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COVID-19 EPIDEMIC MODEL: STUDY OF NUMERICAL METHODS AND SOLVING OPTIMAL CONTROL PROBLEM THROUGH FORWARD-BACKWARD SWEEP METHOD

SRI PURWANI[1,*], FATUH INAYATUROHMAT[2], SANUBARI TANSAH TRESNA[2]

[1]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang 45363, Indonesia

[2]Mathematics Master Program, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang 45363, Indonesia

Abstract: COVID-19 pandemic is still a great challenge for several research fields. A mathematical model is one of the main epidemiology research contributions, which is able to study the pattern of disease spread and its long-term behavior. Many researchers have done this research with their interventions and mathematical approach methods, such as model analysis, use of optimal control theory, data used and numerical methods for solving the model. In this paper, we normalize the basic SIR epidemiological model and estimate the parameters involved based on the COVID-19 data in Indonesia, especially for West Java. The optimal control theory is applied to know the disease behavior by considering vaccination and its cost to prevent the spread of the disease. To see the disease behavior through graphical simulation, the Runge-Kutta Fehlberg method is used for solving the model numerically. The result shows that the spread of COVID-19 can be prevented and controlled due to vaccination.

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*Corresponding author
E-mail address: sri.purwani@unpad.ac.id
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1. INTRODUCTION

Coronavirus is a virus that causes symptoms ranging from the common cold to more severe symptoms. In December 2019, a new type of Coronavirus (2019-nCoV) was discovered in China. This virus causes the emergence of a new, different disease, namely COVID-19, which has different symptoms and severity from previously identified coronavirus types such as SARS-CoV and MERS-CoV [1]. The Indonesian government announced the first case of Covid-19 in Indonesia on 6 March 2020. The virus was transmitted during a visit of Japanese citizens living in Malaysia [2]. Since its initial appearance in Indonesia, the development of COVID-19 cases has continued to increase. It was recorded that until April 4, 2022, there were 6,039,873 people who were confirmed positive for COVID-19 in Indonesia [3]. According to Worldometer, Indonesia is ranked to 18th in the world and 7th in Asia for positive cases of COVID-19 [4]. West Java is one of the central-activity provinces in Indonesia. Based on the population census conducted in 2020, West Java was the province with the largest population in Indonesia, which was 49,935,858 people [5]. As the province with the largest population, West Java is one of the provinces with a relatively large percentage of the total number of COVID-19 cases in Indonesia. It was recorded that until April 4, 2022, there were 1,104,074 people who were confirmed positive for COVID-19 in West Java [3].

Many articles study an epidemiology model to interpret how the COVID-19 spreads with a variety of interventions in an effort to control the disease. Libotte et al. [6] built a SIR epidemic model with treatment solved by optimal control, and used Runge Kutta Fehlberg. There are some authors who built epidemic models by considering the latent period of COVID-19 infection with various interventions [7, 8, 9, 10]. Other research estimates parameter values by using real data of COVID-19 cases from some nations to gain a more accurate simulation [6, 11, 12, 13]. Mungkasi et al. [14] showed two epidemic models and solved the problem numerically with successive approximation, variations iteration, and multistage-analytical method. Marinca et al. [15] used the optimal auxiliary functions method to solve the optimal control problem of the epidemic model. Afrah et al. [16] built a complex model to interpret how COVID-19 spread and solved the problem numerically through the fourth-order Runge-Kutta method, but not written explicitly. Inayaturohmat et al. [17] developed a model in the presence of waning immunity, and then Inayaturohmat et al. [18] used optimal control of treatment to suppress the spread of COVID-19. Over reading the literature, we conclude that none research explicitly shows the numerical solutions through some approach.

Based on the literature review, it is known that many researchers have built various models, such as SIR, SEIR, and SIR-type, considering some interventions as an effort to control the spread of the disease. There is much research about estimating parameters for the model to gain a more accurate prediction for any scenarios. Numerical methods are used to solve this problem which include such as fourth-order Runge-Kutta method, Runge-Kutta Fehlberg, successive approximation, variations iteration, and multistage-analytical method. But no one shows the procedure to get the numerical solution explicitly written. The whole paper only shows how to build the model, estimate its parameters, and serve the graphic of predictions resulted from numerical approaches.

In this paper, we used the SIR model to represent the epidemic phenomenon in West Java, Indonesia. We estimated the parameters using data from the website https://pikobar.jabarprov.go.id/. We normalized the model to simplify the graphical interpretation and numerical solution obtained. The numerical solution is solved by using the Runge-Kutta Fehlberg method. Then, we used optimal control theory to suppress the spreading of COVID-19 by considering vaccination and solving it through the forward-backward sweep method. Finally, we compared the behavior of the disease spread system in conditions with and without vaccination controls.

2. MATERIALS AND METHODS

In this section, the materials such as model formulation and the data of COVID-19 cases in West Java province are written. The Runge-Kutta Fehlberg numerical method and optimal control theory are explained as a method of solving the problem.

2.1. SIR Model

We use a popular epidemic model that describes the transmission of COVID-19 in the human population. The population is divided into three groups, including the Susceptible group (S), Infected group (I), and Recovered group (R). The amount of all groups at time t is given by:

$$N(t) = S(t) + I(t) + R(t)$$

The epidemic model is given by the following system of ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda N(t) - \beta S(t)\frac{I(t)}{N(t)} - \mu S(t) \\
\frac{dI}{dt} &= \beta S(t)\frac{I(t)}{N(t)} - \alpha I(t) - \mu_1 I(t) - \mu I(t) \\
\frac{dR}{dt} &= \alpha I(t) - \mu R(t)
\end{aligned} \tag{1}$$

According to (1), people who are in the susceptible group get infected by the virus when having contact with infected people. The death rate of the infected group is higher than the others because there is a death rate due to the disease (μ_1). Therefore, we need to control the spread of the disease through the vaccination program for the susceptible group in order to prevent the disease infection. Hence, we modified (1) by adding a parameter representing an intervention such as vaccination. The modified model is represented by the diagram shown in Fig. 1.

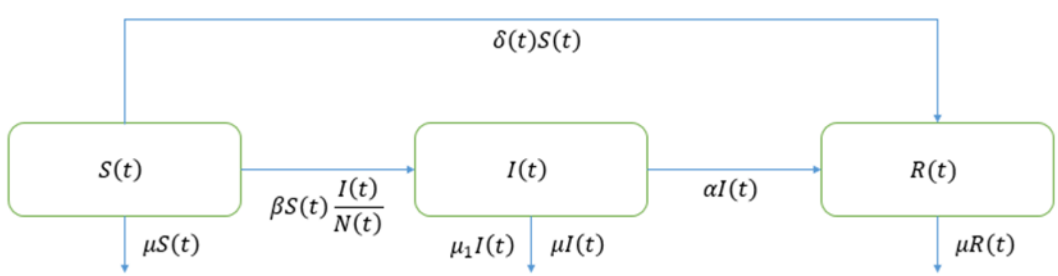


Figure 1. Transmission diagram of COVID-19 spreading

Then we write the modified model (2) and the description of each parameter in Table 1.

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda N(t) - \beta S(t)I(t) - \mu S(t) - \delta S(t) \\
\frac{dI}{dt} &= \beta S(t)I(t) - \alpha I(t) - \mu_1 I(t) - \mu I(t) \\
\frac{dR}{dt} &= \delta S(t) + \alpha I(t) - \mu R(t)
\end{aligned}
\tag{2}
$$

Table 1. Parameters notation

Notation	Description
Λ	The natural birth rate
β	The infection rate
μ	The natural death rate
μ_1	The death rate because of the disease
δ	The vaccination rate
α	The recovered rate

2.2. COVID-19 Data

The data used in this study is the daily confirmed data for COVID-19 in West Java from May 23, 2021, to July 21, 2021 (see Fig. 2), where in this period COVID-19 cases in West Java are experiencing a drastic increase, resulting in a pandemic. The data was obtained from the website https://pikobar.jabarprov.go.id/. The statistical description of the data can be seen in Table 2.

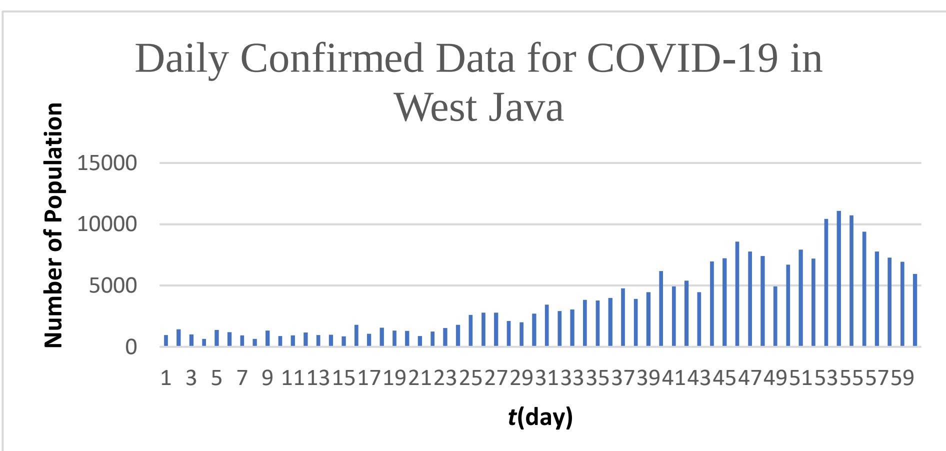


Figure 2. Daily confirmed data for COVID-19 in West Java

Table 2. Statistical description

Minimum	**Maximum**	**Mean**	**Median**
639	11,101	3,872	2,855

Fig. 2 shows that the COVID-19 cases in West Java province fluctuated with a tendency to increase over time. Based on Table 2, the cases on the 54th day of observation reach the highest number of positive cases during the period of observation. Then, we obtained that the mean of this data is 3872, which represents the average number of positive cases per day. In addition, these minimum and median show the lowest number and median of positive cases per day, respectively.

2.3. Runge-Kutta Fehlberg Method

The Runge-Kutta method is one of numerical methods used to solve a differential equation. Runge-Kutta method arose because of the weakness of the Taylor method, which is less efficient in solving differential equations. Moreover, Runge-Kutta Fehlberg uses error control, hence the step size is not fixed [19, 20]. Each step of the process calculates two approximate solutions of different order methods, which are then compared to generate a local error of the lower order. The local error is used to determine the step size. By controlling local errors, global errors will remain under control or sufficiently small. Therefore, the Runge-Kutta Fehlberg method is an alternative to the Taylor method and other iterative methods, such as successive approximation and variational iteration methods [14]. The general formula of the Runge-Kutta Fehlberg method, a pair of methods of orders 4 and 5, can be seen in the following equations.

$$\begin{aligned} y_{n+1} &= y_n + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5 \\ y^*_{n+1} &= y_n + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6 \end{aligned} \tag{3}$$

where

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\left(t_n + \frac{h}{4}, y_n + \frac{1}{4}k_1\right)$$

$$k_3 = hf\left(t_n + \frac{3h}{8}, y_n + \frac{3k_1}{32} + \frac{9k_2}{32}\right)$$

$$k_4 = hf\left(t_n + \frac{12h}{13}, y_n + \frac{1932k_1}{2197} - \frac{7200k_2}{2197} + \frac{7296k_3}{2197}\right)$$

$$k_5 = hf\left(t_n + h, y_n + \frac{439k_1}{216} - 8k_2 + \frac{3680k_3}{513} - \frac{845k_4}{4104}\right)$$

$$k_6 = hf\left(t_n + \frac{h}{2}, y_n - \frac{8k_1}{27} + 2k_2 - \frac{3544k_3}{2565} + \frac{1859k_4}{4104} - \frac{11k_5}{40}\right)$$

2.4. Optimal Control Theory

Optimal control is a tool used to solve a problem with an objective function and obtain an optimal condition due to the cost of control action. The optimal control solution is obtained through the process by considering the constraints and terms [21]. The optimal control problem addresses the control variable $u(t)$ that affects the change from the initial state of the system x_0 at time t_0 to the final state at time T.

Generally, the optimal control problems in the time interval $[t_0, T]$ can be formulated as follows.

$$\begin{aligned} &\max_u \left[\phi\big(x(T)\big) + \int_{t_0}^{T} f\big(t, x(t), u(t)\big)dt\right] \\ &s.t \quad x'(t) = g\big(t, x(t), u(t)\big) \\ &\qquad\;\; x(t_0) = x_0 \end{aligned} \tag{4}$$

where $\phi(x(T))$ is the optimum value of the function at the end of time, otherwise known as Payoff terms.

2.4.1. Pontryagin Maximum Principle

The maximum principle is a condition so that the optimal control solution is obtained that suits the purpose. This principle refers to the determination of the extreme values of the Hamiltonian function (see (5)) so that the control value that optimizes the system is achievable.

$$\begin{aligned} H(t, x, u, \lambda) &= f(t, x, u) + \lambda^T g(t, x, u) \\ &= \text{integrand} + \textit{adjoint}*\text{constraints} \end{aligned} \tag{5}$$

Given λ that is Lagrange multipliers for the constraint $x'(t) = f(x, u)$ and is defined as a function as follows.

$$L = J + \int_{t_0}^{T} \lambda(g(x, u, t) - \dot{x})dt \tag{6}$$

The necessary conditions to optimize L are given as follows:

a. Optimal condition

$$\frac{\partial H}{\partial u} = 0, 0 \leq t \leq T$$

b. *Adjoint* function

$$\lambda'(t) = -\frac{\partial H}{\partial x}, 0 \leq t \leq T$$

c. *Transversal* condition

$$\lambda(T) = 0$$

And the sufficient conditions are given as follows:

a. Minimum condition

$$\frac{\partial^2 H}{\partial u^2} \geq 0$$

b. Maximum condition

$$\frac{\partial^2 H}{\partial u^2} \leq 0$$

2.4.2. Forward-Backward Sweep Method

A rough outline of the Forward-Backward Sweep Method is given in the following figure. We notice that $x = (x_1, x_2, \ldots, x_{N+1})$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{N+1})$ are the vector approximations for the state and *adjoint* respectively. The algorithm that outlines the method is given as follows,

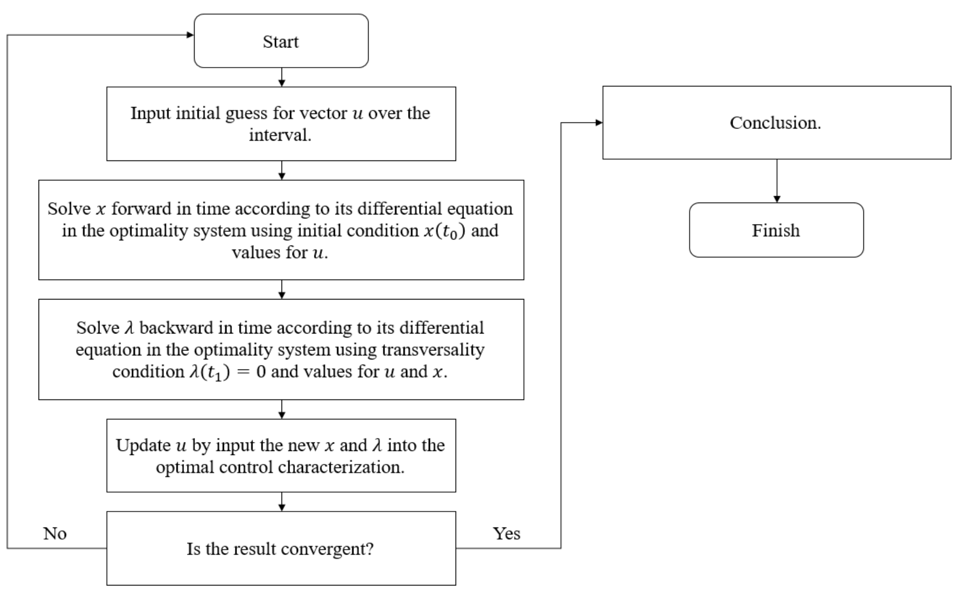


Figure 3. Algorithm of the Forward-Backward Sweep Method

3. MAIN RESULTS

This section elaborates the results and discussion into four subsections, including parameter estimation, model normalization, optimal control problem, and simulation. The process to obtain the results is shown and explained based on the value or graphic produced by simulation.

3.1. Parameter Estimation

The parameters used in the model are obtained from the average of each corresponding data. The average of COVID-19 daily case infection data is used to find the infection rate. The average data on daily recovered from COVID-19 is used to find the recovery rate. Finally, the average data on COVID-19 daily death is used to find the death rate of COVID-19. The result of the parameters estimation and the initial value of the compartments can be seen in Table 3 and Table 4, respectively.

Table 3. Result of parameter estimation

Parameters	**Value**
β	0.075981685
α	0.044286322
μ_1	0.001173315

Table 4. Initial value of the compartments

Compartments	**Value**
$S(t)$	49,630,471
$I(t)$	29,117
$R(t)$	272,187

The basic reproduction number of this model for the estimated parameter is represented as follows,

$$\Re_0 = \frac{\beta}{\alpha} = \frac{0.075981685}{0.044286322} = 1.71569192$$

$\Re_0 > 1$ means that the disease will remain or spread in the population.

3.2. Model Normalization

Normalization of the model is the process of converting the population size to be proportional over time. This proportion represents the composition of each group in a population, ranging in the interval between zero and one $[0,1]$ and the total population size is one. We normalize the model (2) through the following process.

1) Given the total population as follows:

$$N(t) = S(t) + I(t) + R(t)$$

Then we get

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = (\Lambda - \mu)\big(S(t) + I(t) + R(t)\big) - \mu_1 I(t)$$
$$\frac{dN}{dt} = rN(t) - \mu_1 I(t)$$

with $r = \Lambda - \mu$.

2) Let $x(t), y(t)$, and $z(t)$ be new variables representing the proportion of susceptible, infected, and recovered groups in the population, respectively. They are represented by the following equations.

$$x(t) = \frac{S(t)}{N(t)}, y(t) = \frac{I(t)}{N(t)}, z(t) = \frac{R(t)}{N(t)}$$

Taking the derivative of the previous equations with respect to t gives the following equations,

$$\frac{dx}{dt} = \frac{1}{N(t)}\frac{dS}{dt} + \frac{S(t)}{N^2(t)}\frac{dN}{dt}$$
$$\frac{dy}{dt} = \frac{1}{N(t)}\frac{dI}{dt} + \frac{I(t)}{N^2(t)}\frac{dN}{dt}$$
$$\frac{dz}{dt} = \frac{1}{N(t)}\frac{dR}{dt} + \frac{R(t)}{N^2(t)}\frac{dN}{dt}$$

We then obtain the normalized model in the following equations.

$$\frac{dx}{dt} = rx(t) - \beta x(t)y(t) - \mu x(t) - \mu_1 x(t)y(t) - \delta x(t)$$
$$\frac{dy}{dt} = ry(t) + \beta x(t)y(t) - \mu y(t) - \mu_1 y(t) - \mu_1 y^2(t) - \alpha y(t) \tag{7}$$
$$\frac{dz}{dt} = rz(t) + \delta(t)x(t) + \alpha y(t) - \mu z(t) - \mu_1 y(t)z(t)$$

3.3. Optimal Control Problem

We aim to minimize the number of infected people, so that the possibility of the disease spreading is decreased. Moreover, we try to optimize the vaccination used with respect to minimizing the cost. Thus, an objective function was built as follows.

$$J^*(\delta) = \min_{\delta} \int_0^T [Ay(t) + B\delta^2(t)]\, dt \tag{8}$$

Parameters A and B represent the weight of the infected group and the cost of vaccination in the performance function that satisfies $A, B \geq 0$. The Pontryagin Maximum Principle solves the optimal control problem with the variable state $s(t) = [x(t)\ y(t)\ z(t)\ ]^T$ and the constraints (7).

The problem should satisfy the condition: $0 < t < T, 0 < \delta(t) < 1$, and $x(t), y(t), z(t) \geq 0$, where u is the maximum control level. Note that the control $\delta(t)$ represents the percentage of vaccination programs in preventing the disease spreads and suppressing the infected group. The Hamiltonian function defined in (5) is equivalent to the following equations.

$$\begin{aligned} H = Ay(t) + B\delta^2(t) &+ \lambda_1[rx(t) - \beta x(t)y(t) - \mu x(t) - \mu_1 x(t)y(t) - \delta(t)x(t)] \\ &+ \lambda_2[ry(t) + \beta x(t)y(t) - \mu y(t) - \mu_1 y(t) - \mu_1 y^2(t) - \alpha y(t)] \\ &+ \lambda_3[rz(t) + \delta(t)x(t) + \alpha y(t) - \mu z(t) - \mu_1 y(t)z(t)] \end{aligned} \tag{9}$$

where $\lambda_1(t), \lambda_2(t)$, and $\lambda_3(t)$ are the Lagrange multipliers of the optimization problem. The necessary and sufficient conditions as stated in Section 2.4.1, should satisfy the following Pontryagin Maximum Principle:

- Initial state of the system for this model must be non-negative

  $$x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$$

- Lagrange multipliers

  $$\dot{\lambda}_1 = -\lambda_1(t)[-\beta y(t) - \mu - \delta(t) + r - \mu_1 y(t)] - \lambda_2(t)\beta y(t) - \lambda_3(t)\delta(t)$$

  $$\dot{\lambda}_2 = -A - \lambda_1(t)[-\beta x(t) - \mu_1 x(t)] - \lambda_2(t)[\beta x(t) - \mu - \mu_1 - \alpha + r - 2\mu_1 y(t)] - \lambda_3(t)[\alpha - \mu_1 z(t)]$$

  $$\dot{\lambda}_3 = -\lambda_3(t)[-\mu + r - \mu_1(t)y(t)]$$

- Stationer conditions $\partial H/\partial\delta = 0$, then $\delta(t) = [\lambda_1(t) - \lambda_3(t)]\, x(t)/2B$

  Since $0 \leq \delta(t) \leq 1$, we obtained a control

  $$\delta^*(t) = \min\left\{\max\left[0, \frac{[\lambda_1(t) - \lambda_3(t)]x(t)}{2B}\right], 1\right\}$$

- Sufficient condition

Since $\frac{\partial^2 H}{\partial \delta^2} = 2B > 0$ satisfies the criterion of minimization optimal control problem [21] with $\delta^*(t)$ as the optimal control level of the system.

3.4. Simulation

The simulation aims to illustrate the dynamics of all groups and their behavior in the long term. We provide numerical examples with control and without control by using the values of the parameters and initial condition as shown in Tables 3 and 4.

3.4.1. Study of Parameter Estimation

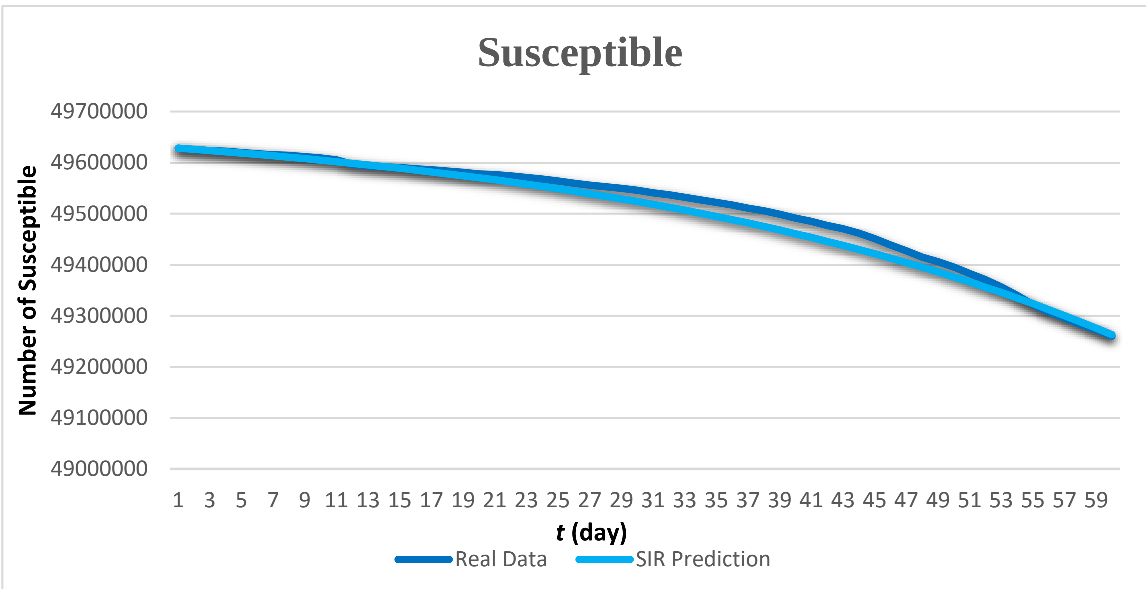


Figure 4. The comparison of the Susceptible

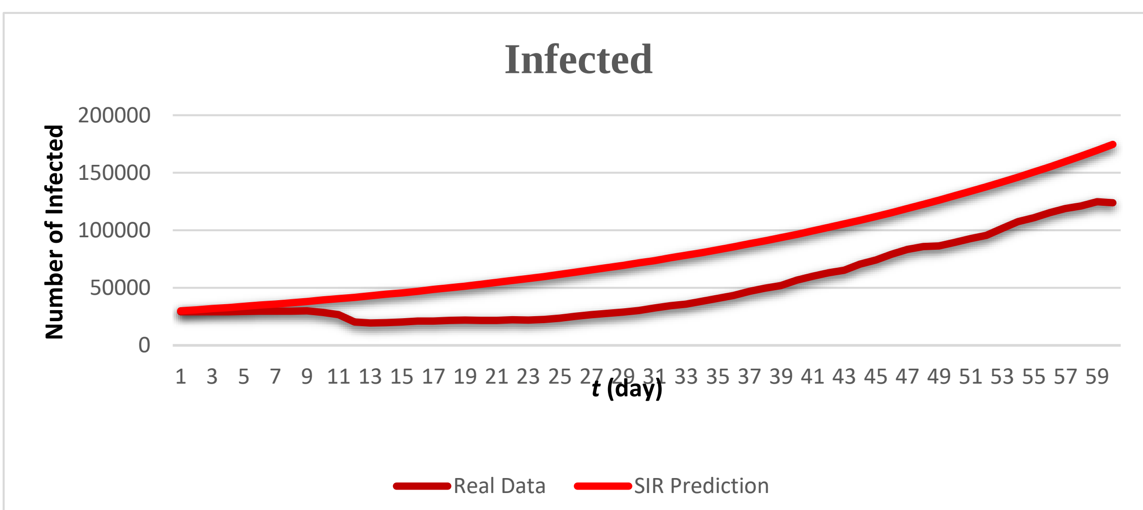


Figure 5. The comparison of the Infected

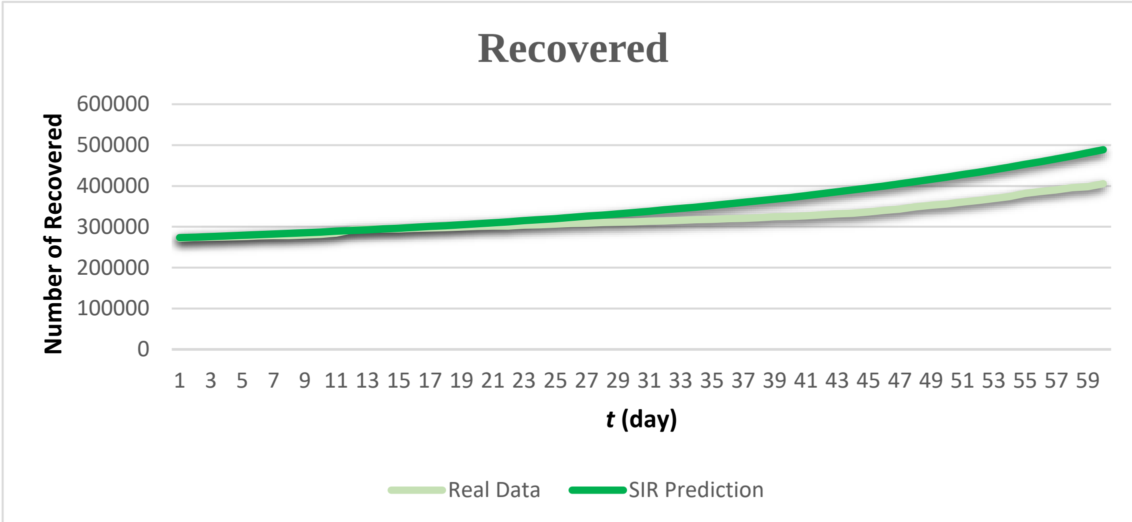


Figure 6. The comparison of the Recovered

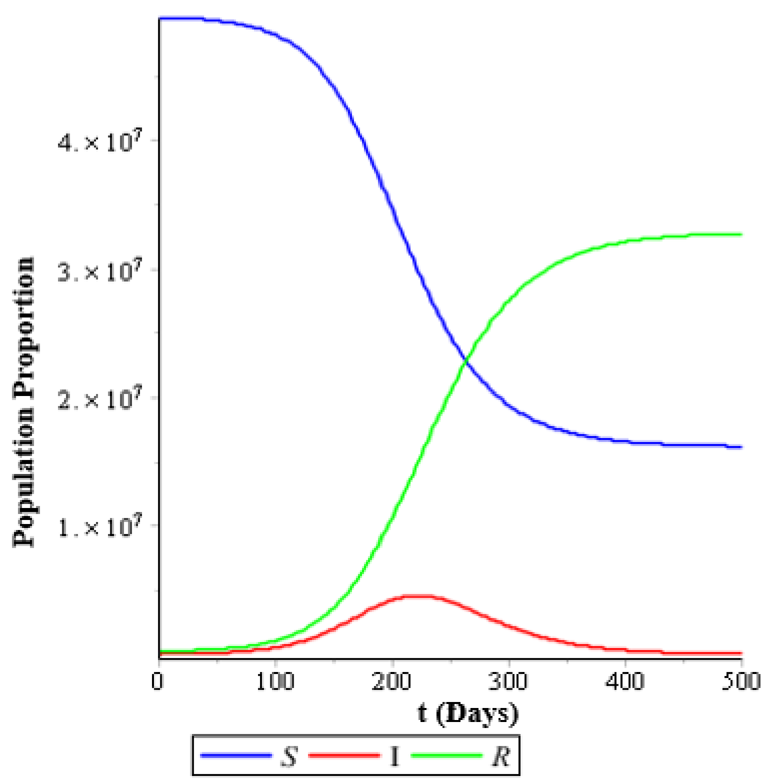
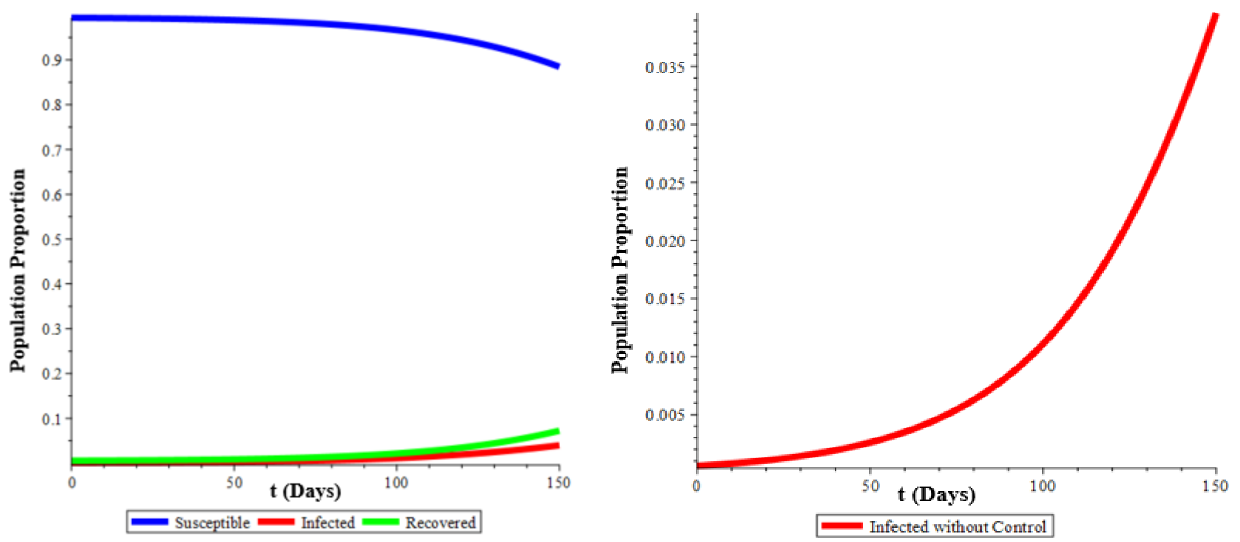


Figure 7. Long-term simulation

The comparison between the numerical result and real data of each compartment can be seen in Fig. 4, Fig. 5, and Fig. 6 respectively. Fig. 4 shows the comparison of the Susceptible individuals. The susceptible individuals decrease over time because of the assumption that the susceptible individuals will move to the infected group if they get infected with COVID-19. Otherwise, the recovered individuals increase over time because of the assumption that the infected individuals will move to the recovered group if they recover from COVID-19. According to the model, the number of infected individuals increases over time because the basic reproduction number is more than one. This means the disease will remain or spread in the population. By using the SIR model, we can see the population dynamics for the long term in Fig. 7. Therefore, we conclude that the population would reach the stability of the equilibrium at one time.

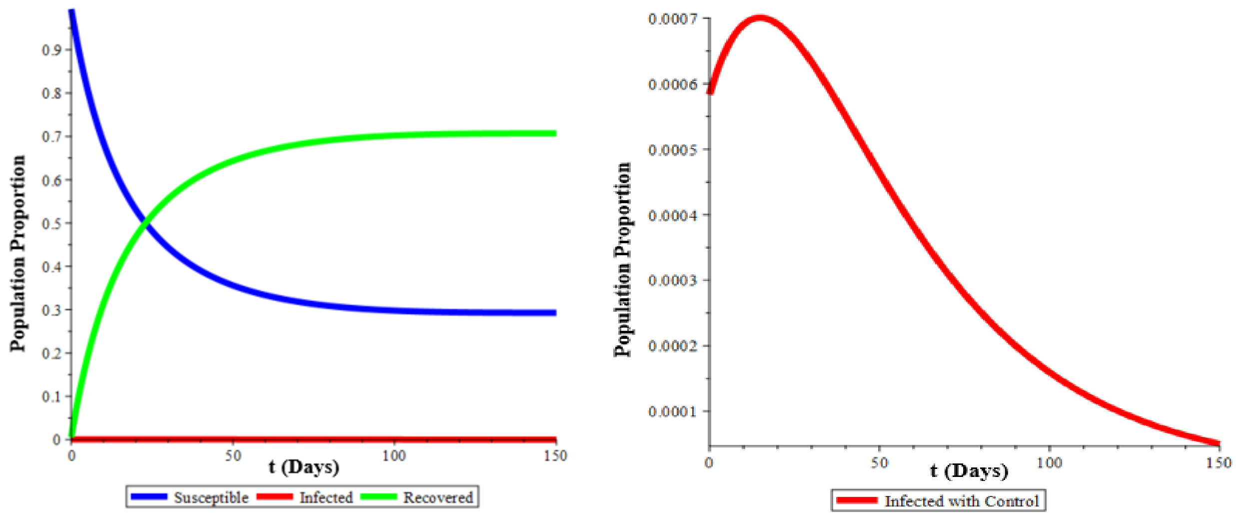
3.4.2. Study of Optimal Control Problem



(a) (b)

Figure 8. Population dynamics without vaccination control; (a) the whole population and (b) the infected

Fig. 8 shows that the proportion of infected group increases in the population over time. This is because there is no vaccination effort as an intervention to prevent the spread of COVID-19 disease. Therefore, preventive action such as vaccination is needed to be applied. Thus, we used optimal control by using vaccination to prevent the disease spread and kept to optimize the effort with minimized cost.



(a) **(b)**

Figure 9. Population dynamics with vaccination control; (a) the whole population and (b) the infected

Fig. 9 shows that the proportion of infected group was suppressed in the population. The susceptible group proportion decreases, while the recovered group proportion increases. This means that the intervention through vaccination to prevent and control the disease spreads successfully achieves the goal. Moreover, the graph of the control level over time is obtained (see Fig. 10).

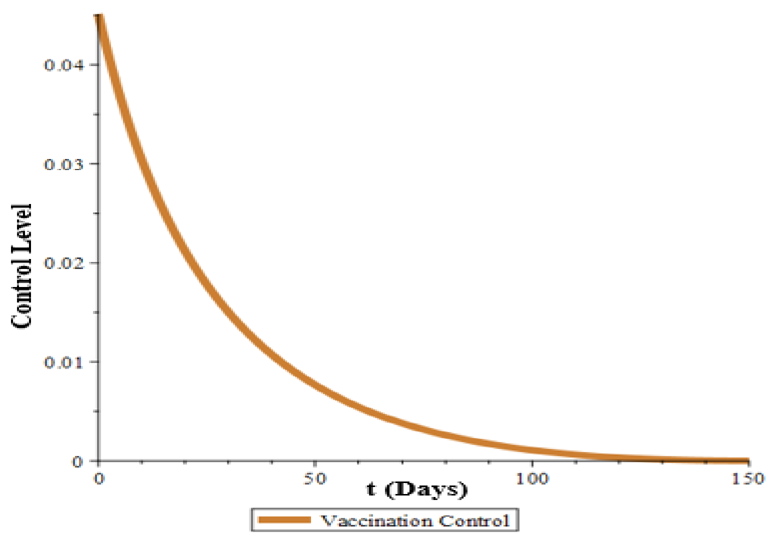


Figure 10. Vaccination control function

Fig. 10 shows that the control level by using vaccination decreases over time. This is related to the decrease of the infected group proportion in the population. It seems that vaccination control indirectly impacts the reduction of infected groups. We then compare both graphs of the infected group, without control and with control to find out their different behavior (see Fig. 11).

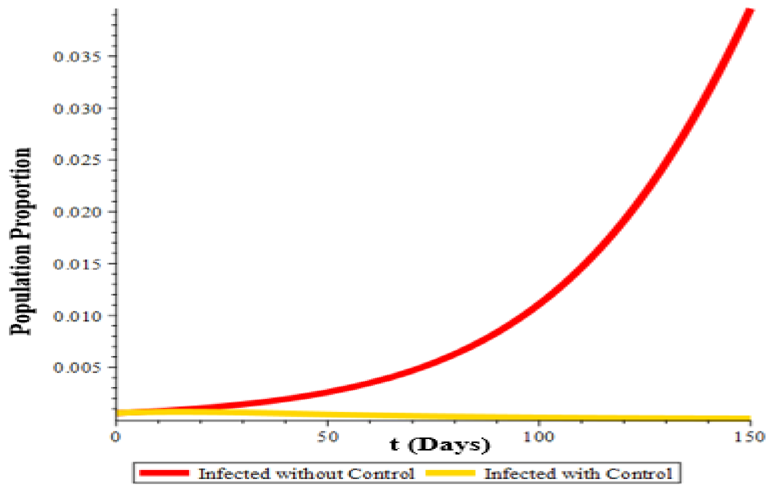


Figure 11. The effect of vaccination control

Fig. 11 shows the different condition between without control and with control of vaccination to the system of disease spreads. It is shown that the vaccination control clearly suppresses the infected group proportion.

4. CONCLUSION

In this article, a study of COVID-19 spread in the West Java province of Indonesia has been done using the SIR epidemic model. We estimated the parameters using the average of each corresponding data to obtain the infection rate, the recovery rate, and the death rate due to the disease. The comparison between the actual data and that of the model shows that the SIR model can fit the data well enough. Then, the SIR model was normalized to simplify the calculating process and obtain a more straightforward interpretation from the simulation of the dynamics population. The simulation shows that the infected group is predicted to increase over time when the vaccination control is not applied. Furthermore, we used a control such as vaccination to prevent the disease spread and regard it as an optimal control problem that can be solved through the Pontryagin Maximum Principle. The result shows that this control impacts the system by reducing the infected group proportion in the population and going extinct over time.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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