)}(x_{01})\right) + \\ \left(\beta_0^{(2)}(x_{02}) + (x_2 - x_{02})\beta_1^{(2)}(x_{02}) + (x_2 - x_{02})^2 \beta_2^{(2)}(x_{02}) + ... + (x_2 - x_{02})^{d_2^{(2)}} \beta_{d_2^{(2)}}^{(2)}(x_{02})\right) \\ + ...\left(\beta_0^{(2)}(x_{0p}) + (x_p - x_{0p})\beta_1^{(2)}(x_{0p}) + (x_p - x_{0p})^2 \beta_2^{(2)}(x_{0p}) + ... + (x_p - x_{0p})^{d_p^{(2)}} \beta_{d_p^{(2)}}^{(2)}(x_{0p})\right) \end{array} \quad (19)$$

If $\underset{\sim}{x}=(x_1,x_2,...,x_p)^T$ and $\underset{\sim}{x}_0=(x_{01},x_{02},...,x_{0p})^T$ then the bi-response multi-predictor nonparametric regression function at a certain point can be expressed in vector notation:

(20) $$f_1(\underset{\sim}{x})=\underset{\sim}{x}_1^*(\underset{\sim}{x}_0)\underset{\sim}{\beta}^{(1)}(\underset{\sim}{x}_0)$$ and

(21) $$f_2(\underset{\sim}{x})=\underset{\sim}{x}_2^*(\underset{\sim}{x}_0)\underset{\sim}{\beta}^{(2)}(\underset{\sim}{x}_0)$$

Based on equations (20) and (21), it can be expressed in matrix notation as follows:

(22) $$\underset{\sim}{\hat{f}}=\underset{\sim}{x}^*(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0)$$

where:

$$\underset{\sim}{x}^*(x_0)=\begin{pmatrix}\underset{\sim}{x}_1^* & \underset{\sim}{0}\\ \underset{\sim}{0} & \underset{\sim}{x}_2^*\end{pmatrix};$$

$$x_1^*(x_0)=\begin{pmatrix}1 & (x_1-x_{01}) & (x_1-x_{01})^2 & \cdots & (x_1-x_{01})^{d_1^{(1)}} & (x_2-x_{02}) & (x_2-x_{02})^2 & \cdots & (x_2-x_{02})^{d_2^{(1)}} & (x_p-x_{0p}) & (x_p-x_{0p})^2 & \cdots & (x_p-x_{0p})^{d_p^{(1)}}\end{pmatrix}$$

$$x_2^*(x_0)=\begin{pmatrix}1 & (x_1-x_{01}) & (x_1-x_{01})^2 & \cdots & (x_1-x_{01})^{d_1^{(2)}} & (x_2-x_{02}) & (x_2-x_{02})^2 & \cdots & (x_2-x_{02})^{d_2^{(2)}} & (x_p-x_{0p}) & (x_p-x_{0p})^2 & \cdots & (x_p-x_{0p})^{d_p^{(2)}}\end{pmatrix}$$

$$\underset{\sim}{\beta}(x_0)=\begin{pmatrix}\underset{\sim}{\beta}^{(1)}(x_0)\\ \underset{\sim}{\beta}^{(2)}(x_0)\end{pmatrix};\underset{\sim}{\beta}^{(1)}(x_0)=\left(\underset{\sim}{\beta}_0^{(1)}(x_0) \quad \underset{\sim}{\beta}_1^{(1)}(x_{01}) \quad \cdots \quad \underset{\sim}{\beta}_p^{(1)}(x_{0p})\right)^T;$$

$$\underset{\sim}{\beta}^{(2)}(x_0)=\left(\underset{\sim}{\beta}_0^{(2)}(x_0) \quad \underset{\sim}{\beta}_1^{(2)}(x_{01}) \quad \cdots \quad \underset{\sim}{\beta}_p^{(2)}(x_{0p})\right)^T$$

By obtaining equation (22), then based on equation (4) can be written a regression model for each response in Equation (23) and (24). By taking n paired samples $\left(\underset{\sim}{y}^{(1)},\underset{\sim}{y}^{(2)},\underset{\sim}{x}_1,\underset{\sim}{x}_2,...,\underset{\sim}{x}_p\right)$, the regression model on the first response with the i^{th} observation and j^{th} predictor is:

(23) $$\left.\begin{aligned}
&y_1^{(1)}=\beta_0^{(1)}(x_0)+\left(\beta_1^{(1)}(x_{01})(x_{11}-x_{01})+\beta_2^{(1)}(x_{01})(x_{11}-x_{01})^2+...+\beta_p^{(1)}(x_{01})(x_{11}-x_{01})^{d_1^{(1)}}\right)+\\
&\left(\beta_1^{(1)}(x_{02})(x_{12}-x_{02})+\beta_2^{(1)}(x_{02})(x_{12}-x_{02})^2+...+\beta_p^{(1)}(x_{02})(x_{12}-x_{02})^{d(1)}\right)\\
&+...\left(\beta_1^{(1)}(x_{0p})(x_{1p}-x_{0p})+\beta_2^{(1)}(x_{0p})(x_{1p}-x_{0p})^2+...+\beta_p^{(1)}(x_{0p})(x_{1p}-x_{0p})^{d_p^{(1)}}\right)+\varepsilon_1^{(1)}\\
&y_2^{(1)}=\beta_0^{(1)}(x_0)+\left(\beta_1^{(1)}(x_{01})(x_{21}-x_{01})+\beta_2^{(1)}(x_{01})(x_{21}-x_{01})^2+...+\beta_p^{(1)}(x_{01})(x_{21}-x_{01})^{d_2^{(1)}}\right)+\\
&\left(\beta_1^{(1)}(x_{02})(x_{22}-x_{02})+\beta_2^{(1)}(x_{02})(x_{22}-x_{02})^2+...+\beta_p^{(1)}(x_{02})(x_{22}-x_{02})^{d_2^{(1)}}\right)\\
&+...\left(\beta_1^{(1)}(x_{0p})(x_{2p}-x_{0p})+\beta_2^{(1)}(x_{0p})(x_{2p}-x_{0p})^2+...+\beta_p^{(1)}(x_{0p})(x_{2p}-x_{0p})^{d_p^{(1)}}\right)+\varepsilon_2^{(1)}\\
&\qquad\vdots\\
&y_n^{(1)}=\beta_0^{(1)}(x_0)+\left(\beta_1^{(1)}(x_{01})(x_{n1}-x_{01})+\beta_2^{(1)}(x_{01})(x_{n1}-x_{01})^2+...+\beta_p^{(1)}(x_{01})(x_{n1}-x_{01})^{d_1^{(1)}}\right)+\\
&\left(\beta_1^{(1)}(x_{02})(x_{n2}-x_{02})+\beta_2^{(1)}(x_{02})(x_{n2}-x_{02})^2+...+\beta_p^{(1)}(x_{02})(x_{n2}-x_{02})^{d_2^{(1)}}\right)\\
&+...\left(\beta_1^{(1)}(x_{0p})(x_{np}-x_{0p})+\beta_2^{(1)}(x_{0p})(x_{np}-x_{0p})^2+...+\beta_p^{(1)}(t_{0p})(t_{np}-t_{0p})^{d_p^{(1)}}\right)+\varepsilon_n^{(1)}
\end{aligned}\right\}$$

Regression model on the 2^{nd} response with the i^{th} observation and j^{th} predictor is:

$$\left.\begin{aligned}
&y_1^{(2)} = \beta_0^{(2)}(x_0) + \left(\beta_1^{(2)}(x_{01})(x_{11}-x_{01}) + \beta_2^{(2)}(x_{01})(x_{11}-x_{01})^2 + \ldots + \beta_p^{(2)}(x_{01})(x_{11}-x_{01})^{d_1^{(2)}}\right) + \\
&\left(\beta_1^{(2)}(x_{02})(x_{12}-x_{02}) + \beta_2^{(2)}(x_{02})(x_{12}-x_{02})^2 + \ldots + \beta_p^{(2)}(x_{02})(x_{12}-x_{02})^{d_2^{(2)}}\right) \\
&+\ldots\left(\beta_1^{(2)}(x_{0p})(x_{1p}-x_{0p}) + \beta_2^{(2)}(x_{0p})(x_{1p}-x_{0p})^2 + \ldots + \beta_p^{(2)}(x_{0p})(x_{1p}-x_{0p})^{d_p^{(2)}}\right) + \varepsilon_1^{(2)} \\
&y_2^{(2)} = \beta_0^{(2)}(x_0) + \left(\beta_1^{(2)}(x_{01})(x_{21}-x_{01}) + \beta_2^{(2)}(x_{01})(x_{21}-x_{01})^2 + \ldots + \beta_p^{(2)}(x_{01})(x_{21}-x_{01})^{d_1^{(2)}}\right) + \\
&\left(\beta_1^{(2)}(x_{02})(x_{22}-x_{02}) + \beta_2^{(2)}(x_{02})(x_{22}-x_{02})^2 + \ldots + \beta_p^{(2)}(x_{02})(x_{22}-x_{02})^{d_2^{(2)}}\right) \\
&+\ldots\left(\beta_1^{(2)}(x_{0p})(x_{2p}-x_{0p}) + \beta_2^{(2)}(x_{0p})(x_{2p}-x_{0p})^2 + \ldots + \beta_p^{(2)}(x_{0p})(x_{2p}-x_{0p})^{d_p^{(2)}}\right) + \varepsilon_2^{(2)} \\
&\qquad\vdots \\
&y_n^{(2)} = \beta_0^{(2)}(x_0) + \left(\beta_1^{(2)}(x_{01})(x_{n1}-x_{01}) + \beta_2^{(2)}(x_{01})(x_{n1}-x_{01})^2 + \ldots + \beta_p^{(2)}(x_{01})(x_{n1}-x_{01})^{d_n^{(2)}}\right) + \\
&\left(\beta_1^{(2)}(x_{02})(x_{n2}-x_{02}) + \beta_2^{(2)}(x_{02})(x_{n2}-x_{02})^2 + \ldots + \beta_p^{(2)}(x_{02})(x_{n2}-x_{02})^{d_2^{(2)}}\right) \\
&+\ldots\left(\beta_1^{(2)}(x_{0p})(x_{np}-x_{0p}) + \beta_2^{(2)}(x_{0p})(x_{np}-x_{0p})^2 + \ldots + \beta_p^{(2)}(x_{0p})(x_{np}-x_{0p})^{d_p^{(2)}}\right) + \varepsilon_n^{(2)}
\end{aligned}\right\} \tag{24}$$

Equations (23) and (24) can be expressed in vector notation as follows:

$$y^{(1)} = \mathbf{X}_1(\underset{\sim}{x}_0)\underset{\sim}{\beta}^{(1)}(\underset{\sim}{x}_0) + \varepsilon^{(1)} \tag{25}$$

$$y^{(2)} = \mathbf{X}_2(\underset{\sim}{x}_0)\underset{\sim}{\beta}^{(2)}(\underset{\sim}{x}_0) + \varepsilon^{(2)} \tag{26}$$

Equations (19) and (20) can be expressed in matrix notation as follows:

$$\underset{\sim}{y} = \mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0) + \underset{\sim}{\varepsilon} \tag{27}$$

where $\underset{\sim}{y}$ is a vector of size 2*n*x1; $\mathbf{X}(\underset{\sim}{x}_0)$ is a vector of size 2*n* x (2+*d*1+*d*2); $\underset{\sim}{\beta}(\underset{\sim}{x}_0)$ is a vector of size (2+*d*1+*d*2) x 1; $d_1 = \sum_{j=1}^{p} d_j^{(1)}$; $d_2 = \sum_{j=1}^{p} d_j^{(2)}$; $\underset{\sim}{\varepsilon}$ is a vector of size 2*n*x1. Hence, we have:

$$\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \end{bmatrix}; \underset{\sim}{y}_r = \begin{bmatrix} \underset{\sim}{y}_1^{(r)} \\ \underset{\sim}{y}_2^{(r)} \\ \vdots \\ \underset{\sim}{y}_n^{(r)} \end{bmatrix}; \quad \mathbf{X}(\underset{\sim}{x}_0) = \begin{pmatrix} \mathbf{X}_1(\underset{\sim}{x}_0) & \underset{\sim}{0} \\ \underset{\sim}{0} & \mathbf{X}_2(\underset{\sim}{x}_0) \end{pmatrix}; \mathbf{X}_r(\underset{\sim}{x}_0) = \begin{pmatrix} 1 & \underset{\sim}{x}_{11} & \underset{\sim}{x}_{12} & \cdots & \underset{\sim}{x}_{1p} \\ 1 & \underset{\sim}{x}_{21} & \underset{\sim}{x}_{22} & \cdots & \underset{\sim}{x}_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underset{\sim}{x}_{n1} & \underset{\sim}{x}_{n2} & \cdots & \underset{\sim}{x}_{np} \end{pmatrix}, r = 1,2;$$

$$\underset{\sim}{x}_{ij} = \left((x_{i1}-x_{01}) \quad (x_{i1}-x_{01})^2 \quad \cdots \quad (x_{i1}-x_{01})^{d_j}\right), i = 1,2,\ldots,n \text{ dan } j = 1,2,\ldots,p$$

$$\hat{\underset{\sim}{\beta}}(\underset{\sim}{x}_0) = \begin{pmatrix} \underset{\sim}{\beta}^{(1)}(\underset{\sim}{x}_0) \\ \underset{\sim}{\beta}^{(2)}(\underset{\sim}{x}_0) \end{pmatrix}; \underset{\sim}{\beta}^{(r)}(\underset{\sim}{x}_0) = \begin{pmatrix} \beta_0^{(r)}(x_0) \\ \beta_1^{(r)}(x_{01}) \\ \vdots \\ \beta_p^{(r)}(x_{0p}) \end{pmatrix}; \quad \underset{\sim}{\varepsilon} = \begin{pmatrix} \underset{\sim}{\varepsilon}^{(1)} \\ \underset{\sim}{\varepsilon}^{(2)} \end{pmatrix}; \underset{\sim}{\varepsilon}^{(r)} = \begin{pmatrix} \varepsilon_1^{(r)} \\ \varepsilon_2^{(r)} \\ \vdots \\ \varepsilon_n^{(r)} \end{pmatrix}.$$

The estimation is based on the local polynomial estimator using the kernel function K$_h$ as weighting. The weight shape is determined by the kernel function, while the weight size is determined by the value of the parameter h called bandwidth. In the nonparametric bi-response regression model, it is assumed that there is a correlation between the first response error ($\varepsilon^{(1)}$) and the second response error ($\varepsilon^{(2)}$) that is $\rho = corr\left(\varepsilon^{(1)}, \varepsilon^{(2)}\right)$, and the variances ($\varepsilon^{(1)}$) and ($\varepsilon^{(2)}$) are different. A correlation ρ that resulted in the addition of a weighted matrix of covariance variance was obtained from the estimation of the unweighted model sample $\left(\mathbf{V}^{-1}\right)$ at the time of estimating $\underset{\sim}{\beta}$.

Estimation of $\underset{\sim}{\beta}$ in equation (21) using the weighted least square (WLS) can be obtained by minimizing the function:

$$\mathbf{Q}(\underset{\sim}{x}_0) = \left(\underset{\sim}{y} - \mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0)\right)^T \mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\left(\underset{\sim}{y} - \mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0)\right) \tag{28}$$

where $\mathbf{V}^{-1} = \begin{pmatrix} \mathbf{\Sigma_{11}} & \mathbf{\Sigma_{12}} \\ \mathbf{\Sigma_{21}} & \mathbf{\Sigma}_{22} \end{pmatrix}^{-1}$ is a weighting matrix of size $2n$x$2n$. $\mathbf{\Sigma}_{rr}$ for r = 1,2 is a diagonal matrix containing the k^{th} response error variance. In contrast, diagonal matrices have the covariance of the first response error and the second response error, and $\mathbf{K_h}$ are a diagonal matrix measuring 2nx2n as in equation (23).

$$\mathbf{K_h} = \begin{pmatrix} diag\left(K_{h_1}^{(1)}(\underset{\sim}{x}_0), K_{h_2}^{(1)}(\underset{\sim}{x}_0), \ldots, K_{h_n}^{(1)}(\underset{\sim}{x}_0)\right) & \mathbf{0} \\ \mathbf{0} & diag\left(K_{h_1}^{(2)}(\underset{\sim}{x}_0), K_{h_2}^{(2)}(\underset{\sim}{x}_0), \ldots, K_{h_n}^{(2)}(\underset{\sim}{x}_0)\right) \end{pmatrix} \tag{29}$$

where $K_h^{(1)}(\underset{\sim}{x}_0) = \prod_{j=1}^{p}\left(K_{h_j}^{(1)}(x_{oj})\right)$ and $K_h^{(2)}(\underset{\sim}{x}_0) = \prod_{j=1}^{p}\left(K_{h_j}^{(2)}(x_{oj})\right)$.

Equation (28) can be described as follows:

$$\begin{aligned} Q(\underset{\sim}{x}_0) &= \left(\underset{\sim}{y} - \mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0)\right)^{\mathrm{T}} \mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\left(\underset{\sim}{y} - \mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0)\right) \\ &= \left(\underset{\sim}{y}^{\mathrm{T}} - \underset{\sim}{\beta}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\right)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\left(\underset{\sim}{y} - \mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0)\right) \\ &= \underset{\sim}{y}^{\mathrm{T}}\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y} - \underset{\sim}{y}^{\mathrm{T}}\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0) - \underset{\sim}{\beta}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y} + \underset{\sim}{\beta}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0) \end{aligned}$$

(30) $$Q(\underset{\sim}{x}_0) = \underset{\sim}{y}^{\mathrm{T}}\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y} - 2\underset{\sim}{\beta}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y} + \underset{\sim}{\beta}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0).$$

The estimated value of $\underset{\sim}{\beta}$ is $\hat{\underset{\sim}{\beta}}$, if substituted in Equation (22), will minimize $\mathbf{Q}(\underset{\sim}{x}_0)$. This estimated value can be obtained by differentiating equation (22) with respect to $\underset{\sim}{\beta}$. The minimum value of $\mathbf{Q}(\underset{\sim}{x}_0)$ is reached when $\frac{\partial \mathbf{Q}(\underset{\sim}{x}_0)}{\partial \underset{\sim}{\beta}} = 0.$ Therefore, it is obtained:

$$\frac{\partial \mathbf{Q}(\underset{\sim}{x}_0)}{\partial \underset{\sim}{\beta}} = 0 \Leftrightarrow -2\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y} + 2\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0) = 0$$

$$\Leftrightarrow -2\left(\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y} - \mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0)\right) = 0$$

$$\Leftrightarrow \mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y} - \mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0) = 0$$

$$\Leftrightarrow \mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\mathbf{X}(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0) = \mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y}.$$

(31) $$\hat{\underset{\sim}{\beta}} = \left(\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\mathbf{X}(\underset{\sim}{x}_0)\right)^{-1}\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y}$$

Based on equations (22) and (31), we estimate $\underset{\sim}{f}(\underset{\sim}{x}_0)$ by taking n samples with the i^{th} observation and j^{th} predictor, and we obtain:

(32) $\hat{\underset{\sim}{f}}(\underset{\sim}{x}_0) = x^*(\underset{\sim}{x}_0)\underset{\sim}{\beta}(\underset{\sim}{x}_0)$, where $\underset{\sim}{x}_0 = \left(x_{01}, x_{02}, ..., x_{0p}\right)$, and it can be written as:

(33) $$\hat{\underset{\sim}{f}}(\underset{\sim}{x}_0) = x^*(\underset{\sim}{x}_0)\left(\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\mathbf{X}(\underset{\sim}{x}_0)\right)^{-1}\mathbf{X}^{\mathrm{T}}(\underset{\sim}{x}_0)\mathbf{K_h}(\underset{\sim}{x}_0)\mathbf{V}^{-1}\underset{\sim}{y}.$$

Lemma. *In bi-responses multi-predictors local polynomial nonparametric regression, there is a correlation between the responses, providing for model estimation using the local polynomial estimator. We also utilize a weighting matrix for the variance and covariance* **(V)** *of the vector* $\underset{\sim}{\varepsilon}$ *in addition to kernel weighting, as follows:*

(34) $$\mathbf{V} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

where $\Sigma_{rr} = diag\left(\sigma_1^{2(r)}, \sigma_2^{2(r)}, ..., \sigma_n^{2(r)}\right)$, $\Sigma_{rs} = \Sigma_{sr} = diag\left(\sigma_1^{(r)(s)}, \sigma_2^{(r)(s)}, ..., \sigma_n^{(r)(s)}\right)$. **V** *is a weighting matrix of size* 2*n*x2*n*. $\mathbf{\Sigma}_{rr}$ For r = 1,2 *is a diagonal matrix containing the* k^{th} *response error variance.*

Proof of Lemma. In general, the bi-response multi-predictor nonparametric regression model can be expressed in the following equation:

(35) $$\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x}) + \underset{\sim}{\varepsilon}$$

where $\underset{\sim}{y} = \left(y^{(1)} \quad y^{(2)}\right)^T$; $\underset{\sim}{\varepsilon} = \left(\varepsilon^{(1)} \quad \varepsilon^{(2)}\right)^T$; $\underset{\sim}{f} = \left(f_1 \quad f_2\right)^T$.

By taking n paired samples $\left(y_i^{(1)}, y_i^{(2)}, x_{i1}, x_{i2}, ..., x_{ip}\right), i = 1, 2, ..., n$, the equation (35) can be written as follows:

(36) $$\underset{\sim}{y}_i = \underset{\sim}{f}(\underset{\sim}{x}_i) + \underset{\sim}{\varepsilon}_i, i = 1, 2, ..., n$$

where error random $\underset{\sim}{\varepsilon}$ follows the following assumptions:

(37) $$E\left(\varepsilon_{ri}\right) = 0, \text{ and } E\left(\varepsilon_{ri}^2\right) = \sigma_{ri}^2, E\left(\varepsilon_{ri}\varepsilon_{si}\right) = \begin{cases} \sigma_{(rs)i}, & i = i' \\ 0, & i \neq i' \end{cases} \text{ for } r \neq s,\ r = s = 1, 2.$$

Based on i^{th} observation and for r responses, we can elaborate the equation (36) as follows:

(38) $$\left.\begin{array}{l} y_1^{(1)} = \sum_{j=1}^{p} f_{1j}\left(x_{1j}\right) + \varepsilon_1^{(1)} \\ y_2^{(1)} = \sum_{j=1}^{p} f_{1j}\left(x_{2j}\right) + \varepsilon_2^{(1)} \\ \vdots \\ y_n^{(1)} = \sum_{j=1}^{p} f_{1j}\left(x_{nj}\right) + \varepsilon_n^{(1)} \\ y_1^{(2)} = \sum_{j=1}^{p} f_{2j}\left(x_{1j}\right) + \varepsilon_1^{(2)} \\ y_2^{(2)} = \sum_{j=1}^{p} f_{2j}\left(x_{2j}\right) + \varepsilon_2^{(2)} \\ \vdots \\ y_n^{(2)} = \sum_{j=1}^{p} f_{2j}\left(x_{nj}\right) + \varepsilon_n^{(2)} \end{array}\right\}$$

Equation (37) can be expressed in the following matrix notation:

(39) $$\underset{\sim}{y} = \underset{\sim}{f} + \underset{\sim}{\varepsilon}$$

where $\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{y}^{(1)} \\ \underset{\sim}{y}^{(2)} \end{bmatrix}; \underset{\sim}{f} = \begin{bmatrix} \underset{\sim}{f}_1 \\ \underset{\sim}{f}_2 \end{bmatrix}; \underset{\sim}{f}_r = \begin{bmatrix} \underset{\sim}{f}_{r1} \\ \underset{\sim}{f}_{r2} \\ \vdots \\ \underset{\sim}{f}_{rp} \end{bmatrix}; f_{rj} = \sum_{j=1}^{p} f_{rj}(x_{ij}); \underset{\sim}{\varepsilon} = \begin{bmatrix} \underset{\sim}{\varepsilon}^{(1)} \\ \underset{\sim}{\varepsilon}^{(2)} \end{bmatrix};$

$$Var\left(\underset{\sim}{\varepsilon}\right)=Var\begin{bmatrix}\underset{\sim}{\varepsilon}^{(1)}\\ \underset{\sim}{\varepsilon}^{(2)}\end{bmatrix};\ \underset{\sim}{\varepsilon}^{(1)}=\begin{bmatrix}\varepsilon_1^{(1)}\\ \varepsilon_2^{(1)}\\ \vdots\\ \varepsilon_n^{(1)}\end{bmatrix};\underset{\sim}{\varepsilon}^{(2)}=\begin{bmatrix}\varepsilon_1^{(2)}\\ \varepsilon_2^{(2)}\\ \vdots\\ \varepsilon_n^{(2)}\end{bmatrix}.$$

Based on equation (39), there is a correlation between the first and the second response, so to get the estimated model using the bi-response multi-predictor local polynomial estimator, we have to use weights either kernel function or kernel function covariance matrix. Statistically, the weight matrix is inverse of the covariance matrix of the error vector $\underset{\sim}{\varepsilon}$. If we notate the covariance matrix as **V**, then we can determine **V** as follows:

$$\mathbf{V}=Var\left(\underset{\sim}{\varepsilon}\right)=E\left(\underset{\sim}{\varepsilon}^T\underset{\sim}{\varepsilon}\right)-\left[E\left(\underset{\sim}{\varepsilon}^T\right)E\left(\underset{\sim}{\varepsilon}\right)\right]=E\left(\underset{\sim}{\varepsilon}^T\underset{\sim}{\varepsilon}\right)$$

$$=E\left[\varepsilon_1^{(1)}\quad \varepsilon_2^{(1)}\quad \cdots\quad \varepsilon_n^{(1)}\quad \varepsilon_1^{(2)}\quad \varepsilon_2^{(2)}\quad \cdots\quad \varepsilon_n^{(2)}\right]^T\left[\varepsilon_1^{(1)}\quad \varepsilon_2^{(1)}\quad \cdots\quad \varepsilon_n^{(1)}\quad \varepsilon_1^{(2)}\quad \varepsilon_2^{(2)}\quad \cdots\quad \varepsilon_n^{(2)}\right]$$

Next, by considering the assumption in (37), the covariance matrix of **V** can be obtained as follows:

$$\mathbf{V}=E\begin{bmatrix}
\varepsilon_1^{(1)}\varepsilon_1^{(1)} & \varepsilon_1^{(1)}\varepsilon_2^{(1)} & \cdots & \varepsilon_1^{(1)}\varepsilon_n^{(1)} & \varepsilon_1^{(1)}\varepsilon_1^{(2)} & \varepsilon_1^{(1)}\varepsilon_2^{(2)} & \cdots & \varepsilon_1^{(1)}\varepsilon_n^{(2)}\\
\varepsilon_2^{(1)}\varepsilon_1^{(1)} & \varepsilon_2^{(1)}\varepsilon_2^{(1)} & \cdots & \varepsilon_2^{(1)}\varepsilon_n^{(1)} & \varepsilon_2^{(1)}\varepsilon_1^{(2)} & \varepsilon_2^{(1)}\varepsilon_2^{(2)} & \cdots & \varepsilon_2^{(1)}\varepsilon_n^{(2)}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\varepsilon_n^{(1)}\varepsilon_1^{(1)} & \varepsilon_n^{(1)}\varepsilon_2^{(1)} & \cdots & \varepsilon_n^{(1)}\varepsilon_n^{(1)} & \varepsilon_n^{(1)}\varepsilon_1^{(2)} & \varepsilon_n^{(1)}\varepsilon_2^{(2)} & \cdots & \varepsilon_n^{(1)}\varepsilon_n^{(2)}\\
\varepsilon_1^{(2)}\varepsilon_1^{(1)} & \varepsilon_1^{(2)}\varepsilon_2^{(1)} & \cdots & \varepsilon_1^{(2)}\varepsilon_n^{(1)} & \varepsilon_1^{(2)}\varepsilon_1^{(2)} & \varepsilon_1^{(2)}\varepsilon_2^{(2)} & \cdots & \varepsilon_1^{(2)}\varepsilon_n^{(2)}\\
\varepsilon_2^{(2)}\varepsilon_1^{(1)} & \varepsilon_2^{(2)}\varepsilon_2^{(1)} & \cdots & \varepsilon_2^{(2)}\varepsilon_n^{(1)} & \varepsilon_2^{(2)}\varepsilon_1^{(2)} & \varepsilon_2^{(2)}\varepsilon_2^{(2)} & \cdots & \varepsilon_2^{(2)}\varepsilon_n^{(2)}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\varepsilon_n^{(2)}\varepsilon_1^{(1)} & \varepsilon_n^{(2)}\varepsilon_2^{(1)} & \cdots & \varepsilon_n^{(2)}\varepsilon_n^{(1)} & \varepsilon_n^{(2)}\varepsilon_1^{(2)} & \varepsilon_n^{(2)}\varepsilon_2^{(2)} & \cdots & \varepsilon_n^{(2)}\varepsilon_n^{(2)}
\end{bmatrix}$$

$$=\begin{bmatrix}
E\left(\varepsilon_1^{(1)}\varepsilon_1^{(1)}\right) & E\left(\varepsilon_1^{(1)}\varepsilon_2^{(1)}\right) & \cdots & E\left(\varepsilon_1^{(1)}\varepsilon_n^{(1)}\right) & E\left(\varepsilon_1^{(1)}\varepsilon_1^{(2)}\right) & E\left(\varepsilon_1^{(1)}\varepsilon_2^{(2)}\right) & \cdots & E\left(\varepsilon_1^{(1)}\varepsilon_n^{(2)}\right)\\
E\left(\varepsilon_2^{(1)}\varepsilon_1^{(1)}\right) & E\left(\varepsilon_2^{(1)}\varepsilon_2^{(1)}\right) & \cdots & E\left(\varepsilon_2^{(1)}\varepsilon_n^{(1)}\right) & E\left(\varepsilon_2^{(1)}\varepsilon_1^{(2)}\right) & E\left(\varepsilon_2^{(1)}\varepsilon_2^{(2)}\right) & \cdots & E\left(\varepsilon_2^{(1)}\varepsilon_n^{(2)}\right)\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
E\left(\varepsilon_n^{(1)}\varepsilon_1^{(1)}\right) & E\left(\varepsilon_n^{(1)}\varepsilon_2^{(1)}\right) & \cdots & E\left(\varepsilon_n^{(1)}\varepsilon_n^{(1)}\right) & E\left(\varepsilon_n^{(1)}\varepsilon_1^{(2)}\right) & E\left(\varepsilon_n^{(1)}\varepsilon_2^{(2)}\right) & \cdots & E\left(\varepsilon_n^{(1)}\varepsilon_n^{(2)}\right)\\
E\left(\varepsilon_1^{(2)}\varepsilon_1^{(1)}\right) & E\left(\varepsilon_1^{(2)}\varepsilon_2^{(1)}\right) & \cdots & E\left(\varepsilon_1^{(2)}\varepsilon_n^{(1)}\right) & E\left(\varepsilon_1^{(2)}\varepsilon_1^{(2)}\right) & E\left(\varepsilon_1^{(2)}\varepsilon_2^{(2)}\right) & \cdots & E\left(\varepsilon_1^{(2)}\varepsilon_n^{(2)}\right)\\
E\left(\varepsilon_2^{(2)}\varepsilon_1^{(1)}\right) & E\left(\varepsilon_2^{(2)}\varepsilon_2^{(1)}\right) & \cdots & E\left(\varepsilon_2^{(2)}\varepsilon_n^{(1)}\right) & E\left(\varepsilon_2^{(2)}\varepsilon_1^{(2)}\right) & E\left(\varepsilon_2^{(2)}\varepsilon_2^{(2)}\right) & \cdots & E\left(\varepsilon_2^{(2)}\varepsilon_n^{(2)}\right)\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
E\left(\varepsilon_n^{(2)}\varepsilon_1^{(1)}\right) & E\left(\varepsilon_n^{(2)}\varepsilon_2^{(1)}\right) & \cdots & E\left(\varepsilon_n^{(2)}\varepsilon_n^{(1)}\right) & E\left(\varepsilon_n^{(2)}\varepsilon_1^{(2)}\right) & E\left(\varepsilon_n^{(2)}\varepsilon_2^{(2)}\right) & \cdots & E\left(\varepsilon_n^{(2)}\varepsilon_n^{(2)}\right)
\end{bmatrix}$$

$$=\begin{bmatrix} \sigma_1^{2(1)} & 0 & \cdots & 0 & \sigma_1^{(1)(2)} & 0 & \cdots & 0 \\ 0 & \sigma_2^{2(1)} & \cdots & 0 & 0 & \sigma_2^{(1)(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^{2(1)} & 0 & 0 & \cdots & \sigma_n^{(1)(2)} \\ \sigma_1^{(2)(1)} & 0 & \cdots & 0 & \sigma_1^{2(2)} & 0 & \cdots & 0 \\ 0 & \sigma_2^{(2)(1)} & \cdots & 0 & 0 & \sigma_2^{2(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^{(2)(1)} & 0 & 0 & \cdots & \sigma_n^{2(2)} \end{bmatrix}.$$

$$(40)\quad \mathbf{V}=\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \text{ where } \Sigma_{rr}=diag\left(\sigma_1^{2(r)},\sigma_2^{2(r)},\ldots,\sigma_n^{2(r)}\right),\ \Sigma_{rs}=\Sigma_{sr}=diag\left(\sigma_1^{(r)(s)},\sigma_2^{(r)(s)},\ldots,\sigma_n^{(r)(s)}\right).$$

3.2. Simulation Study

In this study, a simulation was conducted to determine the performance of bi-response local polynomial regression in predicting two response variables. Performance indicators use the mean square error (MAPE) and mean absolute percentage error (MAPE). This simulation study also creates an algorithm for estimating the bi-response multi-predictor local polynomial regression and its predictive performance.

Algorithms for estimating the bi-response multi-predictor nonparametric regression model based on the local polynomial estimator are as follows:

(a). Algorithm for determining the optimal bandwidth value without a weighting matrix $(\mathbf{V}^{-1})$.

Step 1: Defining the response variable $\underset{\sim}{y}^{(r)}, r=1,2$ and the predictor variables $\underset{\sim}{x}_j, j=1,2,\ldots,p$;

Step 2: Determine the kernel function used, based on the Gaussian Kernel, according to the equation: $\mathbf{K_h}(x)=\frac{1}{h}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{1}{2}\left(-\frac{x}{h}\right)^2\right), -\infty<x<\infty;$

Step 3: Defining the matrix $\mathbf{X}(\underset{\sim}{x}_0)$ according to equation (27);

Step 4: Determine the set of bandwidth values on the order of 1 to d on each predictor for each response, namely (h_c) where *bb* is the lower limit and *ba* is the upper limit of the bandwidth value, and δ is the addition of the bandwidth value for each iteration.

Step 5: Determine the diagonal matrix ($\mathbf{K_h}(x)$) for the response variable $\underset{\sim}{y}^{(r)}, r=1,2$;

Step 6: Calculate the GCV value for each $h_c \in [seq(bb, ba, \delta)]$ on the order of 1 to d using the following equation:

$$GCV(h) = \frac{n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(y_{ij} - \hat{f}(x_{ij})\right)^2}{\left(1 - tr\left[\mathrm{I} - A(h_j)\right]/n\right)^2}$$ where $n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(y_{ij} - \hat{f}(x_{ij})\right)^2$ is a MSE;

Step 7: Repeat steps (4) to (6) for different values of bandwidth (h_c) on the order of 1 to d until the minimum GCV value is obtained;

Step 8: The bandwidth value (h_c) that produces the minimum GCV value is the optimal bandwidth value (h_c) on a predetermined order;

Step 9: Plotting between bandwidth and GCV values.

(b). Algorithm for determining weighting matrix $(\mathbf{V}^{-1})$**.**

Step 1: Defining the response variable $\underset{\sim}{y}^{(r)}, r=1,2$ and the predictor variables $\underset{\sim}{x}_j, j=1,2,...,p$;

Step 2: Determine the kernel function used, based on the Gaussian Kernel, according to the following equation:

$$\mathbf{K_h}(x) = \frac{1}{h}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{1}{2}\left(-\frac{x}{h}\right)^2\right), -\infty < x < \infty;$$

Step 3: Defining the matrix $\mathbf{X}(\underset{\sim}{x}_0)$ according to equation (27);

Step 4: Enter the optimal order and bandwidth values obtained from Algorithm (a).

Step 5: Determine the diagonal matrix ($\mathbf{K_h}(x)$) for the response variable $\underset{\sim}{y}^{(r)}, r=1,2$;

Step 6: Estimating the beta value as in equation (31);

Step 7: Estimating the value of $\hat{y}$;

Step 8: Calculating the error value of the first response and the second response;

Step 9: Calculating the variance and covariance of error of the two responses (equation 40), which $\sigma_{11,}\sigma_{22,}\sigma_{12,}$ and $\sigma_{21,}$ is obtained where σ_{11} the error variance of response 1, σ_{22} is the error variance of response two, and σ_{12} and σ_{21} are the error variance between the first and second response. The calculation of $\widehat{\mathbf{V}}$ based on [8]:

(41) $$\hat{\mathbf{V}} = \begin{bmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} \end{bmatrix}$$

where

$$\Sigma_{11} = diag\left(\hat{\sigma}_1^{2(1)}, \hat{\sigma}_2^{2(1)}, ..., \hat{\sigma}_n^{2(1)}\right) = diag\left(\frac{\hat{\underset{\sim}{Y}}_r' Q_{x_1}^{(11)} \underset{\sim}{Y}_r}{tr\left(Q_{x_1}^{(11)}\right)}, \frac{\hat{\underset{\sim}{Y}}_r' Q_{x_2}^{(11)} \underset{\sim}{Y}_r}{tr\left(Q_{x_2}^{(11)}\right)}, ..., \frac{\hat{\underset{\sim}{Y}}_r' Q_{x_n}^{(11)} \underset{\sim}{Y}_r}{tr\left(Q_{x_n}^{(11)}\right)}\right);$$

$$\Sigma_{22} = diag\left(\hat{\sigma}_1^{2(2)}, \hat{\sigma}_2^{2(2)}, ..., \hat{\sigma}_n^{2(2)}\right) = diag\left(\frac{\hat{\underset{\sim}{Y}}_r' Q_{x_1}^{(22)} \underset{\sim}{Y}_r}{tr\left(Q_{x_1}^{(22)}\right)}, \frac{\hat{\underset{\sim}{Y}}_r' Q_{x_2}^{(22)} \underset{\sim}{Y}_r}{tr\left(Q_{x_2}^{(22)}\right)}, ..., \frac{\hat{\underset{\sim}{Y}}_r' Q_{x_n}^{(22)} \underset{\sim}{Y}_r}{tr\left(Q_{x_n}^{(22)}\right)}\right);$$

$$\Sigma_{12} = \Sigma_{21} = diag\left(\hat{\sigma}_1^{2(1)(2)}, \hat{\sigma}_2^{2(1)(2)}, ..., \hat{\sigma}_n^{2(1)(2)}\right) = diag\left(\frac{\hat{\underset{\sim}{Y}}_r' Q_{x_1}^{(1)(2)} \underset{\sim}{Y}_r}{tr\left(Q_{x_1}^{(1)(2)}\right)}, \frac{\hat{\underset{\sim}{Y}}_r' Q_{x_2}^{(1)(2)} \underset{\sim}{Y}_r}{tr\left(Q_{x_2}^{(1)(2)}\right)}, ..., \frac{\hat{\underset{\sim}{Y}}_r' Q_{x_n}^{(1)(2)} \underset{\sim}{Y}_r}{tr\left(Q_{x_n}^{(1)(2)}\right)}\right)$$

$$Q_{x_0}^{(11)} = (I - A_{x_0}^{(1)})'(I - A_{x_0}^{(1)}); Q_{x_0}^{(22)} = (I - A_{x_0}^{(2)})'(I - A_{x_0}^{(2)}); Q_{x_0}^{(1)(2)} = Q_{x_0}^{(2)(1)} = (I - A_{x_0}^{(1)})'(I - A_{x_0}^{(2)})$$

for $x_0 = x_i, i = 1, 2, ..., n$, we have:

$$A_{x_i}^{(r)} = X_{x_i}^{(r)} \left(X_{x_i}^{(r)'} K_h^{(r)}(x_i) X_{x_i}^{(r)}\right)^{-1} X_{x_0}^{(r)} K_h^{(r)}(x_i), \text{ where } r = 1, 2 \text{ ; } i = 1, 2, ..., n$$

Step 10: Form a diagonal matrix of each element that has been replicated N times where N is the vector length of $\varepsilon^{(r)}, r = 1, 2;$

Step 11: Merge the diagonal matrices obtained in Step (10).

(c). Algorithm for determining the optimal bandwidth value with a weighting matrix $(\mathbf{V}^{-1})$**.**

Step 1: Defining the response variable $\underset{\sim}{y}^{(r)}, r = 1, 2$ and the predictor variables $\underset{\sim}{x}_j, j = 1, 2, ..., p;$

Step 2: Determine the kernel function used, based on the Gaussian Kernel, according to the following equation:

$$\mathbf{K_h}(x) = \frac{1}{h} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{2}\left(-\frac{x}{h}\right)^2\right), -\infty < x < \infty;$$

Step 3: Defining the matrix $\mathbf{X}(\underset{\sim}{x}_0)$ according to equation (27);

Step 4: Determine the weighting matrix ($\mathbf{V^{-1}}$) obtained in equation (41);

Step 5: Determine the set of bandwidth values on the order of 1 to *d* on each predictor for each response, namely (h_c) where *bb* is the lower limit and *ba* is the upper limit of the bandwidth value, and δ is the addition of the bandwidth value for each iteration;

Step 6: Determine the diagonal matrix ($\mathbf{K_h}(x)$) for the response variable $\underset{\sim}{y}^{(r)}, r=1,2;$

Step 7: Calculate the GCV value for each $h_c \in [seq(bb, ba, \delta)]$ on the order of 1 to *d* using the following Equation:

$$GCV(h) = \frac{n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(y_{ij} - \hat{f}(x_{ij})\right)^2}{\left(1 - tr\left[\mathrm{I} - A(h_j)\right]/n\right)^2} \quad \text{where} \quad n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(y_{ij} - \hat{f}(x_{ij})\right)^2 \text{ is a MSE;}$$

Step 8: Repeat steps (4) to (6) for different values of bandwidth (h_c) on the order of 1 to *d* until the minimum GCV value is obtained;

Step 9: The bandwidth value (h_c) that produces the minimum GCV value is the optimal bandwidth value (h_c) on a predetermined order;

Step 10: Plotting between bandwidth and GCVvalues.

(d). The model estimation algorithm uses a weighted matrix **($\mathbf{V^{-1}}$)**.

Step 1: Defining the response variable $\underset{\sim}{y}^{(r)}, r=1,2$ and the predictor variables $\underset{\sim}{x}_j, j=1,2,...,p;$

Step 2: Determine the kernel function used, based on the Gaussian Kernel, according to the following equation:

$$\mathbf{K_h}(x) = \frac{1}{h}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{1}{2}\left(-\frac{x}{h}\right)^2\right), -\infty < x < \infty;$$

Step 3: Defining the matrix $\mathbf{X}(\underset{\sim}{x}_0)$ according to equation (27);

Step 4: Enter the optimal order and bandwidth values obtained from Algorithm (a);

Step 5: Inserting the weighting matrix $(\mathbf{V^{-1}})$ obtained in equation (41);

Step 6: Determine the diagonal matrix ($\mathbf{K_h}$) for the response variable $\underset{\sim}{y}^{(r)}, r = 1,2;$

Step 7: Estimating the beta value as in equation (31);

Step 8: Estimating the value of $(\hat{\boldsymbol{y}})$;

Step 9: Calculating the error value of the first and second responses;

Step 10: Calculating MAPE and MSE values.

(e). Implementation of the nonparametric model for trigonometric function.

Firstly, we generate a uniformly distributed x_i variables, $x_i \sim U(1,4)$. Next, we determine the regression functions $f_r(x_i),\ r = 1,2$, $i = 1,2,...,120$; namely $f_1(x_i) = 4 + 2\sin(2\pi x_i)$ and $f_2(x_i) = 3 + 3\sin(2\pi x_i)$; and then we determine the $\boldsymbol{\Omega}$ matrix by giving values: $\rho_{12} = 0.9;\ \sigma_1^2 = 0.2;\ \sigma_2^2 = 0.25.$

Scatter plots for the first and the second responses of simulation data for trigonometric functions are given in Figure 1.

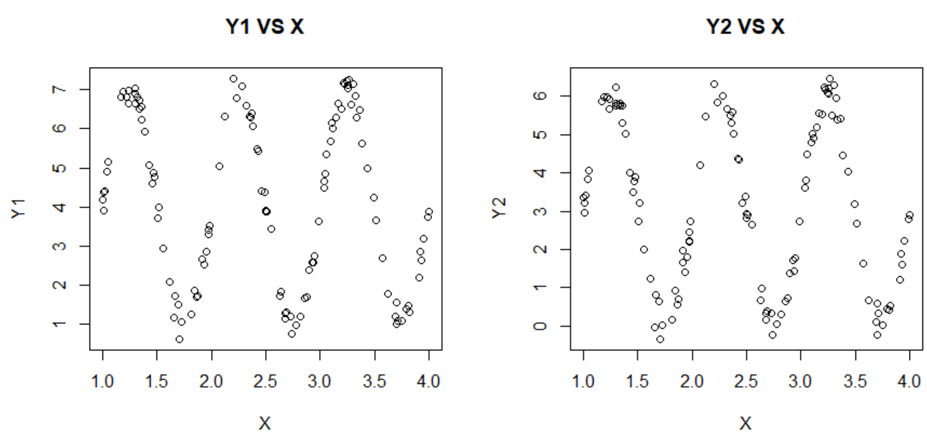


Figure 1. Scatter plots data simulation data for trigonometric functions.

In this step, the data is divided into two randomly, 30 out-sample data and 90 in-sample data. The optimal bandwidth for each response is determined at polynomial degrees one to three (d=1,2,3) on in-sample data. The optimal bandwidth to be used is the bandwidth at the polynomial degree, which has the minimum GCV value. Based on the in-sample data in the simulation, the optimal bandwidth is obtained based on the minimum GCV value on a second degrees polynomial (see Table 1).

Table 1. Bandwidth and GCV values of polynomial degree d=1,2,3 for the first response.

Polynomial Degrees	h_1	GCV
1	0.05	0.06892674
2	**0.10**	**0.05236624**
3	0.11	0.05328136

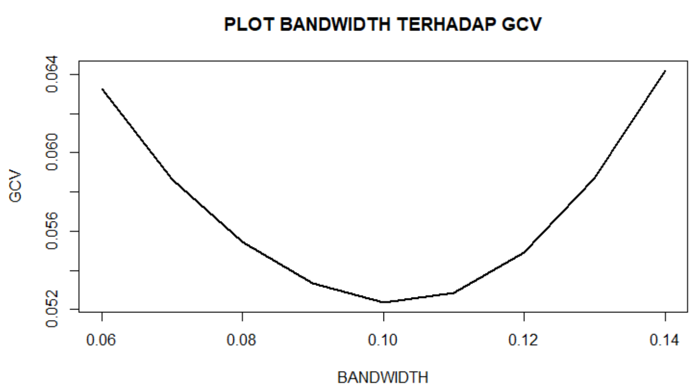


Figure 1. Plot of bandwidth versus GCV of trigonometric function type (d=2, r=1).

Table 2. Bandwidth and GCV values of polynomial degree d=1,2,3 for the second response.

Polynomial Degrees	h_2	GCV
1	0.05	0.10431970
2	**0.10**	**0.08492393**
3	0.12	0.08703441

Based on Table 1 and Table 2, the optimal bandwidth used as a smoothing parameter in the local polynomial regression estimation process is h_1 = 0.10 and h_2 = 0.10 at second degree polynomial

(d = 2) as presented in bold letter of Table 1 and Table 2.

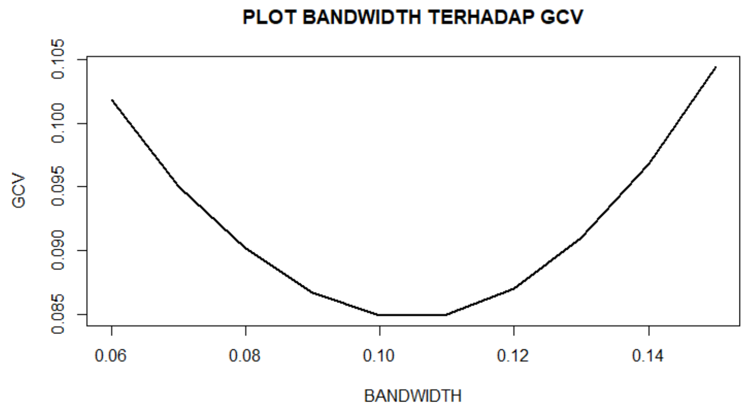


Figure 2. Plot of bandwidth versus GCV of trigonometric function type (d=2, r=2).

Based on the optimal bandwidth (h_1 = 0.10 and h_2 = 0.10) for the second-degree polynomial, the plot of the estimated results of the two responses on in-sample data is shown in Figure 3.

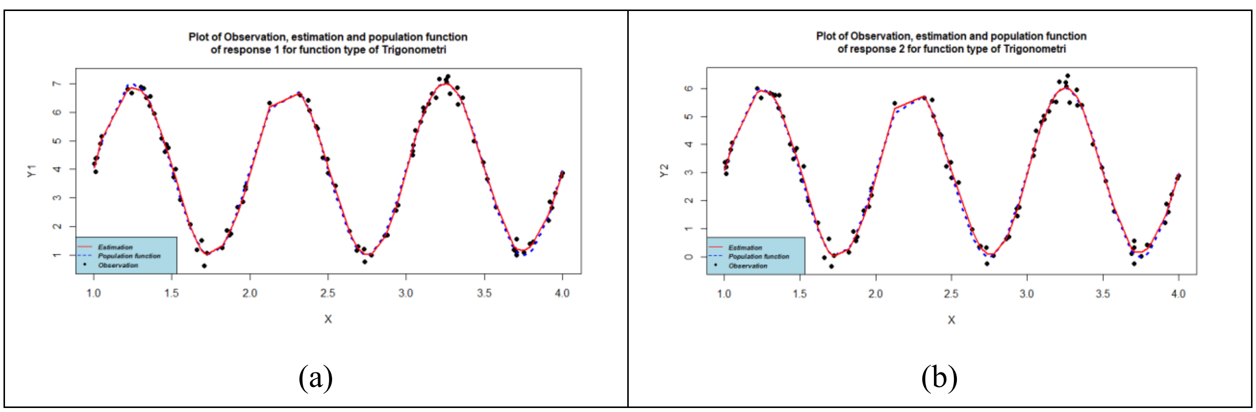


Figure 3. Plots of observation, estimation, and population function of the first response (a) and the second response (b) for function trigonometric using optimal bandwidth values.

We also simulate the estimation of the two responses for in-sample data based on greater than the optimal bandwidth values, *i.e.,* h_1=0.55 and h_2 = 0.6 (Fig. 4a and 4b), or smaller than the optimal bandwidth values, *i.e.,* h_1=0.01 and h_2=0.02 (Fig. 4c and 4d). The results are given in Figure 4.

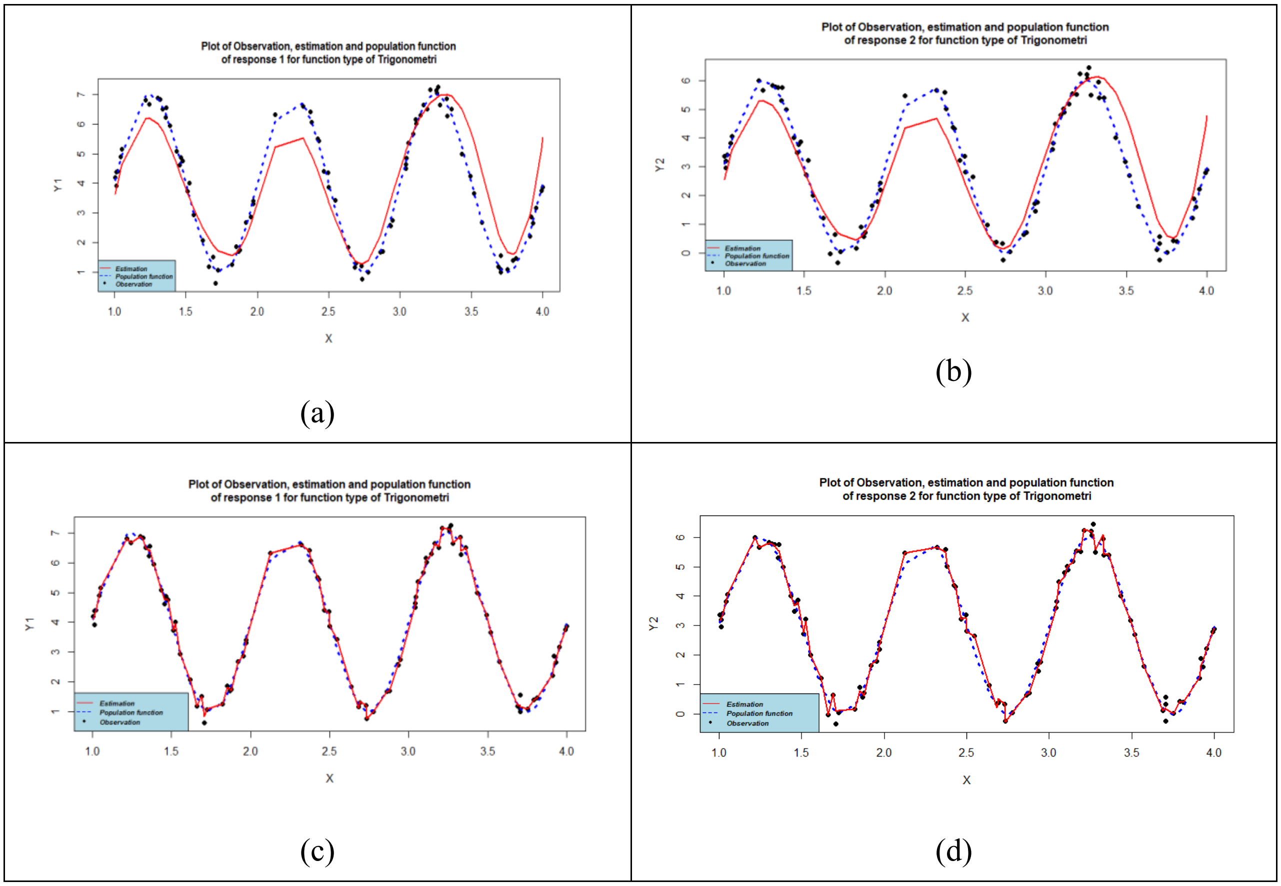


Figure 4. Plots of observation, estimation, and population function of two responses for function type of trigonometric using greater than optimal bandwidth values [plots (a) and (b)], and using smaller than optimal bandwidth values [plots (c) and (d)].

Figure 4 shows that the estimation of the regression function using the optimal bandwidth gives an estimation plot that closes to the population values. It can be seen from the red line (estimated), which coincides with the blue line at optimal bandwidth conditions. While the estimation for greater than optimal bandwidth value, it gives a smoother estimation plot, but the estimated value (red line) is far from the population (blue line). On the other hand, the estimation for smaller than the optimal bandwidth values gives a rough plot, and the estimated value closes to the observed value.

After determining the bandwidth and estimating the in-sample data, the next step is to predict the values of the response variable on the out-sample data. The goodness of fit of prediction values is measured based on two criteria, including the MSE (mean squared error) value and the MAPE (mean absolute percentage error) value. Table 3 shows the MSE and MAPE values for the out-sample data prediction process using the optimal bandwidth, greater than, and smaller than the optimal bandwidth values. While Figure 5 shows an estimation plot for out-sample data at various bandwidth values.

Table 3. Comparison of MSE and MAPE values in the three estimation conditions on trigonometric functions.

Bandwidth	MSE Value	MAPE Value
Optimal bandwidths (h_1=0.1 ; h_2=0.1)	**0.056518**	**10.70283**
Greater than optimal bandwidths (h_1=0.55 ; h_2 = 0.6)	0.581681	41.55267
Smaller than optimal bandwidths (h_1=0.01 ; h_2=0.02)	2.121037	19.07117

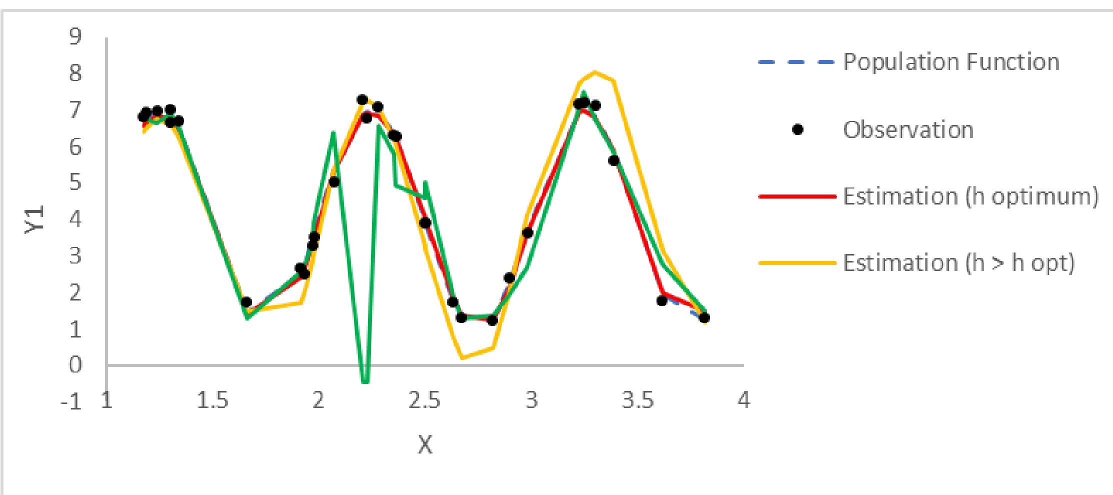


Figure 5. Plots of observation, estimation, and population function of the first response for function trigonometric using optimal bandwidth, greater and smaler than optimal bandwidth values.

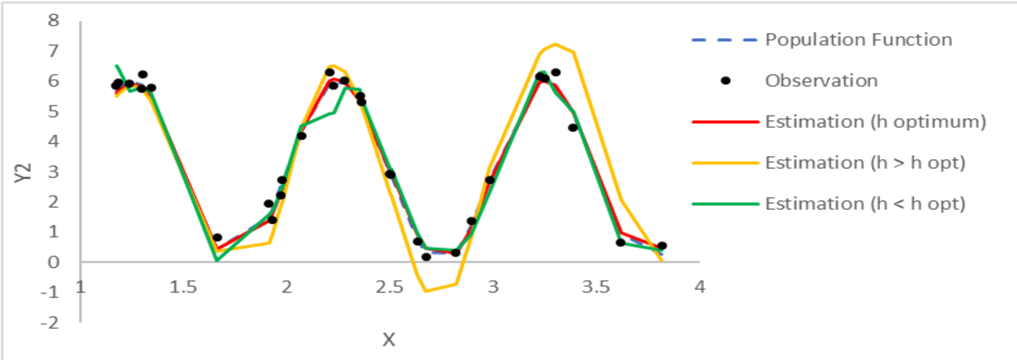


Figure 6. Plots of observation, estimation, and population function of the second response for function type of trigonometric using optimal bandwidth, greater and smaller than optimal bandwidth values.

(f). Implementation of the nonparametric model for the exponential function.

Firstly, we generate a uniformly distributed x_i variables, $x_i \sim U(1,4)$. Next, we determine the regression functions $f_r(x_i)$, $r = 1,2$, $i = 1,2,...,120$: namely $f_1(x_i) = 4\exp(2x_i)$ and $f_2(x_i) == 3\exp(2x_i)$; and then we determine the $\mathbf{\Omega}$ matrix by giving values: $\rho_{12} = 0.8$; $\sigma_1^2 = 0.6$; $\sigma_2^2 = 0.8$.

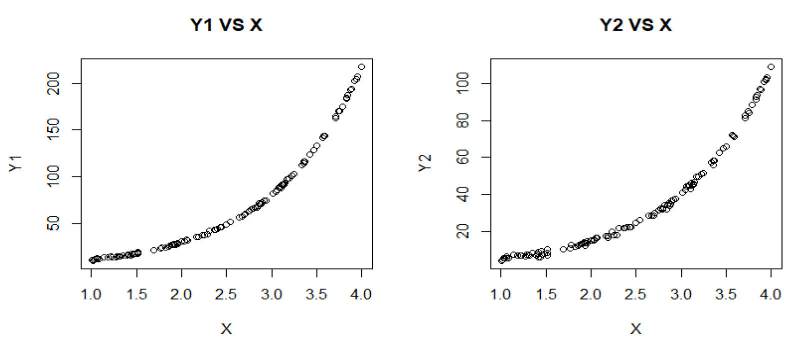


Figure 7. Plots of the first and the second responses on simulation data for the exponential function.

In this step, the data is divided into two randomly, 30 out-sample data and 90 in-sample data. Determination of the optimal bandwidth for each response is carried out at polynomial degrees one to three (d=1,2,3). The optimal bandwidth to be used is the bandwidth at the polynomial degree, which has the minimum GCV value. Based on the in-sample data in the simulation, the optimal bandwidth is obtained based on the minimum GCV value on a second degrees polynomial (see Table 4 and Table 5).

Table 4. Bandwidth and GCV values of polynomial degree d=1,2,3 for the first response.

Polynomial Degrees	h_1	GCV
1	0.07	0.4871139
2	**0.16**	**0.4235946**
3	0.18	0.4322939

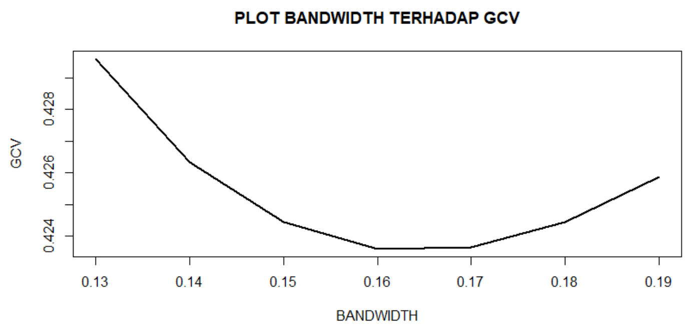


Figure 8. Plot of bandwidth versus GCV for the exponential function type (d=2, r=1).

Table 5. Bandwidth and GCV values of polynomial degree d=1,2,3 for the second response.

Polynomial Degrees	h_2	GCV
1	0.10	0.8520331
2	**0.22**	**0.8017685**
3	0.44	0.7953196

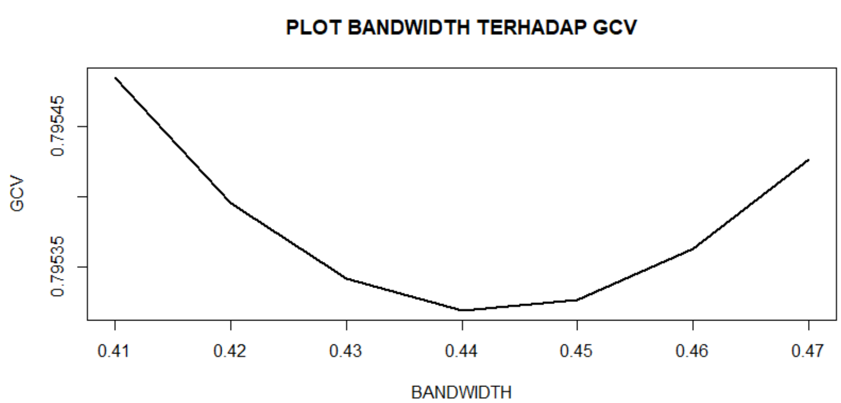


Figure 9. Plot of bandwidth versus GCV for the exponential function type (d=2, r=2).

Based on the optimal bandwidth (h_1 = 0.16 and h_2 = 0.22), we obtain the plot of the estimated results of the two responses on in-sample data which is shown in Figure 10. In addition, we simulate the estimation for in-sample data using non-optimal bandwidth values. Figures 11a and 11b are estimation plots using bandwidth values that greater than the optimal bandwidth values, h_1=3.5, and h_2=3.7. Figures 11c and 11d are estimation plots using bandwidth values that smaller than the optimal values, h_1=0.009, and h_2=0.009.

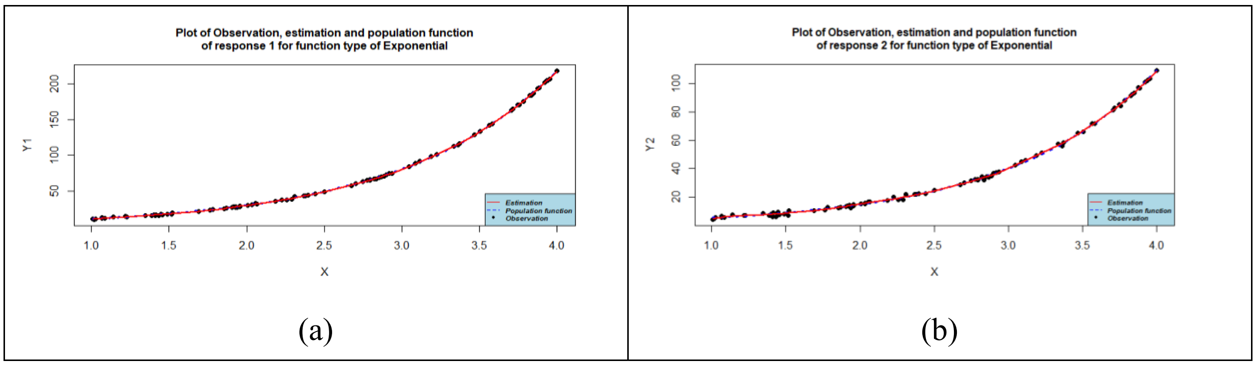
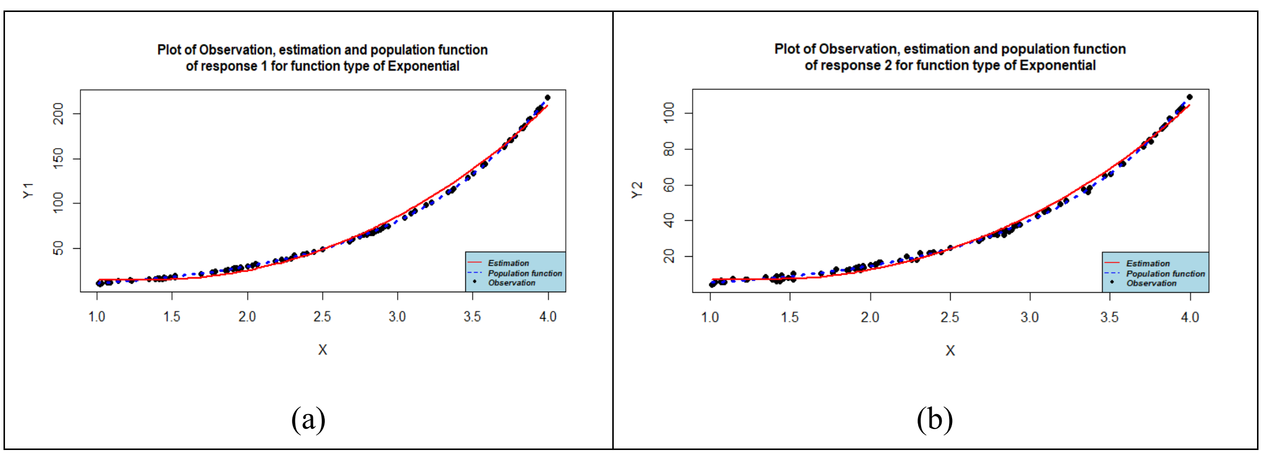


Figure 10. Plots of observation, estimation and population function for the first response (a) and the second response (b) for trigonometric function type using optimal bandwidth values (h_1 = 0.16 and h_2 = 0.22).



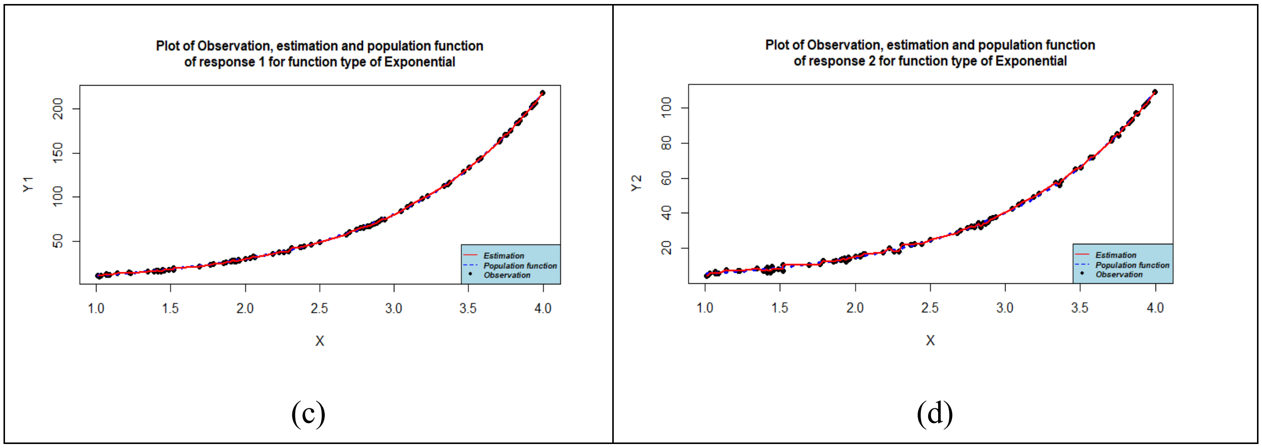


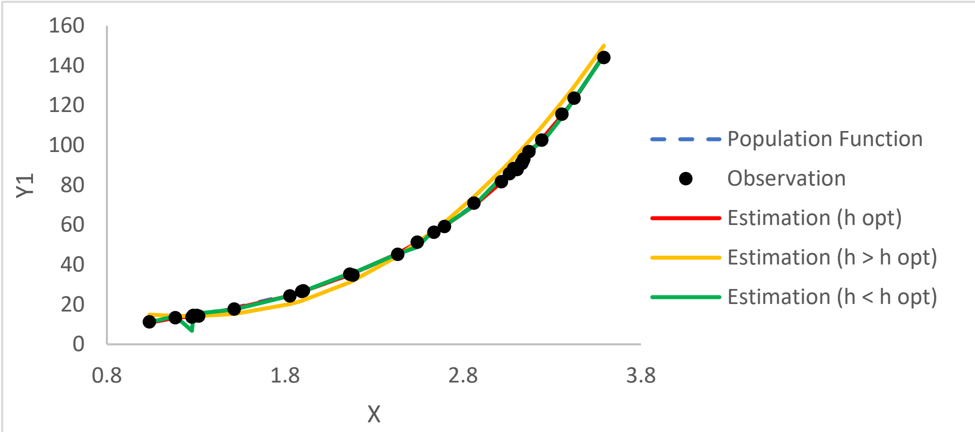
Figure 11. Plots of observation, estimation, and population function of two responses for Exponential function type using greater and smaller than optimal bandwidth values.

Next, we predict the two responses on out-sample data based on the optimal bandwidth on in-sample data. Besides, we also used non-optimal bandwidth values to indicate the two responses on out-sample data. This step can provide that optimal bandwidth is crucial because of affects the roughness of the estimation plot and the MSE and MAPE values (see Table 6).

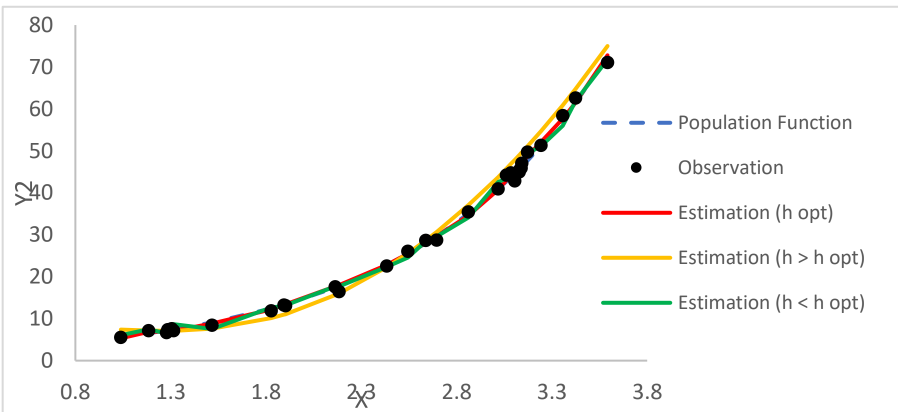
Table 6. Comparison of MSE and MAPE values in the three estimation conditions on exponential functions.

Bandwidth	MSE Value	MAPE Value
Optimal bandwidths ($h_1 = 0.16$; $h_2 = 0.22$)	0.642678	1.977667
Greater than optimal bandwidths (h_1=3.5 ; h_2=3.7)	12.82616	7.368667
Smaller than optimal bandwidths (h_1=0.009 ; h_2=0.009)	1.970840	3.989000

Table 6 shows the MSE and MAPE values in the out-sample data prediction process using the optimal bandwidth, greater and smaller than the optimal bandwidth values. While Figures 12 and 13 provide estimation plots on out-sample data at various bandwidth values.



Gambar 12. The plot of observation, estimation, and population function of first response for exponential function type using optimal bandwidth, above and below optimal bandwidth value.



Gambar 13. The plot of observation, estimation, and population function of second response for exponential function type using optimal bandwidth, above and below optimal bandwidth value.

(g). Implementation of the nonparametric model for the polynomial function.

Firstly, we generate a uniformly distributed $x_i$variables, $x_i \sim U(1,4)$. Next, we determine the regression functions $f_r(x_i),\ r=1,2\ ,\ i=1,2,...,120$: namely $f_1(x_i)=x^2-4x+2$ and $f_2(x_i)==x^2-5x+3;$ and then determine the $\boldsymbol{\Omega}$ matrix by giving values:

$\rho_{12}=0.9;\ \sigma_1^2=0.15;\ \sigma_2^2=0.18.$

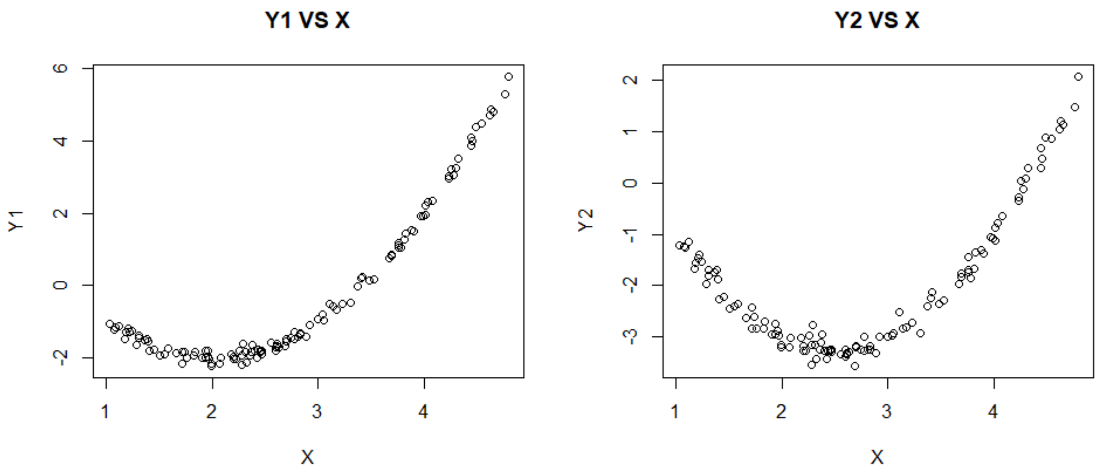


Figure 12. Simulation data for the polynomial function.

We divided the data into two parts, 90 as in-sample data and 30 as out-sample data. The optimal bandwidth is determined based on the minimum GCV value on in-sample data. This process is done at one to three degrees polynomial (see Table 7 and Table 8). Then, we choose the one that has the minimum GCV value.

Table 7. Bandwidth and GCV values at polynomial degree d=1,2,3 in the first response.

Polynomial Degrees	h_1	GCV
1	0.14	0.01635847
2	**2.18**	**0.01487016**
3	7.18	0.01484882

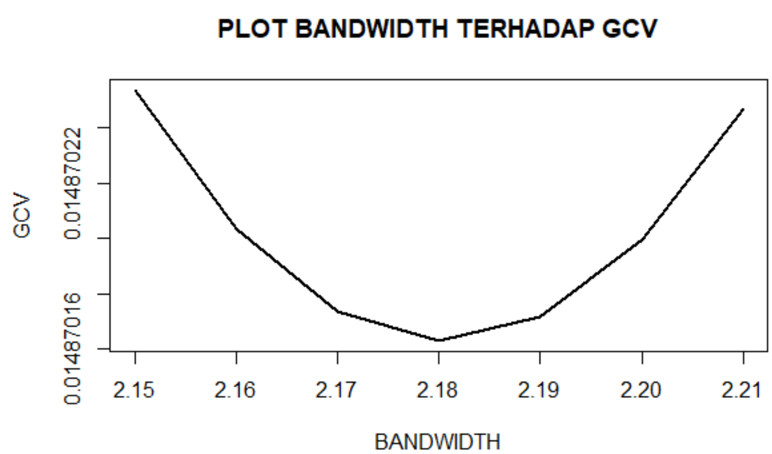


Figure 13. The plot of bandwidth and GCV on the polynomial function type (d=2, r=1).

Table 8. Bandwidth and GCV values at polynomial degree d=1,2,3 in the second response.

Polynomial Degrees	h_1	GCV
1	0.16	0.02550365
2	**2.77**	**0.02271762**
3	20.4	0.02285883

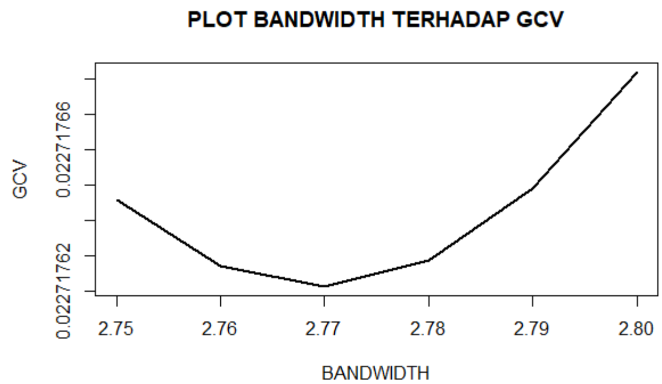
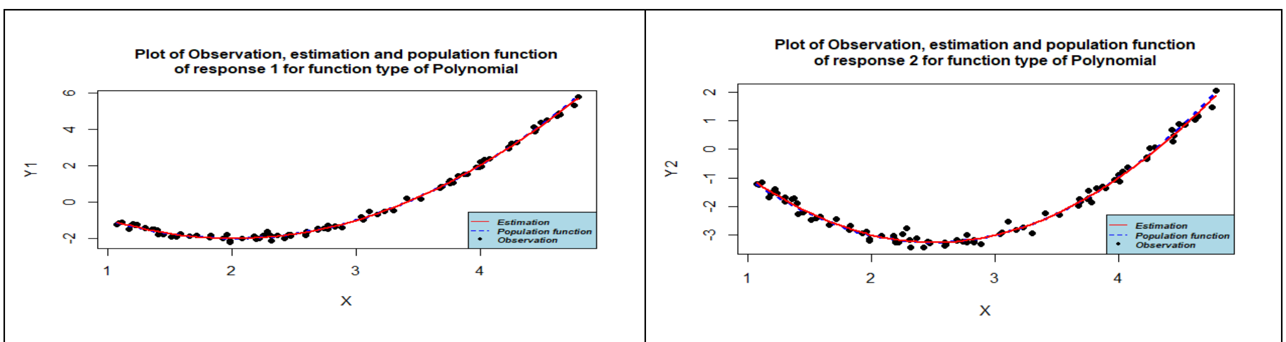


Figure 14. The plot of bandwidth and GCV on the polynomial function type (d=2, r=2).

We estimate in-sample data using the optimal bandwidth (h_1 = 2.18 and h_2 = 2.77) for the polynomial of the second degree. The plot of the estimated results of the two responses is shown in Figure 15.



(a) (b)

Figure 15. Plot of observation, estimation dan population function for first response (a) and second response (b) for trigonometric function type using optimal bandwidth (h_1 = 2.18 and h_2 = 2.77).

In addition, we simulate estimates using non-optimal bandwidth values on in-sample data. Figures 12a and 12b are estimation plots using bandwidth that greater than the optimal values, $h_1 = 3.5$ and $h_2 = 3.8$. Figures 12c and 12d are estimation plots using bandwidth that smaller than the optimal values, $h_1 = 0.01$ and $h_2 = 0.02$.

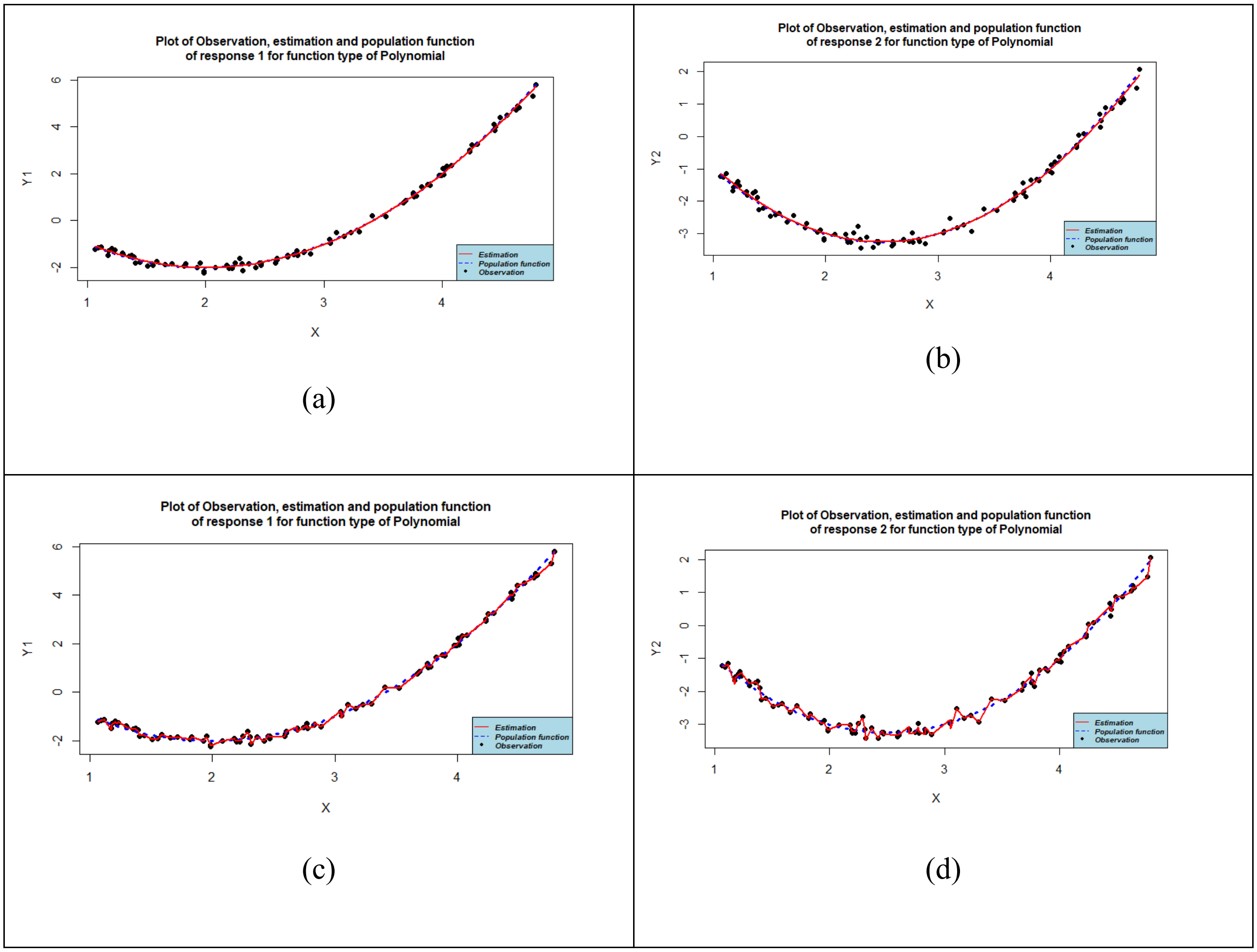


Figure 16. Plots of observation, estimation, and population function of two responses for polynomial function type using greater and smaller than optimal bandwidth value.

We predict the two responses on out-sample data based on the optimal bandwidth on in-sample data. Furthermore, we used non-optimal bandwidth values to represent the two responses on out-of-sample data. This step can provide optimal bandwidth, which is essential because it affects the roughness of the estimation plot and the MSE and MAPE values. The results are presented in Table 9.

Table 9. Comparison of MSE and MAPE values in the three estimation conditions on polynomial functions.

Bandwidth	Nilai MSE	Nilai MAPE
Optimal bandwidths ($h_1 = 2.18$; $h_2 = 2.77$)	0.02006232	12.375
Greater than optimal bandwidths (h_1=3.5 ; h_2=3.8)	0.02035597	12.60083
Smaller than optimal bandwidths (h_1=0.01 ; h_2=0.02)	0.05771129	26.73333

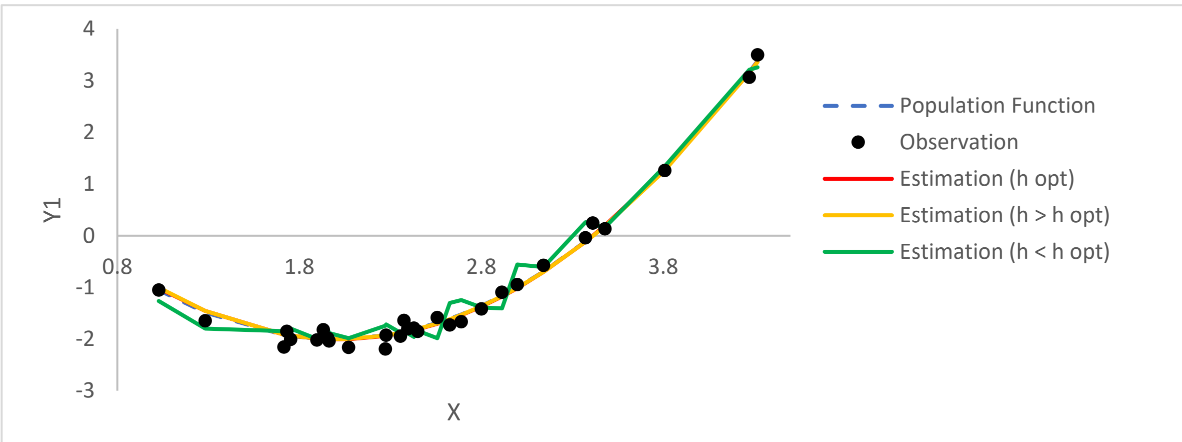


Figure 17. Plots of observation, estimation, and population function of first response for function type of polynomial using optimal bandwidth, greater and smaller than optimal bandwidth values.

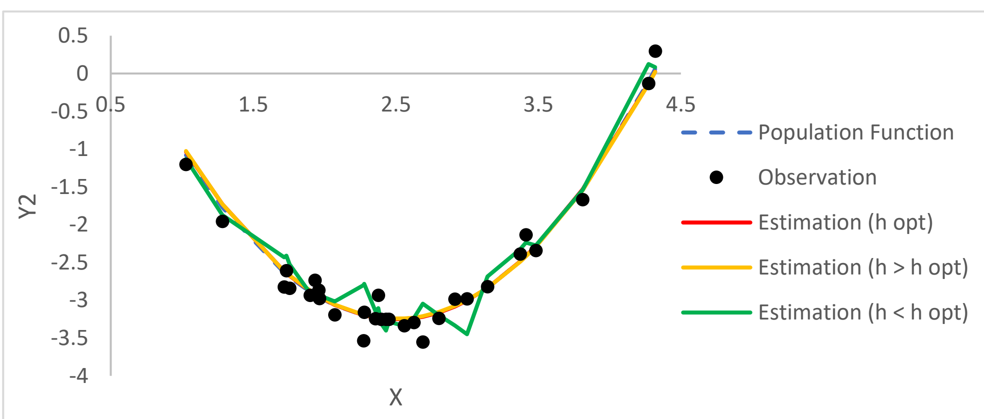


Figure 18. Plots of observation, estimation, and population function of second response for function type of polynomial using optimal bandwidth, greater and smaller than optimal bandwidth values.

Based on the simulation results, it can be seen that the optimal use of bandwidth in the estimation process of local polynomial regression is essential because it affects the roughness or smoothness of the estimation line and the size of the MSE and MAPE values. The plot results show that the regression estimation line (red line) will coincide with the optimal bandwidth's population function line (blue line). It means that our estimation results can approach the population value. It is evidenced by the better MSE and MAPE scores. The optimal bandwidth selection has a significant impact, resulting in smooth regression estimates [39]. In addition, the resulting MAPE value is less than 20% which means it is included in the accurate category [40]. However, the estimation line is too smooth at greater than optimal bandwidth values and does not coincide with the population function line. The MSE and MAPE values are also above the MSE and MAPE values at optimal bandwidth. Likewise, in the estimation using a bandwidth value smaller than optimal bandwidth values, the estimation line is too rough and close to the observed value. It causes the MSE and MAPE values to be smaller than the MSE and MAPE values at the optimal bandwidth. We usually estimate with a small mean squared error (MSE). The selection of bandwidth h is commonly critical due to bias-variance trade-offs. A small value of h gives a lot of weight to data points close to X_0, resulting in a slight bias and a large variance because only a few data points are used for estimation. Estimation with a large value of h, has a significant bias and a minor variance, and vice versa [41].

4. CONCLUSIONS

Theoretically, bi-response multi-predictor local polynomial nonparametric regression can predict the value of the parameters determining the mango maturity, including TSS and the pH value of mango. Simulation results on data with trigonometric, exponential, and polynomial functions indicate that estimates must be made using the optimal bandwidth value to obtain a smooth estimation plot and have accurate prediction with small values of MSE and MAPE.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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