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STABILITY ANALYSIS OF AN SEIS EPIDEMIC MODEL WITH NONLINEAR INCIDENCE FUNCTIONAL AND IMMIGRATION

SARA SOULAIMANI*, ABDELILAH KADDAR

Chouaib Doukkali University of El Jadida, Natl Sch App Sci , Sci Engineer Lab Energy, El Jadida, Morocco

Abstract. In this research, we propose an *SEIS* epidemic model with immigration and nonlinear incidence rates, considering the impact of infectious forces in both the latent and infected periods. The local dynamics of an endemic equilibrium is examined. Using a suitable Lyapunov functional, we established the global asymptotic stability of the endemic equilibrium. For the *SEIS* model without immigration, we calculate the basic reproduction number and establish the global stability of equilibria by means of Lyapunov functionals. Finally, two examples with numerical simulations are given to illustrate the validity of our results.

Keywords: local dynamics; global dynamics; *SEIS* epidemic model; Lyapunov function; immigration.

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1. Introduction

Humans are exposed to a variety of infectious diseases, such as the coronavirus (Covid-19) that emerged in a Chinese city in 2019. Covid-19 is an infectious disease caused by the SARS-CoV-2 virus and the severe acute respiratory condition that leads to this dangerous disease. Covid-19 can be spread from one person to another [1]. Mathematicians and epidemiologists concentrated their efforts on developing mathematical models that could predict the emergence of these undesirable infectious diseases.

*Corresponding author
E-mail address: soulaimani.sara96@gmail.com
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Some infections, such as AIDS, measles, TB, and Covid-19, have a latency or incubation phase during which an individual is infected but not infectious. Covid-19, for example, has a latency period of about 5-6 days [2]. This latency can be represented by the creation of a new class, the exposed class, in which the susceptible individual stays for some time before becoming infected. The resulting models are those of SEI [3], SEIS [4], SEIR [5], SEIRI [6], or SEIRS [7].

The means of transporting people from one place to another have considerably progressed in the modern world. This development was accompanied by a significant increase in the number of immigrants. As a result, certain immigrants may be infected by contagious diseases upon their arrival.

Motivated by the previous discussions, we formulate and study the following *SEIS* model with immigration and nonlinear incidence rates that represents the infectious force in both the latent and infected periods. As a result, we'll concentrate our study on the following epidemic model:

$$\begin{cases} \dfrac{dS}{dt} = T_S + A - \mu_S S - Sh_1(I) - Sh_2(E) + \delta I \\ \dfrac{dE}{dt} = T_E + Sh_1(I) + Sh_2(E) - (\mu_E + \gamma)E \\ \dfrac{dI}{dt} = T_I + \gamma E - (\mu_I + \delta)I, \end{cases} \tag{1}$$

where the variable S denotes the number of susceptible individuals, E denotes the number of exposed individuals, and I is for the number of infected individuals, A is the rate that individuals enter the susceptible class. T_S, T_E, and T_I are the recruitment rates into the classes S, E, and I through immigration, respectively. δ denotes the rate of transmission from the infectious class to the susceptible class. γ is the rate of transmission from the exposed class to the infected class. Hence, $\frac{1}{\gamma}$ represents the average latent period. Individuals with per capita death rates μ_S, μ_E and μ_I leave the susceptible, exposed, and infected classes, respectively. We assume that $A, \mu_S, \mu_E, \mu_I, T_S, T_E, T_I, \gamma, \delta > 0$.
We associate the above system with nonnegative initial conditions $S(0)$, $E(0)$, and $I(0)$.
We pose $A_S = T_S + A$ in all of the following.

To investigate the dynamics of the model (1), we enumerate some hypotheses over the functions h_1 and h_2, as follows:

($\mathscr{H}_0$) The functions h_i $(i = 1,\ 2)$ are continuously differentiable on $\mathbb{R}^+$;

($\mathscr{H}_1$) $h_i'(x) \geq 0$ for all $x \in \mathbb{R}^+$, with $h_i(0) = 0$;

($\mathscr{H}_2$) $\dfrac{h_i(x)}{x}$ is monotone decreasing on $\mathbb{R}^+$.

This article is organized as follows. In Section 'Basic properties and equilibria', we give some fundamental properties and propositions that are needed for this work. Moreover, we established the existence of an endemic equilibrium for the model (1) . In Section 'Local stability of the endemic equilibrium', we investigate the local stability of the model (1). The global asymptotic stability of the unique endemic equilibrium is discussed in Section 'Global stability and the uniqueness of the endemic equilibrium'. In Section 'Analysis of system (1) in the case of $T_S = T_E = T_I = 0$' the model (1) without immigration is presented. Firstly, we calculate the basic reproduction number and prove the existence and uniqueness of two equilibrium points, the disease free and endemic equilibrium points. We have shown the global asymptotic stability of the first one. Furthermore, the global stability of the endemic equilibrium is proved by using a suitable Lyapunov function. Finally, in the last Section 'Numerical Simulations', we present some numerical simulations to verify our theoretical results.

2. Basic Properties and Equilibria

Proposition 1. $0 < \frac{h_i(x)}{x} \leq h_i'(0)$ for all $x > 0$.

Proof. This result can be clearly seen using the hypotheses ($\mathscr{H}_1$)-($\mathscr{H}_2$).

The feasible region for the system (1) is given by

$$\Psi = \left\{ (S,E,I) : S \geq 0, E \geq 0, I \geq 0 \mid S + E + I \leq \frac{T}{\mu} \right\},$$

where $\mu = \min\{\mu_S, \mu_E, \mu_I\}$ and $T = A_S + T_E + T_I$.

Proposition 2. Ψ is positively invariant with respect to system (1).

Proof. Let S, E and I be the solutions of model (1) and $(S(0),E(0),I(0)) \in \Psi$ the initial condition of the system (1).

In system (1), we add the three equations to get the total population size as

$$N = S+E+I,$$

taking the derivative of N with respect to t, we have:

$$\frac{dN}{dt} \leq T - \mu N.$$

Let

$$\frac{dN}{dt} + \mu N \leq T. \tag{2}$$

Multiplying both sides of equation (2) by $e^{\mu t}$, we get

$$d[Ne^{\mu t}] \leq Te^{\mu t}dt. \tag{3}$$

Integrating both sides, we have

$$N = \frac{T}{\mu} + Ce^{-\mu t} \tag{4}$$

At $t = 0$, we have $N(0) \leq \frac{T}{\mu} + C$, which implies

$$N(0) - \frac{T}{\mu} \leq C, \tag{5}$$

then

$$N \leq \frac{T}{\mu}(1-e^{-\mu t}) + N(0)e^{-\mu t}. \tag{6}$$

When $t \to +\infty$, N converges to $\frac{T}{\mu}$.

Remark 1. There is no disease-free equilibrium in the model (1). This is given by the fact that $\left(\frac{dE}{dt} + \frac{dI}{dt}\right)_{E=I=0} = T_E + T_I > 0$. There is no basic reproduction number since there is no disease-free equilibrium [8].

Proposition 3. For system (1), there exists an endemic equilibrium designated (S_*, E_*, I_*).

Proof. Let the right sides of the three differential equations in system (1) be zeros, that is,

$$\begin{cases} A_S - \mu_S S - Sh_1(I) - Sh_2(E) + \delta I = 0, \\ T_E + Sh_1(I) + Sh_2(E) - (\mu_E + \gamma)E = 0, \\ T_I + \gamma E - (\mu_I + \delta)I = 0. \end{cases} \tag{7}$$

Then the equilibrium of system (1) satisfies system (7).

From the third equation of (7), we get

$$E_* = \frac{1}{\gamma}((\mu_I + \delta)I_* - T_I).$$

And summing the first two equations of the system gives

$$S_* = \frac{1}{\mu_S \gamma}(\gamma A_S + \gamma T_E + \gamma \delta I_* - (\mu_E + \gamma)(\mu_I + \delta)I_* + (\mu_E + \gamma)T_I).$$

We define the function $\Im$ as follow

$$\begin{aligned} \Im(I_*) = \quad & -T_E - \tfrac{1}{\mu_S\gamma}(\gamma A_S + \gamma T_E + \gamma\delta I_* - (\mu_E+\gamma)(\mu_I+\delta)I_* + (\mu_E+\gamma)T_I)h_1(I_*) \\ & -\tfrac{1}{\mu_S\gamma}(\gamma A_S + \gamma T_E + \gamma\delta I_* - (\mu_E+\gamma)(\mu_I+\delta)I_* + (\mu_E+\gamma)T_I)h_2(\tfrac{1}{\gamma}((\mu_I+\delta)I_* - T_I)) \\ & +\tfrac{(\mu_E+\gamma)}{\gamma}((\mu_I+\delta)I_* - T_I). \end{aligned}$$

We have $\Im(\frac{T_I}{\mu_I+\delta}) = -T_E - \frac{1}{\mu_S}(A_S + T_E + \frac{\delta T_I}{\mu_I+\delta})h_1(\frac{T_I}{\mu_I+\delta}) \leq 0$, for a chosen I_*, in such a way that $S_* = 0$, we get $I_* = \frac{\gamma A_S + \gamma T_E + (\gamma+\mu_E)T_I}{(\mu_I+\delta)(\gamma+\mu_E)-\gamma\delta}$, then $\Im(I_*) = -T_E + (\gamma + \mu_E)E_*$. Since, $I_* \geq (\frac{T_E\gamma}{\mu_E+\gamma} + T_I)\frac{1}{\mu_I+\delta}$. Thus, $\Im(I_*) \geq 0$, which prove that an endemic equilibrium exists.

The uniqueness is exhibited in section 4.

3. Local Stability of the Endemic Equilibrium

Theorem 3.1. The equilibrium (S_*, E_*, I_*) of model (1) is locally asymptotically stable.

Proof. When the Jacobian matrix of system (1) is evaluated at the positive equilibrium (S_*, E_*, I_*), we get

$$J(S_*,E_*,I_*) = \begin{pmatrix} -\mu_S - h_1(I_*) - h_2(E_*) & -S_* h_2'(E_*) & -S_* h'_1(I_*) + \delta \\ h_1(I_*) + h_2(E_*) & S_* h'_2(E_*) - (\mu_E + \gamma) & S_* h'_1(I_*) \\ 0 & \gamma & -(\mu_I + \delta) \end{pmatrix} \tag{8}$$

The characteristic equation of $J(S_*,E_*,I_*)$ is given by

$$\lambda^3 + K_2\lambda^2 + K_1\lambda + K_0 = 0, \tag{9}$$

where

$$K_2 = \mu_S + h_1(I_*) + h_2(E_*) + (\mu_I + \delta) + \frac{T_E}{E_*} + S_*\left(\frac{h_2(E_*)}{E_*} - h'_2(E_*)\right) + S_*\frac{h_1(I_*)}{E_*},$$

$$\begin{aligned} K_1 = {} & ((\mu_I + \delta) + (\mu_E + \gamma))(h_1(I_*) + h_2(E_*)) + \mu_S((\mu_I + \delta) + S_*\left(\frac{h_2(E_*)}{E_*} - h'_2(E_*)\right) \\ & + \gamma S_*\left(\left(\frac{T_I}{\gamma E_*} + 1\right)\frac{h_1(I_*)}{I_*} - h'_1(I_*)\right) + (\mu_I + \delta)S_*\left(\frac{h_2(E_*)}{E_*} - h'_2(E_*)\right) \\ & + (\mu_S + \mu_I + \delta)\frac{T_E}{E_*} + \mu_S S_*\frac{h_1(I_*)}{E_*}, \end{aligned}$$

and

$$\begin{aligned} K_0 = {} & ((\mu_I + \delta)(\mu_E + \gamma) - \delta\gamma)(h_1(I_*) + h_2(E_*)) + \mu_S(\mu_I + \delta)S_*\left(\frac{h_2(E_*)}{E_*} - h'_2(E_*)\right) \\ & + \mu_S\gamma S_*\left(\left(\frac{T_I}{\gamma E_*} + 1\right)\frac{h_1(I_*)}{I_*} - h'_1(I_*)\right) + \mu_S(\mu_I + \delta)\frac{T_E}{E_*}. \end{aligned}$$

Using $(\mathscr{H}_2)$, $\frac{h_i(x)}{x}$ is monotone decreasing on $\mathbb{R}^+$, so $\left(\frac{h_i(x)}{x}\right)' \leq 0$, hence $\frac{h_i(x)}{x} - (h_i(x))' \geq 0$ for all $x \geq 0$. And we have $(\mu_I + \delta)(\mu_E + \gamma) - \delta\gamma = \mu_I\mu_E + \mu_I\gamma + \delta\mu_E > 0$. Then it is easy to verify that $K_2 > 0, K_1 > 0, K_0 > 0$ and $K_2K_1 > K_0$. Therefore, by the Routh-Hurwitz criterion, the eigenvalues of (9) all have negative real part, and hence the equilibrium is locally asymptotically stable.

4. Global Stability and the Uniqueness of the Endemic Equilibrium

Theorem 4.1. *If $\mu_S S_* - \delta I_* \geq 0$, the endemic equilibrium (S_*,E_*,I_*) of system (1) is globally asymptotically stable in the case of $\mu_S = \mu_E$.*

Proof. Let us consider the Lyapunov function $\mathscr{Y}$, defined as follows:

$$\mathscr{Y} = \mathscr{Y}_1 + \mathscr{Y}_2,$$

where

$$\mathscr{Y}_1 = S_* l(\frac{S}{S_*}) + E_* l(\frac{E}{E_*}) + \frac{S_* h_1(I_*)}{\gamma E_*} I_* l(\frac{I}{I_*}),$$

and

$$\mathscr{Y}_2 = \frac{\delta}{\gamma S_*}\frac{(I-I_*)^2}{2} + \frac{\delta}{(\mu_S+\mu_I)S_*}\frac{(S+E+I-S_*-E_*-I_*)^2}{2}.$$

with $l(x) = -1 + x - \ln x$, for all $x > 0$.

The differentiation of $\mathscr{Y}_1$ with respect to t is:

$$\begin{aligned}
\frac{d\mathscr{Y}_1}{dt} &= \left(1-\frac{S_*}{S}\right)S' + \left(1-\frac{E_*}{E}\right)E' + \frac{S_*h_1(I_*)}{\gamma E_*}\left(1-\frac{I_*}{I}\right)I' \\
&= \left(1-\frac{S_*}{S}\right)(-\mu_S(S-S_*) + S_*h_1(I_*) - Sh_1(I) + S_*h_2(E^*) - Sh_2(E) + \delta(I-I_*)) \\
&\quad + \left(1-\frac{E_*}{E}\right)(T_E + Sh_1(I) + Sh_2(E) - (\mu_E+\gamma)E) \\
&\quad + \frac{S_*h_1(I_*)}{\gamma E_*}\left(1-\frac{I_*}{I}\right)(T_I + \gamma E - (\mu_I+\delta)I) \\
&= -\mu_S\frac{(S-S_*)^2}{S} + S_*h_1(I_*)\left(1-\frac{S_*}{S}-\frac{Sh_1(I)}{S_*h_1(I_*)}+\frac{h_1(I)}{h_1(I_*)}\right) \\
&\quad + S_*h_2(E_*)\left(1-\frac{S_*}{S}-\frac{Sh_2(E)}{S_*h_2(E_*)}+\frac{h_2(E)}{h_2(E_*)}\right) + \delta\left(1-\frac{S_*}{S}\right)(I-I_*) \\
&\quad + \left(1-\frac{E_*}{E}\right)\left(T_E\left(1-\frac{E}{E_*}\right) + Sh_1(I) - S_*h_1(I_*)\frac{E}{E_*} + Sh_2(E) - S_*h_2(E_*)\frac{E}{E_*}\right) \\
&\quad + \frac{S_*h_1(I_*)}{\gamma E_*}\left(1-\frac{I_*}{I}\right)\left(T_I\left(1-\frac{I}{I_*}\right) + \gamma E_*\left(\frac{E}{E_*}-\frac{I}{I_*}\right)\right) \\
&= -\mu_S\frac{(S-S_*)^2}{S} - T_E\frac{(E-E_*)^2}{EE_*} - T_I\frac{S_*h_1(I_*)}{\gamma E_*}\frac{(I-I_*)^2}{II_*} + \delta\left(1-\frac{S_*}{S}\right)(I-I_*) \\
&\quad + S_*h_1(I_*)\left(1-\frac{S_*}{S}-\frac{Sh_1(I)}{S_*h_1(I_*)}+\frac{h_1(I)}{h_1(I_*)}\right) + S_*h_1(I_*)\left(1-\frac{E}{E_*}+\frac{Sh_1(I)}{S_*h_1(I_*)}-\frac{Sh_1(I)E_*}{S_*h_1(I_*)E}\right) \\
&\quad + S_*h_1(I_*)\left(\frac{E}{E_*}-\frac{I}{I_*}-\frac{I_*E}{IE_*}+1\right) + S_*h_2(E_*)\left(1-\frac{S_*}{S}-\frac{Sh_2(E)}{S_*h_2(E_*)}+\frac{h_2(E)}{h_2(E_*)}\right)
\end{aligned}$$

$$+S_*h_2(E_*)\left(1-\frac{E}{E_*}+\frac{Sh_2(E)}{S_*h_2(E_*)}-\frac{Sh_2(E)E_*}{S_*h_2(E_*)E}\right).$$

Then, we have that

$$\begin{aligned}\frac{d\mathscr{Y}_1}{dt}=&-\mu_S\frac{(S-S_*)^2}{S}-T_E\frac{(E-E_*)_2}{EE_*}-T_I\frac{S_*h_1(I_*)}{\gamma E_*}\frac{(I-I_*)^2}{II_*}\\&+S_*h_1(I_*)\left(\left(-1+\frac{h_1(I)}{h_1(I_*)}-\frac{I}{I_*}+\frac{Ih_1(I_*)}{I_*h_1(I)}\right)+\left(4-\frac{I_*E}{IE_*}-\frac{S_*}{S}-\frac{Sh_1(I)E_*}{S_*h_1(I_*)E}-\frac{Ih_1(I_*)}{I_*h_1(I)}\right)\right)\\&+S_*h_2(E_*)\left(\left(-1+\frac{h_2(E)}{h_2(E_*)}-\frac{E}{E_*}+\frac{Eh_2(E_*)}{E_*h_2(E)}\right)+\left(3-\frac{S_*}{S}-\frac{SE_*h_2(E)}{S_*Eh_2(E_*)}-\frac{Eh_2(E_*)}{E_*h_2(E)}\right)\right)\\&+\delta(I-I_*)\left(1-\frac{S_*}{S}\right).\end{aligned}$$

The differentiation of $\mathscr{Y}_2$ with respect to t is:

$$\begin{aligned}\frac{d\mathscr{Y}_2}{dt}=&\frac{\delta}{\gamma S_*}(I-I_*)(T_I+\gamma E-(\mu_I+\delta)I)\\&+\frac{\delta}{(\mu_S+\mu_I)S_*}(S-S_*+E-E_*+I-I_*)(-\mu_S(S-S_*)-\mu_E(E-E_*)-\mu_I(I-I_*))\end{aligned}$$

If $\mu_S=\mu_E$, we have

$$\begin{aligned}\frac{d\mathscr{Y}_2}{dt}=&\frac{\delta}{S_*}(I-I_*)(E-E_*)-\frac{\delta(\mu_I+\delta)}{\gamma S_*}(I-I_*)^2-\frac{\mu_I\delta}{(\mu_S+\mu_I)S_*}(I-I_*)^2\\&+\frac{\delta}{(\mu_S+\mu_I)S_*}(-\mu_S(S-S_*+E-E_*)^2-(\mu_S+\mu_I)((S-S_*)(I-I_*)+(E-E_*)(I-I_*))\\=&-\frac{\delta(\mu_I+\delta)}{\gamma S_*}(I-I_*)^2-\frac{\mu_I\delta}{(\mu_S+\mu_I)S_*}(I-I_*)^2-\frac{\mu_S\delta}{(\mu_S+\mu_I)S_*}(S-S_*+E-E_*)^2\\&-\delta(\frac{S}{S_*}-1)(I-I_*).\end{aligned}$$

The result of adding these two functions is then

$$\begin{aligned}\frac{d\mathscr{Y}}{dt}=&-((\mu_SS_*-\delta I_*)+\delta I)\frac{(S-S_*)^2}{SS_*}-T_E\frac{(E-E_*)^2}{EE_*}-T_I\frac{S_*h_1(I_*)}{\gamma E_*}\frac{(I-I_*)^2}{II_*}\\&-\frac{\delta}{S_*}(\frac{1}{\gamma}+\frac{\mu_I}{\mu_S+\mu_I})(I-I_*)^2-\frac{\mu_S\delta}{(\mu_S+\mu_I)S_*}(S-S_*+E-E_*)^2\\&+S_*h_1(I_*)\left(\left(-1+\frac{h_1(I)}{h_1(I_*)}-\frac{I}{I_*}+\frac{Ih_1(I_*)}{I_*h_1(I)}\right)+\left(4-\frac{I_*E}{IE_*}-\frac{S_*}{S}-\frac{Sh_1(I)E_*}{S_*h_1(I_*)E}-\frac{Ih_1(I_*)}{I_*h_1(I)}\right)\right)\\&+S_*h_2(E_*)\left(\left(-1+\frac{h_2(E)}{h_2(E_*)}-\frac{E}{E_*}+\frac{Eh_2(E_*)}{E_*h_2(E)}\right)+\left(3-\frac{S_*}{S}-\frac{SE_*h_2(E)}{S_*Eh_2(E_*)}-\frac{Eh_2(E_*)}{E_*h_2(E)}\right)\right).\end{aligned}$$

Using the hypotheses $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$, we obtain

$$-1+\frac{h_1(I)}{h_1(I_*)}-\frac{I}{I_*}+\frac{Ih_1(I_*)}{I_*h_1(I)}=\frac{1}{I_*}\left(\frac{I_*}{h_1(I_*)}-\frac{I}{h_1(I)}\right)(h_1(I)-h_1(I_*))\leq 0,$$

$$-1+\frac{h_2(E)}{h_2(E_*)}-\frac{E}{E_*}+\frac{Eh_2(E_*)}{E_*h_2(E)}=\frac{1}{E_*}\left(\frac{E_*}{h_2(E_*)}-\frac{E}{h_2(E)}\right)(h_2(E)-h_2(E_*))\leq 0,$$

by the expression of l and properties of logarithms, we have

$$4-\frac{I_*E}{IE_*}-\frac{S_*}{S}-\frac{Sh_1(I)E_*}{S_*h_1(I_*)E}-\frac{Ih_1(I_*)}{I_*h_1(I)}\leq 0,$$

$$3-\frac{S_*}{S}-\frac{SE_*h_2(E)}{S_*Eh_2(E)}-\frac{Eh_2(E_*)}{E_*h_2(E)}\leq 0,$$

That is $\frac{d\mathscr{Y}}{dt}\leq 0$ for all $S,E,I\geq 0$. Furthermore, the singleton $\{(S_*,E_*,I_*)\}$ is the largest compact invariant region in $\left\{(S,E,I):(S,E,I)\in\mathbb{R}^3_+,\frac{d\mathscr{Y}}{dt}=0\right\}$. Then we conclude that the equilibrium (S_*,E_*,I_*) of system (1) is globally asymptotically stable by applying LaSalle's invariance principle [9]. The uniqueness can be asserted using the fact that $\frac{d\mathscr{Y}}{dt}=0$ has only one solution $S=S_*$.

5. Analysis of System (1) in the Case of $T_S=T_E=T_I=0$

In this section, we shall investigate the existence and the global stability of the equilibria of system (1) in the case of $T_S=T_E=T_I=0$ and $\mu_S=\mu_E=\mu_I=\mu$. Let us consider the following system:

$$\begin{cases}\dfrac{dS}{dt}=A-\mu S-Sh_1(I)-Sh_2(E)+\delta I\\ \dfrac{dE}{dt}=Sh_1(I)+Sh_2(E)-(\mu+\gamma)E\\ \dfrac{dI}{dt}=\gamma E-(\mu+\delta)I.\end{cases}\tag{10}$$

System (10) always has a disease-free equilibrium $E_0(S_0,0,0)$, where $S_0=\frac{A}{\mu}$. According to van den Driessche and Watmough's [10] definition of the basic reproduction number for ODE systems, the basic reproduction number for system (10) is

$$R_0=S_0\frac{\gamma h'_1(0)+(\mu+\delta)h'_2(0)}{(\mu+\gamma)(\mu+\delta)}.$$

Proposition 4. For system (10), there exists a unique endemic equilibrium designated (S_{**},E_{**},I_{**}) if and only if $R_0>1$.

Proof. Let $R_0 > 1$. At an endemic state, system (10) becomes:

$$\begin{cases} A - \mu S - Sh_1(I) - Sh_2(E) + \delta I = 0 \\ Sh_1(I) + Sh_2(E) - (\mu+\gamma)E = 0 \\ \gamma E - (\mu+\delta)I = 0, \end{cases} \tag{11}$$

The third equation of system (11), gives

$$E = \frac{(\mu+\delta)}{\gamma} I. \tag{12}$$

Moreover, by adding the three equations of the system (11), we see that

$$S = \frac{A - \omega I}{\mu}, \tag{13}$$

where $\omega = \frac{\mu(\mu+\delta)}{\gamma} + \mu$. And $S \geq 0$ implies that $I \in \left(0, \frac{A}{\omega}\right]$. As a result, if $I \geq \frac{A}{\omega}$, there is no positive equilibrium. And it follows from the second equation of (11) and equation (12) that

$$S = \frac{(\mu+\gamma)(\mu+\delta)}{\gamma} \frac{1}{\frac{h_1(I)}{I} + \frac{h_2(\frac{(\mu+\delta)}{\gamma} I)}{I}}, \tag{14}$$

As a result of equations (13) and (14), we define the function H as follow:

$$H(I) = A - \omega I - \frac{\mu(\mu+\gamma)(\mu+\delta)}{\gamma} \frac{1}{\frac{h_1(I)}{I} + \frac{h_2(\frac{(\mu+\delta)}{\gamma} I)}{I}}, I \in \left(0, \frac{A}{\omega}\right]. \tag{15}$$

Unsig the hypothesis $(\mathscr{H}_2)$, we get the the function H is strictly monotone decreasing on $\left(0, \frac{A}{\omega}\right]$ with

$$\lim_{I \to 0^+} H(I) = A - \frac{\mu(\mu+\gamma)(\mu+\delta)}{\gamma} \frac{1}{h_1'(0) + \frac{(\mu+\delta)}{\gamma} h_2'(0)} = A\left(1 - \frac{1}{R_0}\right) > 0,$$

and $H(\frac{A}{\omega}) < 0$. Thus, there is only one endemic equilibrium if $R_0 > 1$.

Now, we discuss the global stability of the equilibria in model (10). Let us consider the stability of disease-free equilibrium $E_0(S_0, 0, 0)$ first.

Theorem 5.1. If $R_0 \leq 1$, then the disease-free equilibrium E_0 is globally asymptotically stable.

Proof. We define the following Lyapunov function as:

$$L = L_1 + L_2,$$

where

$$L_1 = S_0 l(\frac{S}{S_0}) + E + \frac{(\mu+\gamma)h_1'(0)}{\gamma h_1'(0) + (\mu+\delta)h_2'(0)} I,$$
$$L_2 = \frac{\delta}{\gamma S_0}\frac{I^2}{2} + \frac{\delta}{2\mu S_0}\frac{(S - S_0 + E + I)^2}{2},$$
$$l(x) = -1 + x - \ln x, \text{ for all } x > 0.$$

The differentiation of L_1 with respect to t is:

$$\begin{aligned}
\frac{dL_1}{dt} =& \left(1 - \frac{S_0}{S}\right) S' + E' + \frac{(\mu+\gamma)h_1'(0)}{\gamma h_1'(0) + (\mu+\delta)h_2'(0)} I' \\
=& \left(1 - \frac{S_0}{S}\right)(A - \mu S - Sh_1(I) - Sh_2(E) + \delta I) + (Sh_1(I) + Sh_2(E) - (\mu+\gamma)E) \\
&+ \frac{(\mu+\gamma)h_1'(0)}{\gamma h_1'(0) + (\mu+\delta)h_2'(0)}(\gamma E - (\mu+\delta)I) \\
=& -\mu\frac{(S-S_0)^2}{S} + (h_1(I) + h_2(E))S_0 - \frac{(\mu+\gamma)(\mu+\delta)}{\gamma h_1'(0) + (\mu+\delta)h_2'(0)}(h_1'(0)I + h_2'(0)E) \\
&+ \delta I\left(1 - \frac{S_0}{S}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{dL_2}{dt} =& \frac{\delta}{\gamma S_0} I(\gamma E - (\mu+\delta)I) \\
&+ \frac{\delta}{2\mu S_0}(-\mu(S - S_0 + E)^2 - \mu I^2 - 2\mu I(S - S_0) - 2\mu EI) \\
=& -\frac{\delta}{S_0}\left(\frac{\mu+\delta}{\gamma} + \frac{1}{2}\right) I^2 - \frac{\delta}{2S_0}(S - S_0 + E)^2 - \delta I(\frac{S}{S_0} - 1).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{dL}{dt} =& -\mu\frac{(S-S_0)^2}{S} - \frac{\delta}{S_0}\left(\frac{\mu+\delta}{\gamma} + \frac{1}{2}\right) I^2 - \frac{\delta}{2S_0}(S - S_0 + E)^2 - \delta I\frac{(S-S_0)^2}{SS_0} \\
&+ (h_1(I) + h_2(E))S_0 - \frac{(\mu+\gamma)(\mu+\delta)}{\gamma h_1'(0) + (\mu+\delta)h_2'(0)}(h_1'(0)I + h_2'(0)E).
\end{aligned}$$

Using Proposition 1, we get that $h_1(I) \leq h_1'(0)I$ and $h_2(E) \leq h_2'(0)E$. Then

$$\begin{aligned}
\frac{dL}{dt} \leq& -\frac{\mu S_0 + \delta}{SS_0}(S - S_0)^2 - \frac{\delta}{S_0}\left(\frac{\mu+\delta}{\gamma} + \frac{1}{2}\right) I^2 - \frac{\delta}{2S_0}(S - S_0 + E)^2 \\
&+ S_0(h_1'(0)I + h_2'(0)E)\left(1 - \frac{1}{R_0}\right),
\end{aligned}$$

since $R_0 \leq 1$, then $\frac{dL}{dt} \leq 0$. Furthermore, the singleton E_0 represents the largest compact invariant region in $\left\{(S,E,I): S \geq 0, E \geq 0, I \geq 0, \frac{dL}{dt} = 0\right\}$. By LaSalle's invariance principle, we conclude that the disease-free equilibrium E_0 of system (10) is globally asymptotically stable.

Theorem 5.2. The equilibrium (S_{**}, E_{**}, I_{**}) is globally stable.

Proof. Here, we choose the following Lyapunov function:

$$\mathscr{L}(t) = \frac{1}{2}(S+E+I-S_{**}-E_{**}-I_{**})^2.$$

The positive equilibrium (S_{**}, E_{**}, I_{**}) of the system (10) satisfies the relations:

$$A = \mu S_{**} + S_{**}h_1(I_{**}) + S_{**}h_2(E_{**}) - \delta I_{**},\ (\mu+\gamma)E_{**} = S_{**}h_1(I_{**}) + S_{**}h_2(E_{**}),\ (\mu+\delta)I_{**} = \gamma E_{**},$$

which can be utilized in the differentiation of $\mathscr{L}$ with respect to t as follows

$$\begin{aligned}
\frac{d\mathscr{L}}{dt} &= (S+E+I-S_{**}-E_{**}-I_{**})(\frac{dS}{dt}+\frac{dE}{dt}+\frac{dI}{dt}) \\
&= (S+E+I-S_{**}-E_{**}-I_{**})(A-\mu S-\mu E-\mu I) \\
&= (S+E+I-S_{**}-E_{**}-I_{**})(-\mu(S+E+I)-\mu(-S_{**}-E_{**}-I_{**})) \\
&= -\mu(S+E+I-S_{**}-E_{**}-I_{**})^2.
\end{aligned}$$

Then we have $\frac{d\mathscr{L}}{dt} \leq 0$. As a result, we deduce from the Lyapunov theorem that (S_{**}, E_{**}, I_{**}) is globally stable.

6. Numerical Simulation

To see the applicability of our theory, we will study two examples in this section by giving a particular nonlinear incidence rate.

Example 1

Let us consider the following model

$$\begin{cases}
\dfrac{dS}{dt} = T_S + A - \mu_S S - \dfrac{\beta_1 SI}{1+d_1 I} - \dfrac{\beta_2 SE}{1+d_2 E} + \delta I \\
\dfrac{dE}{dt} = T_E + \dfrac{\beta_1 SI}{1+d_1 I} + \dfrac{\beta_2 SE}{1+d_2 E} - (\mu_E+\gamma)E \\
\dfrac{dI}{dt} = T_I + \gamma E - (\mu_I+\delta)I,
\end{cases} \tag{16}$$

with the initial conditions listed in Table 1.

$S_1(0) = 0$	$S_2(0) = 200$	$S_3(0) = 400$	$S_4(0) = 600$	$S_5(0) = 800$
$E_1(0) = 0$	$E_2(0) = 60$	$E_3(0) = 120$	$E_4(0) = 180$	$E_5(0) = 240$
$I_1(0) = 0$	$I_2(0) = 50$	$I_3(0) = 100$	$I_4(0) = 150$	$I_5(0) = 200$
$S_6(0) = 1000$	$S_7(0) = 1200$	$S_8(0) = 1400$	$S_9(0) = 1600$	$S_{10}(0) = 1800$
$E_6(0) = 300$	$E_7(0) = 360$	$E_8(0) = 420$	$E_9(0) = 480$	$E_{10}(0) = 540$
$I_6(0) = 250$	$I_7(0) = 300$	$I_8(0) = 350$	$I_9(0) = 400$	$I_{10}(0) = 450$
$S_{11}(0) = 2000$	$S_{12}(0) = 2200$	$S_{13}(0) = 2400$	$S_{14}(0) = 2600$	$S_{15}(0) = 2800$
$E_{11}(0) = 600$	$E_{12}(0) = 660$	$E_{13}(0) = 720$	$E_{14}(0) = 780$	$E_{15}(0) = 840$
$I_{11}(0) = 500$	$I_{12}(0) = 550$	$I_{13}(0) = 600$	$I_{14}(0) = 650$	$I_{15}(0) = 700$
$S_{16}(0) = 3000$	$S_{17}(0) = 3200$	$S_{18}(0) = 3400$	$S_{19}(0) = 3600$	$S_{20}(0) = 3800$
$E_{16}(0) = 900$	$E_{17}(0) = 960$	$E_{18}(0) = 1020$	$E_{19}(0) = 1080$	$E_{20}(0) = 1140$
$I_{16}(0) = 750$	$I_{17}(0) = 800$	$I_{18}(0) = 850$	$I_{19}(0) = 900$	$I_{20}(0) = 950$

Table 1 Initial conditions

where the parameters β_1, β_2, d_1, and d_2 are defined as follows:

β_1 : is the rate of the efficient contact in infected period.

β_2 : is the rate of the efficient contact in latent period.

d_1, d_2 : are used to evaluate the inhibitory or saturation effect.

System (16) is a particular case of model (1) by choosing $h_1(I) = \frac{\beta_1 I}{1+d_1 I}$ and $h_2(E) = \frac{\beta_2 E}{1+d_2 E}$. Hence, the hypotheses $(\mathscr{H}_0) - (\mathscr{H}_2)$ are satisfied.

Now, we give some numerical simulations with the parameter values as shown in Table 2, in order to show the applicability of our theoretical results.

Parameter	A	T_S	T_E	T_I	μ_S	μ_E	μ_I	δ	γ	β_1	β_2	d_1	d_2
Value	30	40	20	10	0.02	0.02	0.03	0.01	0.04	0.02	0.03	7	9

Table 2 Parameter values for model (10)

By calculating, we have $\mu_S S_* - \delta I_* = 51.3390 \geq 0$. And we see that after a specific period of time, the solutions of model (16) converge to the equilibrium point $(S_*, E_*, I_*) = (3014, 644.2, 894.1)$. The unique equilibrium (S_*, E_*, I_*) is therefore globally asymptotically stable (see Figure 4). This numerical conclusion and our primary theoretical findings are fairly consistent.

Example 2

Consider the following system

$$\begin{cases} \dfrac{dS}{dt} = A - \mu S - \dfrac{\beta_1 SI}{1+d_1 I} - \dfrac{\beta_2 SE}{1+d_2 E} + \delta I \\ \dfrac{dE}{dt} = \dfrac{\beta_1 SI}{1+d_1 I} + \dfrac{\beta_2 SE}{1+d_2 E} - (\mu+\gamma)E \\ \dfrac{dI}{dt} = \gamma E - (\mu+\delta)I, \end{cases} \tag{17}$$

with the initial conditions listed in Table 1.

Parameter	A	μ	δ	γ	β_1	β_2	d_1	d_2
Value	30	0.08	0.01	0.04	0.0001	0.0002	7	9

Table 3 Parameter values for model (17)

Parameter	A	μ	δ	γ	β_1	β_2	d_1	d_2
Value	30	0.02	0.01	0.04	0.02	0.03	7	9

Table 4 Parameter values for model (17)

The basic reproduction number of model (17) is

$$R_0 = \frac{\gamma\beta_1 A}{\mu(\mu+\gamma)(\mu+\delta)} + \frac{\beta_2 A}{\mu(\mu+\gamma)}.$$

In order to confirm our theoretical findings, we provide some numerical simulations.

For this purpose, we take into account model (17) with the parameter values listed in Table 3, we have $R_0 = 0.7639 \leq 1$. Its can be seen clearly from Figure 5 that after a specific period of time, the solutions of model (17) converge to the disease-free equilibrium $E_0 = (375, 0, 0)$. The unique equilibrium E_0 is therefore globally asymptotically stable.

Now, we take into account the model (17) with the parameter values listed in Table 4, in this case we have $R_0 = 1.4167e+03 \geq 1$. As a result, according to Proposition 4, model (17) has a unique equilibrium (S_{**}, E_{**}, I_{**}). Figure 6 shows that after a specific period of time, the solutions of model (17) converge to the equilibrium point $(S_{**}, E_{**}, I_{**}) = (1209, 124.6, 166)$. The unique equilibrium (S_{**}, E_{**}, I_{**}) is therefore globally stable.

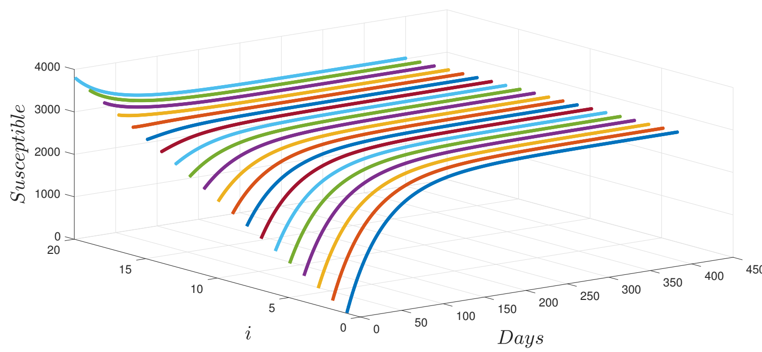

FIGURE 1. Dynamics of S the solution of system (16) with the parameter values listed in Table 2 and initial conditions $S_i(0)$ with $1 \leq i \leq 20$ listed in Table 1.

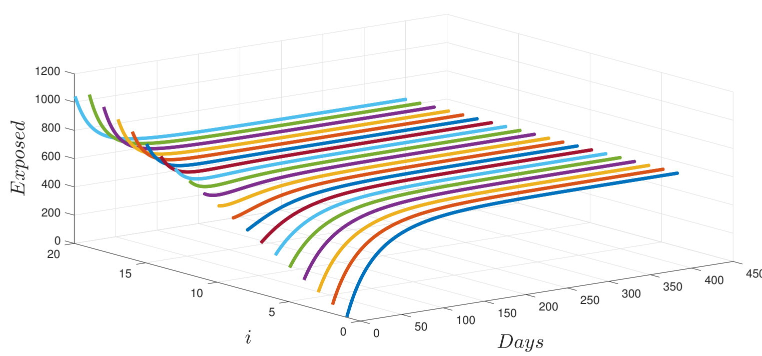

FIGURE 2. Dynamics of E the solution of system (16) with the parameter values listed in Table 2 and initial conditions $E_i(0)$ with $1 \leq i \leq 20$ listed in Table 1.

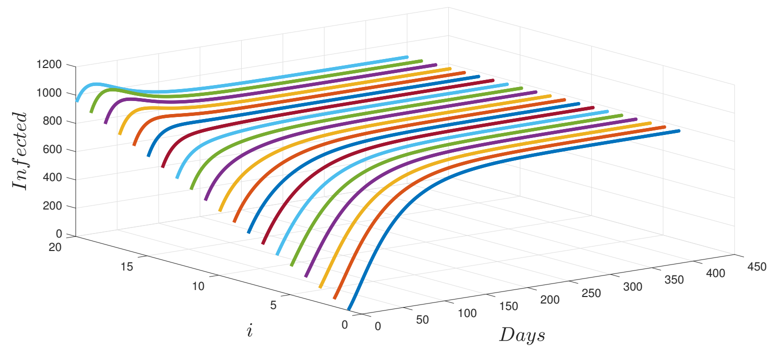

FIGURE 3. Dynamics of I the solution of system (16) with the parameter values listed in Table 2 and initial conditions $I_i(0)$ with $1 \leq i \leq 20$ listed in Table 1.

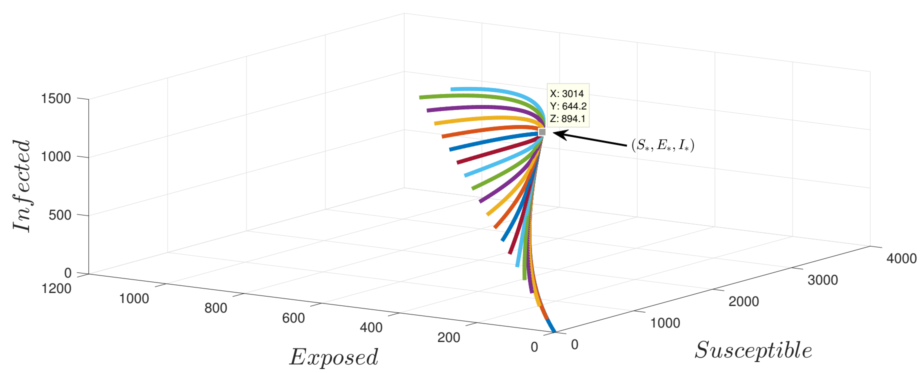

FIGURE 4. Dynamics of (S,E,I) the solution of system (16) with the parameter values listed in Table 2 and initial conditions $(S_i(0),E_i(0),I_i(0))$ with $1 \leq i \leq 20$ listed in Table 1.

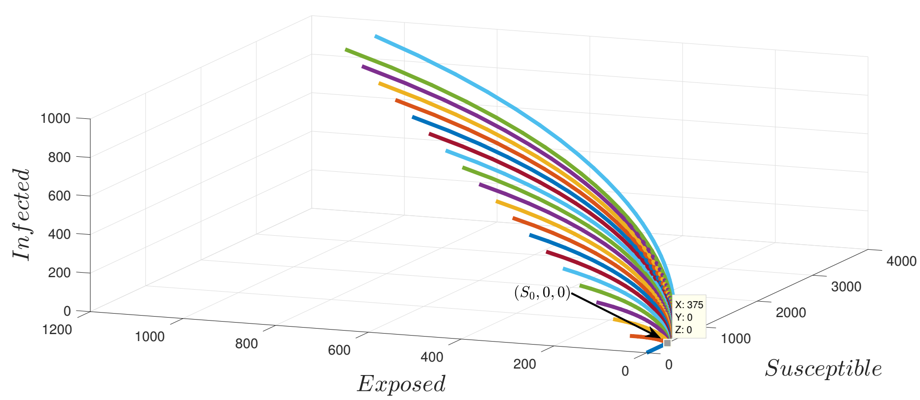

FIGURE 5. Dynamics of (S,E,I) the solution of system (17) with the parameter values listed in Table 3 and initial conditions $(S_i(0),E_i(0),I_i(0))$ with $1 \leq i \leq 20$ listed in Table 1.

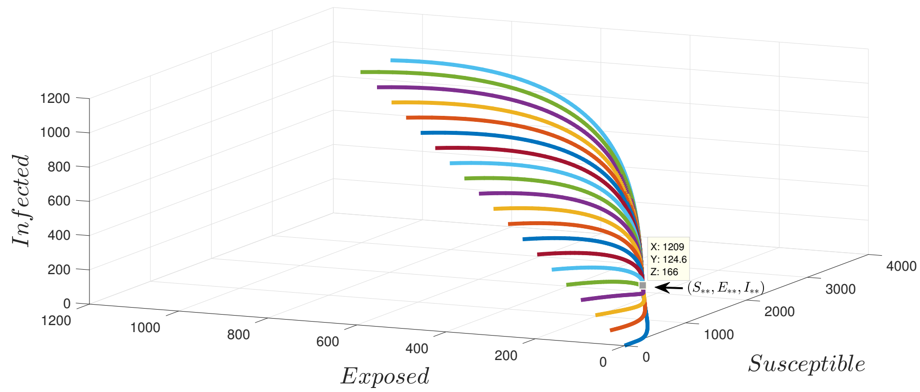

FIGURE 6. Dynamics of (S,E,I) the solution of system (17) with the parameter values listed in Table 4 and initial conditions $(S_i(0),E_i(0),I_i(0))$ with $1 \leq i \leq 20$ listed in Table 1.

7. CONCLUDING REMARKS

In this research, an *SEIS* epidemic model with immigration and nonlinear incidence rates, considering the impact of infectious forces in both the latent and infected periods, is considered. There is no disease-free equilibrium or basic reproduction number for this model. In fact, it was demonstrated that models with immigration had a unique equilibrium for all parameter values. The local asymptotic stability of the endemic equilibrium (S_*,E_*,I_*) is proved using stability methods of differential equations. Furthermore, global asymptotic stability has been demonstrated when $\mu_S S_* - \delta I_* \geq 0$, so that if there is an immigration of exposed or infected people into a region, the disease will persist there and will be difficult to eradicate. Moreover, a particular case is given when $T_S = T_E = T_I = 0$. In this case, the basic reproduction number R_0 is calculated. So that the model without immigration has two equilibria: the disease-free equilibrium E_0 and the endemic equilibrium (S_{**},E_{**},I_{**}). It was found that if $R_0 \leq 1$, then the disease-free equilibrium E_0 is globally asymptotically stable, which indicates that there is no chance for the disease to spread among the population. And if $R_0 > 1$, the unique positive equilibrium (S_{**},E_{**},I_{**}) is globally stable. Consequently, if the infection is initially present, it will persist at its unique endemic equilibrium level. To confirm our finding, two examples are given with numerical simulations.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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