Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2022, 2022:97
https://doi.org/10.28919/cmbn/7625
ISSN: 2052-2541

GLOBAL DYNAMICS OF p-LAPLACIAN REACTION-DIFFUSION EQUATIONS WITH APPLICATION TO VIROLOGY

BENYOUNES BETTIOUI[1,2], ABDELAZIZ EL HASSANI[1], KHALID HATTAF[1,2,*], NACEUR ACHTAICH[1]

[1]Laboratory of Analysis, Modeling and Simulation (LAMS), Faculty of Sciences Ben M'Sick,
Hassan II University of Casablanca, P.O Box 7955 Sidi Othman, Casablanca, Morocco
[2]Centre Régional des Métiers de l'Education et de la Formation (CRMEF), 20340 Derb Ghalef,
Casablanca, Morocco

Abstract. In this paper, we propose a method to investigate the global stability of reaction-diffusion equations involving the p-Laplacian operator with and without delay. The proposed method is based on the direct Lyapunov method which consists to construct an appropriate Lyapunov functional. Furthermore, the method is applied to two biological systems from virology one without delay and the other with both delays in the infection and viral production.

Keywords: reaction-diffusion; p-Laplacian; virology; global stability.

2010 AMS Subject Classification: 35B35, 35B40, 92D25.

1. Introduction

The p-Laplacian operator is a generalization of the classical Laplacian operator. It has been used to study some turbulent fluids through porous media. For example, Diaz and De Thelin [1] focused on a nonlinear parabolic problem arising in some models related to turbulent flows. Ahmed and Sunada [2] dealt with nonlinear flow in porous media. Volker [3] investigated

*Corresponding author
E-mail address: k.hattaf@yahoo.fr
Received July 24, 2022

nonlinear flow in porous media by finite elements. In addition, the p-Laplacian operator is also used in the modeling of non-Newtonian fluids [4, 5, 6].

Reaction-diffusion equations involving p-Laplacian operator have been studied by many authors. For instance, Gmira [7] proved the existence of nontrivial solution of the quasilinear parabolic equations under some conditions. Kamin and Vázquez [8] studied the existence and uniqueness of singular solutions for some nonlinear parabolic equations. Peletier and Wang [9] established the existence of a very singular solution of a quasilinear degenerate diffusion equation with absorption. Bettioui and Gmira [10] proved, under suitable conditions on the parameters, the existence, uniqueness as well as the qualitative behavior of radial solutions of a degenerate quasilinear elliptic equation in $\mathbb{R}^n$. The results presented in [10] have been extended by Bidaut-Véron [11].

On the other hand, the stability of reaction-diffusion equations with the classical Laplacian operator has been investigated by several researchers. However, to our knowledge there is no work for the global stability of reaction-diffusion equations involving the p-Laplacian operator. Therefore, the main purpose of this paper is to extend the method presented in [12] in order to study the global stability of p-Laplacian reaction-diffusion equations with and without delay. To do this, Section 2 is devoted to the description of the extended method. Finally, Section 3 deals with an application in virology.

2. Description of the Method

Let $u = (u_1, \ldots, u_m)$ be the non-negative solution of the ordinary differential equation

$$\dot{u} = f(u), \tag{1}$$

where $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a C^1 function.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $D = diag(d_1, \ldots, d_m)$ with $d_i \geq 0$ for all $i = 1, \ldots, m$.

Suppose u^* is a non-negative equilibrium of (1), then u^* is also a spatially homogeneous steady state of the following reaction-diffusion system with Neumann boundary condition

(2)
$$\begin{cases} \dfrac{\partial u}{\partial t} & = D\Delta_p u + f(u) \quad in\ \Omega \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu} & = 0 \qquad\qquad\qquad on\ \partial\Omega \times (0,+\infty), \\ u(x,0) & = u_0(x) \qquad\qquad in\ \Omega, \end{cases}$$

where $p \geq 2$, $\Delta_p u = div\big(|\nabla u|^{p-2}\nabla u\big)$ is the p-Laplacian operator and ∇u is the gradient of u. Finally, $\dfrac{\partial u}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$.

Let $V(u)$ be a C^1 function defined on some domain in $\mathbb{R}^m_+$. If we put

(3)
$$W = \int_\Omega V(u(x,t))dx,$$

we get

$$\begin{aligned} \frac{dW}{dt} &= \int_\Omega \nabla V(u)\frac{\partial u}{\partial t}dx \\ &= \int_\Omega \nabla V(u)\cdot\big(D\Delta_p u + f(u)\big)dx \\ &= \int_\Omega \nabla V(u)\cdot f(u)dx + \int_\Omega \nabla V(u)\cdot D\Delta_p u\, dx. \end{aligned}$$

Then

(4)
$$\frac{dW}{dt} = \int_\Omega \nabla V(u)\cdot f(u)dx + \sum_{i=1}^m d_i \int_\Omega \frac{\partial V}{\partial u_i}(u)\Delta_p u_i\, dx.$$

On the other hand, we have

$$\int_\Omega \frac{\partial V}{\partial u_i}\Delta_p u_i\, dx = \int_{\partial\Omega} \frac{\partial V}{\partial u_i}|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu}d\sigma - \int_\Omega |\nabla u_i|^{p-2}\nabla u_i \nabla\left(\frac{\partial V}{\partial u_i}\right)dx.$$

Since $\frac{\partial u_i}{\partial \nu} = 0$ on $\partial\Omega$, then

$$\int_\Omega \frac{\partial V}{\partial u_i}\Delta_p u_i\, dx = -\int_\Omega |\nabla u_i|^{p-2}\nabla u_i \nabla\left(\frac{\partial V}{\partial u_i}\right)dx.$$

Hence,

(5)
$$\frac{dW}{dt} = \int_\Omega \nabla V(u)\cdot f(u)dx - \sum_{i=1}^m d_i \int_\Omega |\nabla u_i|^{p-2}\nabla u_i \nabla\left(\frac{\partial V}{\partial u_i}\right)dx.$$

Therefore, we construct the function V such that

(6)
$$\int_\Omega |\nabla u_i|^{p-2}\nabla u_i \nabla\left(\frac{\partial V}{\partial u_i}\right)dx \geq 0, \quad \text{for all } i = 1,\ldots,m.$$

In the literature, many authors like in [13, 14] constructed the explicit Lyapunov functions of the form :

$$V(u)=\sum_{i=1}^{m} a_i(u_i-u_i^*-u_i^*\ln\frac{u_i}{u_i^*}). \tag{7}$$

In this case, we have

$$\frac{\partial V}{\partial u_i}=a_i\left(1-\frac{u_i^*}{u_i}\right).$$

Thus,

$$\int_\Omega |\nabla u_i|^{p-2}\nabla u_i\nabla\left(\frac{\partial V}{\partial u_i}\right)dx=a_iu_i^*\int_\Omega \frac{|\nabla u_i|^p}{u_i^2}dx\geq 0. \tag{8}$$

We summarize the above results in the following proposition.

Proposition 2.1.

(i) *If the Lyapunov function V for the ordinary differential equation* (1) *verifies the condition* (6)*, then the function W defined by* (3) *is a Lyapunov functional for the reaction-diffusion system* (2)*.*

(ii) *If the Lyapunov function V for the ordinary differential equation* (1) *is of the form described by* (7)*, then W is a Lyapunov functional for the reaction-diffusion system* (2)*.*

As in [12], consider the following delayed reaction-diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t} & = D\Delta_p u+f(u)+g(u,u_t) \quad in\, \Omega\times(0,+\infty),\\ \frac{\partial u}{\partial \nu} & =0 \quad on\ \partial\Omega\times(0,+\infty),\\ u(x,t) & =u_0(x,t) \quad in\ \Omega\times[-\tau,0], \end{cases} \tag{9}$$

where $\tau\geq 0$, the function u_t is defined on $\Omega\times[-\tau,0]$ by $u_t(x,\theta)=u(x,t+\theta)$ and g is a functional of u,u_t. In this case, the time derivative of the function W defined by (3) along the positive solution of (9) satisfies

$$\begin{aligned}\frac{dW}{dt}&=\int_\Omega \nabla V(u)\cdot\big(D\Delta_p u+f(u)+g(u,u_t)\big)dx\\ &=\int_\Omega \nabla V(u)\cdot f(u)dx+\int_\Omega \nabla V(u)\cdot D\Delta_p u\,dx+\int_\Omega \nabla V(u).g(u,u_t)dx.\end{aligned}$$

Therefore,

$$\frac{dW}{dt} = \int_\Omega \nabla V(u) \cdot f(u) dx - \sum_{i=1}^{m} d_i \int_\Omega |\nabla u_i|^{p-2} \nabla u_i \nabla \left(\frac{\partial V}{\partial u_i} \right) dx + \int_\Omega \nabla V(u) \cdot g(u, u_t) dx. \tag{10}$$

Like in [12], the integrands of the first and second terms are already calculated. By means idea of Kajiwara et al. [15], the integrand of the third term can be modified to show the negativeness of the time derivative of a Lyapunov function for (9).

3. Application to Virology

In this section, we apply the method described in the previous section to a virological system with and without delay.

Example 1: Consider the following reaction system:

$$\begin{cases} \dot{U} &= \lambda - dU - \beta VU, \\ \dot{I} &= \beta VU - aI, \\ \dot{V} &= kI - \mu V, \end{cases} \tag{11}$$

where the infected target cells (U) are produced at a constant rate λ, die at a rate dU and become infected by virus at a rate βVU. Infected cells (I) die at rate aI. Free virus (V) is produced by infected cells at a rate kI and decays at a rate μV.

To model the mobility of virus, we propose the following system :

$$\begin{cases} \frac{\partial U}{\partial t} = \lambda - dU(x,t) - \beta V(x,t)U(x,t), \\ \frac{\partial I}{\partial t} = \beta V(x,t)U(x,t) - aI(x,t), \\ \frac{\partial V}{\partial t} = d_V \Delta_p V(x,t) + kI(x,t) - \mu V(x,t), \end{cases} \tag{12}$$

where $U(x,t), I(x,t)$ and $V(x,t)$ denote the densities of infected target cells, infected cells and free virus at position x and time t, respectively. In addition, the parameter d_V is the diffusion coefficient.

We consider the system (12) with Neumann boundary condition

$$\frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, +\infty),$$

and initial conditions

$$U(x,0) = U_0(x) \geq 0, \quad I(x,0) = I_0(x) \geq 0, \quad V(x,0) = V_0(x) \geq 0 \quad \text{in } \Omega.$$

Clearly, the system (12) has an infection-free equilibrium $Q^0(U^0,0,0)$ with $U^0=\frac{\lambda}{d}$ and the basic reproduction number is given by

$$R_0=\frac{\lambda\beta k}{da\mu}.$$

In addition, system (12) has another equilibrium named chromic equilibrium of the form $Q^*(U^*,I^*,V^*)$ where

$$U^*=\frac{\lambda}{dR_0},\quad I^*=\frac{d\mu}{\beta k}(R_0-1)\text{ and }\quad V^*=\frac{k}{\mu}I^*.$$

Let $u=(U,I,V)$ be a solution of (11). To establish the stability of the infection-free equilibrium Q^0, we consider the following Lyapunov functional

$$L_0(u)=U^0\phi\left(\frac{U}{U^0}\right)+I+V,$$

where $\phi(z)=z-1-\ln(z)$ for $z>0$.

By a simple computation, we find

$$\begin{aligned}\nabla L_0(u)\cdot f(u)&=\left(1-\frac{U^0}{U}\right)(\lambda-dU-\beta VU)+\beta VU-aI+\frac{a}{k}(kI-\mu V)\\&=-\frac{d}{U}(U-U^0)^2+\frac{a\mu V}{k}(R_0-1).\end{aligned}$$

Since $R_0\leq 1$, we have $\nabla L_0(u)\cdot f(u)\leq 0$.

By applying Proposition 2.1, we construct a Lyapunov functional for reaction-diffusion system (12), as follows

$$W_0=\int_\Omega L_0(u(x,t))dx.$$

Then

$$\begin{aligned}\frac{dW_0}{dt}&=\int_\Omega\left[\frac{-d}{U}(U-U^0)^2+\frac{a\mu}{k}V(R_0-1)\right]dx+\frac{ad_V}{k}\int_\Omega\Delta_p V dx\\&=\int_\Omega\left[\frac{-d}{U}(U-U^0)^2+\frac{a\mu}{k}V(R_0-1)\right]dx.\end{aligned}$$

As $R_0\leq 1$, we have $\frac{dW_0}{dt}\leq 0$. So, W_0 is a Lyapunov functional of (12) at equilibrium Q^0 when $R_0\leq 1$.

For $R_0 > 1$, we consider the following Lyapunov functional

$$L_1(u) = U^*\phi\left(\frac{U}{U^*}\right) + I^*\phi\left(\frac{I}{I^*}\right) + \frac{a}{k}V^*\phi\left(\frac{V}{V^*}\right).$$

Similar calculations give

$$\nabla L_1(u)\cdot f(u) = -\frac{d}{U}(U-U^*)^2 + aI^*\left(3 - \frac{V^*I}{VI^*} - \frac{VUI^*}{IV^*U^*} - \frac{U^*}{U}\right).$$

Since

$$3 - \frac{V^*I}{VI^*} - \frac{VUI^*}{IV^*U^*} - \frac{U^*}{U} \le 0,$$

we have

$$\nabla L_1(u)\cdot f(u) \le 0.$$

Let

$$W_1 = \int_\Omega L_1(u(x,t))dx.$$

Hence,

$$\begin{aligned}\frac{dW_1}{dt} &= \int_\Omega \left[-\frac{d}{U}(U-U^*)^2 + aI^*\left(3 - \frac{V^*I}{VI^*} - \frac{VUI^*}{IV^*U^*} - \frac{U^*}{U}\right)\right]dx \\ &\quad + \frac{ad_V}{k}\int_\Omega \left(1 - \frac{V^*}{V}\right)\Delta_p V dx \\ &= \int_\Omega \left[-\frac{d}{U}(U-U^*)^2 + aI^*\left(3 - \frac{V^*I}{VI^*} - \frac{VUI^*}{IV^*U^*} - \frac{U^*}{U}\right)\right]dx \\ &\quad - \frac{ad_V}{k}V^*\int_\Omega \frac{|\nabla V|^p}{V^2}dx.\end{aligned}$$

Then $\frac{dW_1}{dt} \le 0$. Therefore, W_1 is a Lyapunov functional of (12) at equilibrium Q^*.

Example 2: To describe both delays in the infection and viral production as in [16], system (11) becomes

$$\begin{cases} \dot{U}(t) = \lambda - dU(t) - \beta V(t)U(t), \\ \dot{I}(t) = e^{-m_1\tau_1}\beta V(t-\tau_1)U(t-\tau_1) - aI(t), \\ \dot{V}(t) = ke^{-m_2\tau_2}I(t-\tau_2) - \mu V(t). \end{cases} \tag{13}$$

Here, the first delay τ_1 is the time needed for infected cells to produce virions after viral entry. We assume that virus production lags by a delay τ_1 behind the infection of a cell. This implies that recruitment of virus-production cells at time t is given by the number of cells that were newly infected at time $t-\tau_1$ and are still alive at time t. We assume that the death rate for infected but not yet virus-production cells is m_1. Hence, the probability of surviving from time $t-\tau_1$ to time t is $e^{-m_1\tau_1}$. Further, the delay τ_2 denotes the time necessary for the newly produced virions to become mature and infectious particles. The probability of survival of immature virions is given by $e^{-m_2\tau_2}$ and the average lifetime of an immature virus is given by $\frac{1}{m_2}$, where m_1 and m_2 are positive constants.

To study the impact of diffusion of free virus on the dynamics of viral infection, we propose the following model

$$
(14)\qquad \begin{cases} \frac{\partial U}{\partial t} = \lambda - dU(x,t) - \beta V(x,t)U(x,t), \\ \frac{\partial I}{\partial t} = e^{-m_1\tau_1}\beta V(x,t-\tau_1)U(x,t-\tau_1) - aI(x,t) \\ \frac{\partial V}{\partial t} = d_V\Delta_p V(x,t) + ke^{-m_2\tau_2}I(x,t-\tau_2) - \mu V(x,t). \end{cases}
$$

The system (14) has an infection-free equilibrium $Q^0(\frac{\lambda}{d},0,0)$ and the basic reproduction number

$$
\widetilde{R}_0 = \frac{k\beta\lambda}{ad\mu}e^{-m_1\tau_1-m_2\tau_2}.
$$

On the other hand, system (14) has another equilibrium $\widetilde{Q}^*(\widetilde{U}^*,\widetilde{I}^*,\widetilde{V}^*)$ where

$$
\widetilde{U}^* = \frac{\lambda}{d\widetilde{R}_0}, \quad \widetilde{I}^* = \frac{d\mu e^{m_1\tau_1}}{\beta k}\left(\widetilde{R}_0 - 1\right), \quad \widetilde{V}^* = \frac{k}{\mu}e^{-m_2\tau_2}\widetilde{I}^*.
$$

For $u=(U,I,V)$ a solution of (13), consider the following Lyapunov functional

$$
\begin{aligned} H_0(u) =& U^0\phi\left(\frac{U}{U^0}\right) + e^{m_1\tau_1}I + \frac{a}{k}e^{m_1\tau_1+m_2\tau_2}V + \int_{t-\tau_1}^{t}\beta U(s)V(s)ds \\ &+ ae^{m_1\tau_1}\int_{t-\tau_2}^{t} I(s)ds, \end{aligned}
$$

and let

$$
\widetilde{W}_0 = \int_\Omega H_0(u(x,t))dx.
$$

A similar calculations as in [16] , we get

$$
\nabla H_0(u)\cdot f(u) = -\frac{d}{U}(U-U^0)^2 + \frac{a\mu}{k}e^{m_1\tau_1+m_2\tau_2}V(\widetilde{R}_0 - 1),
$$

and

$$\frac{d\widetilde{W}_0}{dt} = \int_\Omega \left[-\frac{d}{U}(U-U^0)^2 + \frac{a\mu}{k} e^{m_1\tau_1+m_2\tau_2} V(\widetilde{R}_0 - 1) \right] dx.$$

If $\widetilde{R}_0 \leq 1$, then $\frac{d\widetilde{W}_0}{dt} \leq 0$ and the disease-free equilibrium Q^0 is globally asymptotically stable.

When $\widetilde{R}_0 > 1$, we consider the following Lyapunov functional

$$H_1(u) = \widetilde{U}^*\phi\left(\frac{U}{\widetilde{U}^*}\right) + e^{m_1\tau_1}\widetilde{I}^*\phi\left(\frac{I}{\widetilde{I}^*}\right) + \frac{a}{k}e^{m_1\tau_1+m_2\tau_2}\widetilde{V}^*\phi\left(\frac{V}{\widetilde{V}^*}\right)$$
$$+\beta\widetilde{U}^*\widetilde{V}^*\int_{t-\tau_1}^{t}\phi\left(\frac{U(s)V(s)}{\widetilde{U}^*\widetilde{V}^*}\right)ds + ae^{m_1\tau_1}\widetilde{I}^*\int_{t-\tau_2}^{t}\phi\left(\frac{I(s)}{\widetilde{I}^*}\right)ds.$$

Hence,

$$\nabla H_1(u)\cdot f(u) = -\frac{d}{U}(U-\widetilde{U}^*)^2 - a\widetilde{I}^*e^{m_1\tau_1}\left[\phi\left(\frac{\widetilde{U}^*}{U}\right) + \phi\left(\frac{I_{\tau_2}\widetilde{U}^*}{\widetilde{I}^*V}\right)\right.$$
$$\left. +\phi\left(\frac{\widetilde{I}^*V_{\tau_1}U_{\tau_1}}{\widetilde{V}^*\widetilde{U}^*I_{\tau_2}}\right)\right].$$

If we put $\widetilde{W}_1 = \int_\Omega H_1(u(x,t))dx$, we obtain

$$\frac{d\widetilde{W}_1}{dt} = \int_\Omega \frac{-d}{U}(U-\widetilde{U}^*)^2 - a\widetilde{I}^*e^{m_1\tau_1}\left[\phi\left(\frac{\widetilde{U}^*}{U}\right) + \phi\left(\frac{I_{\tau_2}\widetilde{U}^*}{\widetilde{I}^*.V}\right) + \phi\left(\frac{\widetilde{I}^*V_{\tau_1}U_{\tau_1}}{\widetilde{V}^*\widetilde{U}^*I_{\tau_2}}\right)\right]dx$$
$$-\frac{a}{k}e^{m_1\tau_1+m_2\tau_2}d_V\widetilde{V}^*\int_\Omega \frac{|\nabla V|^p}{V^2}dx.$$

Thus, $\frac{d\widetilde{W}_1}{dt} \leq 0$ and $\widetilde{W}_1$ is a Lyapunov functional of (14) at $\widetilde{Q}^*$ when $\widetilde{R}_0 > 1$.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] J.I. Diaz, F. De Thelin, On a nonlinear parabolic problem arising in some models related to turbulent flows, SIAM J. Math. Anal. 25 (1994), 1085–1111. https://doi.org/10.1137/s0036141091217731.

[2] N. Ahmed, D.K. Sunada, Nonlinear flow in porous media, J. Hydr. Div. 95 (1969), 1847–1858. https://doi.org/10.1061/jyceaj.0002193.

[3] R.E. Volker, Nonlinear Flow in Porous Media by Finite Elements, J. Hydr. Div. 95 (1969), 2093–2114. https://doi.org/10.1061/jyceaj.0002207.

[4] G. Astarita, G. Marrucci, Principles of non-Newtonian fluid mechanics, McGraw-Hill, New York, USA, (1974).

[5] J. Benedikt, P. Girg, L. Kotrla, et al. Origin of the p-Laplacian and A. Missbach, Electron. J. Differ. Equ. 2018 (2018), 16. https://digital.library.txstate.edu/handle/10877/15071.

[6] N.E. Mastorakis, H. Fathabadi, On the solution of p-Laplacian for non-Newtonian fluid flow, WSEAS Trans. Math. 6 (2009), 238–245.

[7] A. Gmira, On quasilinear parabolic equations involving measure data, Asymptotic Anal. 3 (1990), 43–56. https://doi.org/10.3233/asy-1990-3103.

[8] S. Kamin, J.L. Vázquez, Singular solutions of some nonlinear parabolic equations, J. Anal. Math. 59 (1992), 51–74. https://doi.org/10.1007/bf02790217.

[9] L.A. Peletier, J.Y. Wang, A very singular solution of a quasilinear degenerate diffusion equation with absorption, Trans. Amer. Math. Soc. 307 (1988), 813–826. https://doi.org/10.1090/s0002-9947-1988-0940229-6.

[10] B. Bettioui, A. Gmira, On the radial solutions of a degenerate quasilinear elliptic equation in $\mathbb{R}^n$, Annales de la faculté des sciences de Toulouse 6^e série. 8 (1999), 411–438. http://www.numdam.org/item?id=AFST_1999_6_8_3_411_0.

[11] M. F. Bidaut-Véron, The p-Laplace heat equation with a source term: Self-similar solutions revisited, Adv. Nonlinear Stud. 6 (2006), 69–108. https://doi.org/10.1515/ans-2006-0105.

[12] K. Hattaf, N. Yousfi, Global stability for reaction–diffusion equations in biology, Computers Math. Appl. 66 (2013), 1488–1497. https://doi.org/10.1016/j.camwa.2013.08.023.

[13] K. Hattaf, N. Yousfi, Dynamics of SARS-CoV-2 infection model with two modes of transmission and immune response, Math. Biosci. Eng. 17 (2020), 5326–5340. https://doi.org/10.3934/mbe.2020288.

[14] K. Hattaf, On the stability and numerical scheme of fractional differential equations with application to biology, Computation. 10 (2022), 97. https://doi.org/10.3390/computation10060097.

[15] T. Kajiwara, T. Sasaki, Y. Takeuchi, Construction of Lyapunov functionals for delay differential equations in virology and epidemiology, Nonlinear Anal.: Real World Appl. 13 (2012), 1802–1826. https://doi.org/10.1016/j.nonrwa.2011.12.011.

[16] K. Hattaf, N. Yousfi, A. Tridane, Stability analysis of a virus dynamics model with general incidence rate and two delays, Appl. Math. Comput. 221 (2013), 514–521. https://doi.org/10.1016/j.amc.2013.07.005.