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STABILITY, BIFURCATION, AND CHAOS CONTROL OF PREDATOR-PREY SYSTEM WITH ADDITIVE ALLEE EFFECT

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Abstract. The current investigation focuses on the dynamics of a discrete-time predator-prey system with additive Allee effect. Discretization is accomplished by the use of a piecewise constant argument approach of differential equations. Firstly, we studied the existence and topological classification of equilibrium points. We then investigated existence and direction of period-doubling and Neimark-Sacker bifurcations in the system. Moreover, to control the chaos caused by bifurcation, we employ a hybrid control technique. Finally, all theoretical results are justified numerically.

Keywords: predator-prey; Allee effect; stability; bifurcation; hybrid control.

2020 AMS Subject Classification: 39A28, 39A30.

1. Introduction

Many mathematicians and ecologists are fascinated by the topic of prey-predator interactions. A prominent trend in relevant theoretical research is the development of increasingly realistic predator-prey dynamics systems. Its significance may be observed in the many proposed systems that describe the interaction between prey and predator in various scenarios. Lotka [1], and Volterra [2] system the basic system for the interaction of two species. Numerous scholars

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have modified this system over the years to provide a more realistic explanation and enhance comprehension since it ignores many real-world scenarios and complexity. Numerous components affect the dynamical properties of the predator-prey systems, such as functional response, fear, refuge, harvesting, and Allee effect [3, 4, 5, 6, 7].

Leslie [8, 9] introduced the following famous Leslie predator-prey system where the carrying capacity of the predator is proportional to the number of prey:

$$\begin{cases} \frac{dx}{dt} = x(t)(a - by(t)), \\ \frac{dy}{dt} = y(t)(c - \frac{my(t)}{x(t)}), \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ represent population densities of prey and predator at time t, respectively. The parameters a and c are the intrinsic growth rates of prey and predator, respectively. The parameter b measures the strength of competition among individuals of species x. The parameter m measures the food quantity the prey provides and is converted to predator birth.

In systemling the predator-prey systems, a key factor in consideration is the Allee effect. The Allee effect is a biological phenomenon that describes the relationship between population size or density and growth rate. Generally, it happens when a species' population has a very low density, making reproduction and survival difficult. The Allee effect, named after Allee [10], significantly contributes to population dynamics. Several works in the literature explore the Allee effect in various population systems [11, 12, 13, 14, 15, 16] and find that it may significantly influence system dynamics.

After incorporating the Allee effect in system (1), we obtain the following system:

$$\begin{cases} \frac{dx}{dt} = x(t)(a - by(t)), \\ \frac{dy}{dt} = y(t)\left(c - \frac{my(t)}{x(t)}(\frac{y(t)}{\alpha + y(t)})\right), \end{cases} \tag{2}$$

where $\frac{y(t)}{\alpha + y(t)}$ represents the Allee effect and $\alpha > 0$ is called the Allee effect constant. We understand that when α increases, the existing Allee effect on the population becomes stronger, and the species' population expansion slows. The Allee effect causes the system's solutions to take substantially longer to achieve a stable equilibrium point.

It is important to note that in the case of populations with nonoverlapping generations, discrete-time systems controlled by difference equations are preferable to continuous ones.

Existing research indicates that the discrete-time system exhibits more complicated dynamic behaviors and produces more effective numerical simulation results. [17, 18, 19, 20, 21, 22].

The authors of [23] investigated the bifurcation and chaos control of the discrete-time version of system (2) by using the forward Euler approach with step size δ as the bifurcation parameter. The numerical results in [23] reveal that when a large step size is taken in Euler's approach, Neimark-Sacker bifurcation occurs; this fact contradicts the precision of the numerical method for discretization. To address this shortcoming, we employ the constant piecewise argument technique [24, 25, 26, 27] to construct the discrete version of system (2), as shown below:

$$\begin{cases} x_{n+1} = x_n exp(a - by_n), \\ y_{n+1} = y_n exp\left(c - \frac{my_n^2}{x_n(\alpha+y_n)}\right), \end{cases} \tag{3}$$

The objective of the present work is to explore the stability, bifurcations, and chaos in a discrete predator-prey system with the piecewise-constant argument method (3). More precisely, our main findings in this paper are as follows:

- The existence and topological classification of equilibrium points are discussed.
- At the interior equilibrium point, we investigate period-doubling(PD) and Neimark-Sacker(NS) bifurcation.
- About interior equilibrium, the direction and existence conditions for both kinds of bifurcations are investigated.
- To control chaos in the system, a hybrid control strategy is used.
- Numerical simulations are performed to illustrate that a discrete system has rich dynamics.

The paper is organized as follows: In Section 2, the existence and topological classification of equilibrium points are investigated. The existence and direction of the period-doubling Neimark-Sacker bifurcations are proved analytically in Section 3. The chaos control system is developed in Section 4. Detailed numerical simulations and computation analysis are developed to support the analytical findings in Section 5. Finally, Section 6 draws the conclusion to this paper.

2. Topological Classification of Equilibrium Points

Equilibrium points of the system (3) are determined by solving the following system of equations:

$$\begin{cases} x = xexp(a-by), \\ y = yexp\left(c - \frac{my^2}{x(\alpha+y)}\right). \end{cases} \tag{4}$$

Simple computation yields the system (3) has one nontrivial equilibrium point $E = (\frac{a^2m}{bc(a+b\alpha)}, \frac{a}{b})$.

To analyze the local stability properties of the equilibria, we need the Jacobian matrix J at an arbitrary equilibrium (x,y), which is given as follows:

$$J(x,y) = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix},$$

where

$$j_{11} = e^{a-by},\ j_{12} = -be^{a-by}x,\ j_{21} = \frac{e^{c-\frac{my^2}{x(y+\alpha)}}my^3}{x^2(y+\alpha)},$$

$$j_{22} = \frac{e^{c-\frac{my^2}{x(y+\alpha)}}\left(x(y+\alpha)^2 - my^2(y+2\alpha)\right)}{x(y+\alpha)^2}.$$

The following lemma describes the various conditions associated with the local stability analysis of equilibrium points.

Lemma 2.1. [28]

Let $F(\theta) = \theta^2 + A_1\theta + A_0$. Assume that $F(1) > 0$. If θ_1, θ_2 are two roots of $F(\theta) = 0$, then

(1) $|\theta_1| < 1$ and $|\theta_2| < 1$ iff $F(-1) > 0$ and $A_0 < 1$,

(2) $|\theta_1| < 1$ and $|\theta_2| > 1$ (or $|\theta_1| > 1$ and $|\theta_2| < 1$) iff $F(-1) < 0$,

(3) $|\theta_1| > 1$ and $|\theta_2| > 1$ iff $F(-1) > 0$ and $A_0 > 1$,

(4) $\theta_1 = -1$ and $|\theta_2| \neq 1$ iff $F(-1) = 0$ and $A_1 \neq 0, 2$,

(5) $\theta_1, \theta_2 \in \mathbb{C}$ and $|\theta_{1,2}| = 1$ iff $A_1^2 - 4A_0 < 0$ and $A_0 = 1$.

Let θ_1, θ_2 be eigenvalues of $J(x,y)$, then following topological classifications are considered:

(i) (x,y) is a sink iff $|\theta_1| < 1$ and $|\theta_2| < 1$,

(ii) (x,y) is a source iff $|\theta_1| > 1$ and $|\theta_2| > 1$,

(iii) (x,y) is a saddle point iff $|\theta_1| < 1$ and $|\theta_2| > 1$ (or $|\theta_1| > 1$ and $|\theta_2| < 1$),

(iv) (x,y) is non-hyperbolic point iff either $|\theta_1| = 1$ or $|\theta_2| = 1$.

The Jacobian matrix J and its characteristic polynomial $F(\theta)$ for system (3) evaluated at $E = (\frac{a^2m}{bc(a+b\alpha)}, \frac{a}{b})$ are computed as follows:

$$J(x,y) = \begin{bmatrix} 1 & -\frac{a^2m}{ac+bc\alpha} \\ \frac{c^2(a+b\alpha)}{am} & \frac{a-ac+b\alpha-2bc\alpha}{a+b\alpha} \end{bmatrix}$$

and

$$F(\theta) = \theta^2 + \frac{(a(-2+c)+2b(-1+c)\alpha)\theta}{a+b\alpha} + \frac{a+a^2c+b(1-2c)\alpha+ac(-1+b\alpha)}{a+b\alpha}$$

By simple computations, we obtain

$$F(1) = ac,\ F(0) = \frac{a+a^2c+b(1-2c)\alpha+ac(-1+b\alpha)}{a+b\alpha},$$
$$F(-1) = \frac{a^2c-4b(-1+c)\alpha+a(4+c(-2+b\alpha))}{a+b\alpha}.$$

Using Lemma 2.1, we discuss the topological classification of E by stating the following result.

Theorem 2.2. *The following holds true for equilibrium point E of system* (3)*:*

(1) E is a sink if $c < \frac{-4a-4b\alpha}{-2a+a^2-4b\alpha+ab\alpha}$ and if one of the requirements listed below is satisfied:

(a) $0 < a \leq 1$,

(b) $1 < a < 2$ and $b > \frac{a-a^2}{-2\alpha+a\alpha}$,

(2) E is a saddle point if $c > \frac{-4a-4b\alpha}{-2a+a^2-4b\alpha+ab\alpha}$ and if one of the requirements listed below is satisfied:

(a) $0 < a \leq 2$,

(b) $2 < a < 4$ and $b > \frac{2a-a^2}{-4\alpha+a\alpha}$,

(3) E is a source if one of the requirements listed below is satisfied:

(a) $a \geq 4$,

(b) $1 < a < 2,\ b < \frac{a-a^2}{-2\alpha+a\alpha}$, and $c < \frac{-4a-4b\alpha}{-2a+a^2-4b\alpha+ab\alpha}$,

(c) $2 \leq a < 4$ and $b \leq \frac{2a-a^2}{-4\alpha+a\alpha}$,

(d) $2 \leq a < 4,\ b > \frac{2a-a^2}{-4\alpha+a\alpha}$, and $c < \frac{-4a-4b\alpha}{-2a+a^2-4b\alpha+ab\alpha}$,

(4) E is a non-hyperbolic point if one of the requirements listed below is satisfied:
(a) $1 < c \leq 2,\ a < \frac{-4(1-c)}{c}$, *and* $b = \frac{-4a+2ac-a^2c}{4\alpha-4c\alpha+ac\alpha}$,
(b) $c > 2,\ \frac{-4+2c}{c} < a < \frac{-4(1-c)}{c}$, *and* $b = \frac{-4a+2ac-a^2c}{4\alpha-4c\alpha+ac\alpha}$,
(c) $c \leq 2,\ 1 < a < 2$, *and* $b = \frac{a(1-a)}{\alpha(-2+a)}$,
(d) $2 < c < 4,\ 1 < a < \frac{4}{c}$, *and* $b = \frac{a(1-a)}{\alpha(-2+a)}$.

It is clear that if $b = \frac{-4a+2ac-a^2c}{4\alpha-4c\alpha+ac\alpha}$ or $b = \frac{-4a+2ac-a^2c}{4\alpha-4c\alpha+ac\alpha}$, then one of the eigenvalues of $J(E)$ is -1. As a result, there is the potential for period-doubling bifurcation to take place if the parameters are allowed to change in a close neighborhood of Λ_1 or Λ_2, where

$$\Lambda_1 = \left\{ (a,b,c,m,\alpha) \in \mathbb{R}^5_+ \middle| 1 < c \leq 2,\ a < \frac{-4(1-c)}{c},\ b = b_1 = \frac{-4a+2ac-a^2c}{4\alpha-4c\alpha+ac\alpha} \right\},$$

$$\Lambda_2 = \left\{ (a,b,c,m,\alpha) \in \mathbb{R}^5_+ \middle| c > 2,\ \frac{-4+2c}{c} < a < \frac{-4(1-c)}{c},\ b = b_2 = \frac{-4a+2ac-a^2c}{4\alpha-4c\alpha+ac\alpha} \right\}.$$

Furthermore, if $b = \frac{a(1-a)}{\alpha(-2+a)}$ or $b = \frac{a(1-a)}{\alpha(-2+a)}$, the eigenvalues of $J(E)$ are unit-modulus complex. Thus, the system experiences Neimark-Sacker bifurcation if the parameters are varied in a close neighborhood of Λ_3 or Λ_4, where

$$\Lambda_3 = \left\{ (a,b,c,m,\alpha) \in \mathbb{R}^5_+ \middle| c \leq 2,\ 1 < a < 2,\ b = b_3 = \frac{a(1-a)}{\alpha(-2+a)} \right\},$$

$$\Lambda_4 = \left\{ (a,b,c,m,\alpha) \in \mathbb{R}^5_+ \middle| 2 < c < 4,\ 1 < a < \frac{4}{c},\ b = b_4 = \frac{a(1-a)}{\alpha(-2+a)} \right\}.$$

3. Bifurcation Analysis

In this section, we discuss period-doubling and Neimark-Sacker bifurcations for equilibrium point E of system (3) by taking b as bifurcation parameter. We began by investigating the period-doubling bifurcation at E when parameters vary in a small neighborhood of Λ_1. Similar investigations can be done for Λ_2.

If b varies in a small neighborhood of b_1, then system (3) takes the following form:

$$\begin{cases} x_{n+1} = x_n exp(a - b_1 y_n), \\ y_{n+1} = y_n exp\left(c - \frac{m y_n^2}{x_n(\alpha+y_n)}\right). \end{cases} \tag{5}$$

Next, we consider a perturbation of system (5) as follows:

$$(6)\qquad \begin{cases} x_{n+1} = x_n exp\left(a-(b_1+\varepsilon)y_n\right), \\ y_{n+1} = y_n exp\left(c-\frac{my_n^2}{x_n(\alpha+y_n)}\right), \end{cases}$$

where ε is a perturbation in bifurcation parameter b and $|\varepsilon| \ll 1$. Now, it is noted that by using transformation $u_n = x_n - \frac{a^2m}{(b+\varepsilon)c(a+(b+\varepsilon)\alpha)}$, $v_n = y_n - \frac{a}{b+\varepsilon}$, one can translate the equilibrium point E to origin. Under this translation, the system (6) becomes

$$(7)\qquad \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{a(4+(-4+a)c)m}{2c^2} \\ -\frac{2c^3}{(4+(-4+a)c)m} & -1-\frac{ac}{2} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} F(u_n,v_n,\varepsilon) \\ G(u_n,v_n,\varepsilon) \end{bmatrix},$$

where

$$\begin{aligned} F(u_n,v_n,\varepsilon) &= a_1u_nv_n^2 + a_2v_n^3 + a_3u_nv_n + a_4v_n^2 + a_5u_nv_n\varepsilon + a_6v_n^2\varepsilon + O((|u_n|+|v_n|+|\varepsilon|)^4), \\ G(u_n,v_n,\varepsilon) &= b_1u_n^3 + b_2u_n^2v_n + b_3u_nv_n^2 + b_4v_n^3 + b_5u_n^2 + b_6u_nv_n + b_7v_n^2 + b_8u_n^2\varepsilon + b_9u_nv_n\varepsilon \\ &\quad + b_{10}v_n^2\varepsilon + b_{11}u_n\varepsilon + b_{12}v_n\varepsilon + b_{13}\varepsilon + b_{14}u_n\varepsilon^2 + b_{15}v_n\varepsilon^2 + b_{16}\varepsilon^2 + b_{17}\varepsilon^3 \\ &\quad + O((|u_n|+|v_n|+|\varepsilon|)^4), \end{aligned}$$

where the values of $a_1,a_2,a_3,a_4,a_5,a_6,b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,b_{10},b_{11},b_{12},b_{13},b_{14},b_{15}$, b_{16},b_{17} are provided in Appendix A.

Next, we use the following transformation:

$$(8)\qquad \begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} -\frac{\left(4a-4ac+a^2c\right)m}{4c^2} & -\frac{4m-4cm+acm}{c^3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix}$$

As a result, the system (7) becomes

$$(9)\qquad \begin{bmatrix} e_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1-\frac{ac}{2} \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix} + \begin{bmatrix} \phi(e_n,f_n,\varepsilon) \\ \varphi(e_n,f_n,\varepsilon) \end{bmatrix},$$

where

$$\begin{aligned}\phi(e_n,f_n,\varepsilon) &= c_1e_n^3+c_2e_n^2f_n+c_3e_nf_n^2+c_4f_n^3+c_5e_n^2+c_6e_nf_n+c_7f_n^2+c_8e_n^2\varepsilon+c_9e_nf_n\varepsilon\\ &\quad +c_{10}f_n^2\varepsilon+c_{11}e_n\varepsilon+c_{12}f_n\varepsilon+c_{13}\varepsilon+c_{14}e_n\varepsilon^2+c_{15}f_n\varepsilon^2+c_{16}\varepsilon^2+c_{17}\varepsilon^3\\ &\quad +O((|e_n|+|f_n|+|\varepsilon|)^4),\\ \varphi(e_n,f_n,\varepsilon) &= d_1e_n^3+d_2e_n^2f_n+d_3e_nf_n^2+d_4f_n^3+d_5e_n^2+d_6e_nf_n+d_7f_n^2+d_8e_n^2\varepsilon+d_9e_nf_n\varepsilon\\ &\quad +d_{10}f_n^2\varepsilon+d_{11}e_n\varepsilon+d_{12}f_n\varepsilon+d_{13}\varepsilon+d_{14}e_n\varepsilon^2+d_{15}f_n\varepsilon^2+d_{16}\varepsilon^2+d_{17}\varepsilon^3\\ &\quad +O((|e_n|+|f_n|+|\varepsilon|)^4),\end{aligned}$$

where the values of $c_1,c_2,c_3,c_4,c_5,c_6,c_7,c_8,c_9,c_{10},c_{11},c_{12},c_{13},c_{14},c_{15},c_{16},c_{17}$, and $b_1,b_2,b_3,b_4,b_5,b_6, b_7,b_8,b_9,b_{10},b_{11},b_{12},b_{13},b_{14},b_{15},b_{16},b_{17}$ are provided in Appendix B.

Next, we determine the center manifold $W^C(0,0)$ of the system (9) at the equilibrium point $(0,0)$ in a small neighborhood of $\varepsilon=0$. It can be expressed as follows:

$$W^C(0,0,0)=\left\{(e_n,f_n,\gamma)\in\mathbb{R}^3\,\middle|\,f_n=M_0\varepsilon+M_1e_n^2+M_2e_n\varepsilon+M_3\varepsilon^2+O((|e_n|+|\varepsilon|)^3)\right\},$$

where

$$\begin{aligned}&M_0=-\frac{d_{13}}{-1+\lambda},\ M_1=-\frac{d_5}{-1+\lambda},\ M_2=\frac{2c_{13}d_5+d_{11}-\lambda d_{11}+d_6d_{13}}{-1+\lambda^2},\\ &M_3=\frac{1}{(-1+\lambda)^3(1+\lambda)}\Big(-c_{13}^2d_5+2\lambda c_{13}^2d_5-\lambda^2c_{13}^2d_5-c_{13}d_{11}+2\lambda c_{13}d_{11}-\lambda^2c_{13}d_{11}\\ &\quad -c_{13}d_6d_{13}+\lambda c_{13}d_6d_{13}-d_{12}d_{13}+\lambda^2d_{12}d_{13}-d_7d_{13}^2-\lambda d_7d_{13}^2-d_{16}+\lambda d_{16}+\lambda^2d_{16}\\ &\quad -\lambda^3d_{16}\Big).\end{aligned}$$

The system (9) restricted to the center manifold $W^C(0,0,0)$ is provided by

$$\begin{aligned}\tilde{F}: e_{n+1} &= -e_n+e_n^2c_5+\varepsilon c_{13}+e_n^3(c_1-\frac{c_6d_5}{-1+\lambda})+e_n\varepsilon(c_{11}-\frac{c_6d_{13}}{-1+\lambda})\\ &\quad +\varepsilon^2(c_{16}+\frac{d_{13}(-(-1+\lambda)c_{12}+c_7d_{13})}{(-1+\lambda)^2})+e_n^2\varepsilon(c_8-\frac{c_{12}d_5}{-1+\lambda}-\frac{c_2d_{13}}{-1+\lambda}+\frac{2c_7d_5d_{13}}{(-1+\lambda)^2}\\ &\quad +\frac{c_6(2c_{13}d_5-(-1+\lambda)d_{11}+d_6d_{13})}{-1+\lambda^2})+e_n\varepsilon^2(c_{14}-\frac{c_9d_{13}}{-1+\lambda}+\frac{c_3d_{13}^2}{(-1+\lambda)^2}\end{aligned}$$

$$+\frac{c_{12}(2c_{13}d_5-(-1+\lambda)d_{11}+d_6d_{13})}{-1+\lambda^2}-\frac{2c_7d_{13}(2c_{13}d_5-(-1+\lambda)d_{11}+d_6d_{13})}{(-1+\lambda)^2(1+\lambda)}$$
$$-\frac{1}{(-1+\lambda)^3(1+\lambda)}c_6((-1+\lambda)^2c_{13}^2d_5+(-1+\lambda)c_{13}((-1+\lambda)d_{11}-d_6d_{13})$$
$$+(1+\lambda)(-(-1+\lambda)d_{12}d_{13}+d_7d_{13}^2+(-1+\lambda)^2d_{16})))$$
$$+\varepsilon^3(c_{17}-\frac{1}{(-1+\lambda)^3(1+\lambda)}c_{12}((-1+\lambda)^2c_{13}^2d_5+(-1+\lambda)c_{13}((-1+\lambda)d_{11}-d_6d_{13})$$
$$+(1+\lambda)(-(-1+\lambda)d_{12}d_{13}+d_7d_{13}^2+(-1+\lambda)^2d_{16}))$$
$$+\frac{1}{(-1+\lambda)^4(1+\lambda)}d_{13}(-(-1+\lambda)^3(1+\lambda)c_{15}$$
$$+(-1+\lambda^2)d_{13}((-1+\lambda)c_{10}-c_4d_{13})+2c_7((-1+\lambda)^2c_{13}^2d_5$$
$$+(-1+\lambda)c_{13}((-1+\lambda)d_{11}-d_6d_{13})+(1+\lambda)(-(-1+\lambda)d_{12}d_{13}+d_7d_{13}^2$$
$$+(-1+\lambda)^2d_{16})))).$$

Now for period-doubling bifurcation, we require that the following two quantities ξ and η are non-zero, where

$$\xi=\tilde{F}_\varepsilon\tilde{F}_{e_ne_n}+2\tilde{F}_{e_n\varepsilon}\bigg|_{(0,0)}=2\left(c_{11}+c_5c_{13}-\frac{c_6d_{13}}{-1+\lambda}\right), \tag{10}$$

$$\eta=\frac{1}{2}(\tilde{F}_{e_ne_n})^2+\frac{1}{3}\tilde{F}_{e_ne_ne_n}\bigg|_{(0,0)}=2\left(c_1+c_5^2-\frac{c_6d_5}{-1+\lambda}\right). \tag{11}$$

As a consequence of the above study, we reach the following conclusion:

Theorem 3.1. *Suppose that* $(a,b,c,m,\alpha)\in\Lambda_1$. *The system* (3) *undergoes period-doubling bifurcation at equilibrium point E if* ξ,η *defined in* (10) *and* (11) *are nonzero and b differs in a small neighborhood of* $b_1=\frac{-4a+2ac-a^2c}{4\alpha-4c\alpha+ac\alpha}$. *Moreover, if* $\eta>0$ *(respectively* $\eta<0$*), then the period-2 orbits that bifurcate from E are stable (respectively, unstable).*

Next, we investigate Neimark-Sacker bifurcation at E when parameters vary in a small neighborhood of Λ_3. Similar investigations can be done for Λ_4.

If b varies in a small neighborhood of b_3, then system (3) takes the following form:

$$\begin{cases} x_{n+1}=x_n exp(a-b_3y_n),\\ y_{n+1}=y_n exp\left(c-\frac{my_n^2}{x_n(\alpha+y_n)}\right). \end{cases} \tag{12}$$

Next, we consider a perturbation of system (12) as follows:

$$\begin{cases} x_{n+1} = x_n exp\left(a-(b_3+\gamma)y_n\right), \\ y_{n+1} = y_n exp\left(c-\frac{my_n^2}{x_n(\alpha+y_n)}\right), \end{cases} \tag{13}$$

where γ is a perturbation in bifurcation parameter b and $|\gamma| \ll 1$. Now, it is noted that by using transformation $u_n = x_n - \frac{a^2m}{(b+\gamma)c(a+(b+\gamma)\alpha)}$, $v_n = y_n - \frac{a}{b+\gamma}$, one can translate the equilibrium point E to origin. Under this translation, the system (13) becomes

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{(-2+a)am}{c} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} F(u_n,v_n,\varepsilon) \\ G(u_n,v_n,\varepsilon) \end{bmatrix}, \tag{14}$$

where

$$k_{21} = \frac{ac^2 e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}}{(-2+a)m(-2\alpha\gamma+a(-1+\alpha\gamma))},$$

$$k_{22} = \frac{e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}\left(-a^3c+4\alpha^2\gamma^2-4a\alpha\gamma(-1+c+\alpha\gamma)+a^2\left(1+2(-1+c)\alpha\gamma+\alpha^2\gamma^2\right)\right)}{(-2\alpha\gamma+a(-1+\alpha\gamma))^2},$$

$$F(u_n,v_n,\varepsilon) = a_1u_nv_n + a_2u_nv_n^2 + a_3v_n^2 + a_4v_n^3 + O((|u_n|+|v_n|)^4),$$

$$G(u_n,v_n,\varepsilon) = b_1u_n^2 + b_2u_nv_n + b_3v_n^2 + b_4u_n^3 + b_5u_n^2v_n + b_6u_nv_n^2 + b_7v_n^3 + O((|u_n|+|v_n|)^4),$$

where

$$a_1 = \frac{(-1+a)a}{(-2+a)\alpha} - \gamma,\ a_2 = \frac{1}{2}\left(-\frac{(-1+a)a}{(-2+a)\alpha}+\gamma\right)^2,$$

$$a_3 = \frac{am\left(a^2+2\alpha\gamma-a(1+\alpha\gamma)\right)}{2c\alpha},\ a_4 = \frac{am\left(a^2+2\alpha\gamma-a(1+\alpha\gamma)\right)^2}{6(-2+a)c\alpha^2},$$

and values of $b_1,b_2,b_3,b_4,b_5,b_6,b_7$ are provided in Appendix C.

Let

$$\theta^2 - p(\gamma)\theta + q(\gamma) = 0 \tag{15}$$

be the characteristic equation of the Jacobian matrix of system (14) evaluated at $(0,0)$, where

$$p(\gamma) = -\frac{1}{(-2\alpha\gamma+a(-1+\alpha\gamma))^2}\Bigg(a^3ce^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}} - 4(1+e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}})\alpha^2\gamma^2 + 4a\alpha\gamma(-1+\alpha\gamma$$
$$+e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}(-1+c+\alpha\gamma)) - a^2((-1+\alpha\gamma)^2 + e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}(1+2(-1+c)\alpha\gamma+\alpha^2\gamma^2))\Bigg),$$

$$q(\gamma) = \frac{e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}\left(-a^3c\alpha\gamma+4\alpha^2\gamma^2-4a\alpha\gamma(-1+c+\alpha\gamma)+a^2\left(1+2(-1+2c)\alpha\gamma+\alpha^2\gamma^2\right)\right)}{(-2\alpha\gamma+a(-1+\alpha\gamma))^2}.$$

The roots of (15) are given by

$$\theta_{1,2} = \frac{p(\gamma)}{2} \pm \frac{i}{2}\sqrt{4q(\gamma)-p^2(\gamma)}. \tag{16}$$

Since $(a,b,c,m,\alpha) \in \Lambda_3$, we have, $|\theta_{1,2}| = 1$ and

$$\left(\frac{d|\theta_1|}{d\gamma}\right)_{\gamma=0} = \left(\frac{d|\theta_2|}{d\gamma}\right)_{\gamma=0} = -\frac{(a-2)(a-1)c\alpha}{2a} \neq 0.$$

Additionally, at $\gamma = 0$ it is required that $\theta_{1,2}^k \neq 1$ for $k \in \{1,2,3,4\}$ which is equivalent to $p(0) \neq -2,-1,0,2$. Since $(a,b,c,m,\alpha) \in \Lambda_3$, it follows that $p(0) = 2-ac \neq \pm 2$. Moreover, $p(0) \neq -1$ and $p(0) \neq 0$ implies that $ac \neq 3$ and $ac \neq 1$, respectively.

The following similarity transformation is considered in order to convert the linear part of (14) into canonical form at $\gamma = 0$:

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} \frac{(-2+a)am}{c} & 0 \\ -\frac{ac}{2} & -\frac{1}{2}\sqrt{-ac(-4+ac)} \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix}. \tag{17}$$

Under the transformation (17), the system (14) becomes

$$\begin{bmatrix} e_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} 1-\frac{ac}{2} & -\frac{1}{2}\sqrt{-ac(-4+ac)} \\ \frac{1}{2}\sqrt{-ac(-4+ac)} & 1-\frac{ac}{2} \end{bmatrix} \begin{bmatrix} e_n \\ f_n \end{bmatrix} + \begin{bmatrix} \Phi(e_n,f_n) \\ \Psi(e_n,f_n) \end{bmatrix}, \tag{18}$$

where

$$\Phi(e_n,f_n) = C_1e_n^3+C_2e_n^2f_n+C_3e_nf_n^2+C_4f_n^3+C_5e_n^2+C_6e_nf_n+C_7f_n^2+O((|e_n|+|f_n|)^4),$$

$$\Psi(e_n,f_n) = D_1e_n^3+D_2e_n^2f_n+D_3e_nf_n^2+D_4f_n^3+D_5e_n^2+D_6e_nf_n+D_7f_n^2+O((|e_n|+|f_n|)^4),$$

where values of $C_1,C_2,C_3,C_4,C_5,C_6,C_7,D_1,D_2,D_3,D_4,D_5,D_6,D_7$ are provided in Appendix D.

To examine the direction of the Neimark-Sacker bifurcation, we consider the first Lyapunov exponent derived as follows:

$$L = \left(\left[-Re\left(\frac{(1-2\theta_1)\theta_2^2}{1-\theta_1}m_{20}m_{11}\right)-\frac{1}{2}|m_{11}|^2-|m_{02}|^2+Re(\theta_2 m_{21})\right]\right)_{\gamma=0}, \tag{19}$$

where

$$m_{20} = \frac{1}{8}\left[\Phi_{e_ne_n} - \Phi_{f_nf_n} + 2\Psi_{e_nf_n} + i(\Psi_{e_ne_n} - \Psi_{f_nf_n} - 2\Phi_{e_nf_n})\right],$$
$$m_{11} = \frac{1}{4}\left[\Phi_{e_ne_n} + \Phi_{f_nf_n} + i(\Psi_{e_ne_n} + \Psi_{f_nf_n})\right],$$
$$m_{02} = \frac{1}{8}\left[\Phi_{e_ne_n} - \Phi_{f_nf_n} - 2\Psi_{e_nf_n} + i(\Psi_{e_ne_n} - \Psi_{f_nf_n} + 2\Phi_{e_nf_n})\right],$$
$$m_{21} = \frac{1}{16}\left[\Phi_{e_ne_ne_n} + \Phi_{e_nf_nf_n} + \Psi_{e_ne_nf_n} + \Psi_{f_nf_nf_n} + i(\Psi_{e_ne_ne_n} + \Psi_{e_nf_nf_n} - \Phi_{e_ne_nf_n} - \Phi_{f_nf_nf_n})\right].$$

Due to the computations mentioned above, we have the following theorem:

Theorem 3.2. *Suppose that* $(a,b,c,m,\alpha) \in \Lambda_3$. *If* $ac \neq 1,3$ *and* $L \neq 0$, *then equilibrium point* E *of system* (3) *undergoes Neimark-Sacker bifurcation when the bifurcation parameter* b *varies in a small neighborhood of* $b_3 = \frac{a(1-a)}{\alpha(-2+a)}$. *In addition, if* $L < 0$ (*or* $L > 0$), *then an attracting (or repelling) invariant closed curve bifurcates from* E *for* $b > b_3$ (*or* $b < b_3$).

4. Chaos Control

In this section, we implement the hybrid control approach in system (3) for controlling the chaos generated by the period-doubling bifurcation and the Neimark-Sacker bifurcation. We consider the following controlled system, which corresponds to system (3):

$$\begin{cases} x_{n+1} = \beta x_n exp(a - by_n) + (1-\beta)x_n, \\ y_{n+1} = \beta y_n exp\left(c - \frac{my_n^2}{x_n(\alpha+y_n)}\right) + (1-\beta)y_n, \end{cases} \tag{20}$$

where $0 < \beta < 1$ is the control parameter for the hybrid control method. The equilibrium points of the controlled system (20) and the uncontrolled system (3) are identical. By appropriate choice of β, the bifurcation for E of system (3) can be advanced or delayed or even entirely obliterated. One can compute Jacobian matrix at E for system (20) as follows:

$$J(E) = \begin{bmatrix} 1 & -\frac{a^2m\beta}{ac+bc\alpha} \\ \frac{c^2(a+b\alpha)\beta}{am} & \frac{a+b\alpha-ac\beta-2bc\alpha\beta}{a+b\alpha} \end{bmatrix}.$$

The trace τ and determinant σ of $J(E)$ are

$$\tau = \frac{2a+2b\alpha - ac\beta - 2bc\alpha\beta}{a+b\alpha},$$
$$\sigma = \frac{a+a^2c\beta^2+b\alpha(1-2c\beta)+ac\beta(-1+b\alpha\beta)}{a+b\alpha}.$$

The Jury condition states that the equilibrium point E of system (20) is stable if and only if the following requirements are met:

$$\begin{cases} \tau+\sigma+1>0, \\ -\tau+\sigma+1>0, \\ \sigma-1<0. \end{cases} \tag{21}$$

5. Numerical Examples

Some intriguing numerical examples are offered in this section to validate our theoretical discussions on various qualitative characteristics of the system.

Example 5.1. *We select the parameter values and initial values as follows:*

$$a = 1.5, c = 1.8, m = 2, \alpha = 0.8, x(0) = 0.01, y(0) = 0.1.$$

For these values, the equilibrium point of (3) *is* $E = (0.0199124, 0.129032)$. *The eigenvalues of* $J(E)$ *for these values are* $\lambda_1 = -1, \lambda_2 = -0.35$*, which confirms that the system* (3) *undergoes period-doubling bifurcation at* $(0.0199124, 0.129032)$ *as bifurcation parameter passes through* $b_1 = 11.625$*. We plot bifurcation diagrams for both prey and predator populations for* $b \in [11.5, 12.5]$ *(see Figure 1a and Figure 1b).*

For controlled system (20)*, we consider the same parameter values with* $\beta = 0.98$*. The bifurcation diagram for the controlled system depicts that period-doubling bifurcation has been delayed. See figures 1c,1d. The controlled system is experiencing Neimark-Sacker bifurcation when b passes through* $b_1 = 12.4147$ *(See Figure 1c and Figure 1d). In the original system* (3)*, the equilibrium point E is stable for* $b < 11.625$*, whereas in the controlled system* (20)*, the equilibrium point E is stable for* $b < 12.4147$*.*

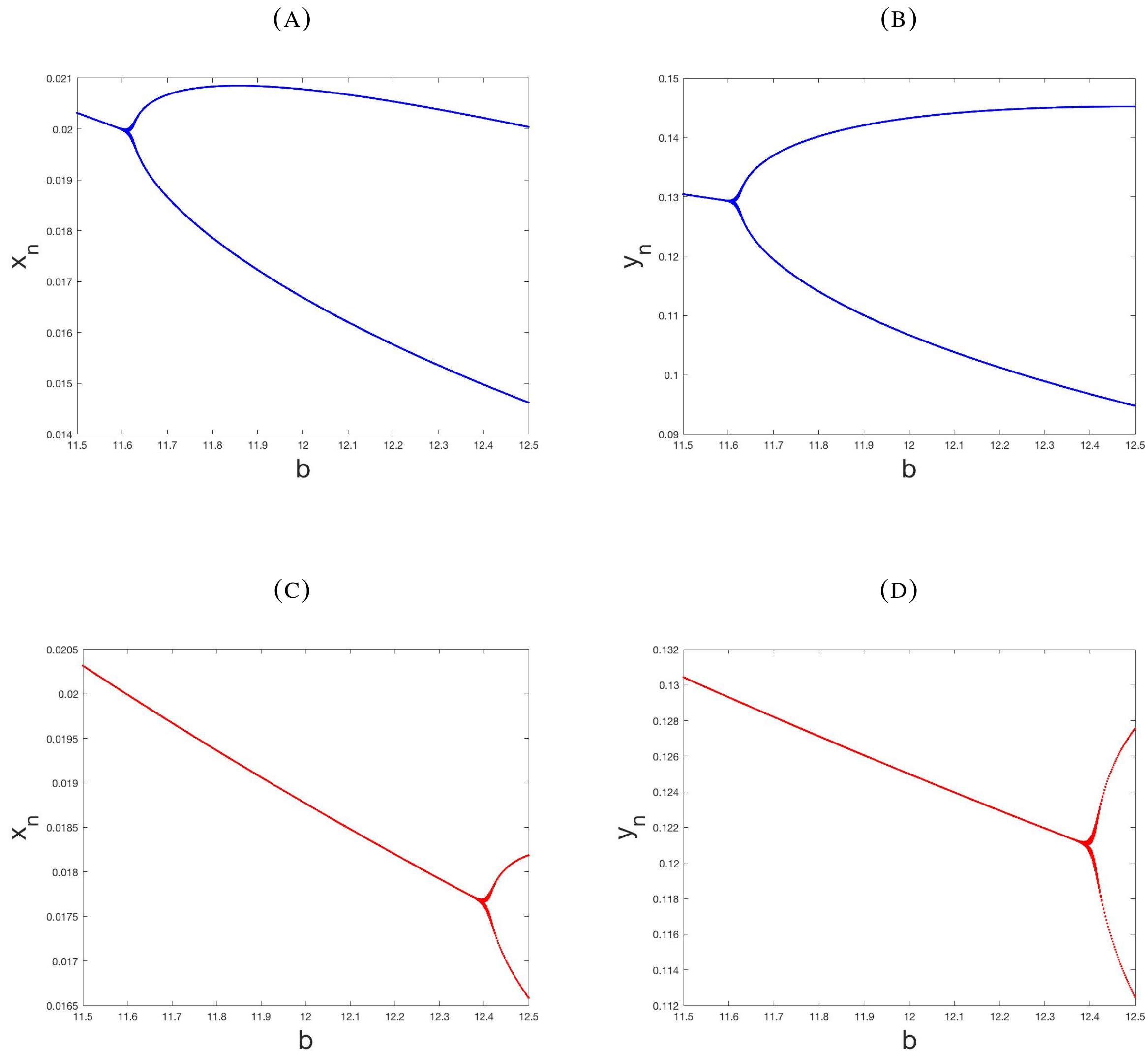

FIGURE 1. Bifurcation diagrams of (3) and (20) for $\beta = 0.98, a = 1.5, c = 1.8, m = 2, \alpha = 0.8, x(0) = 0.01, y(0) = 0.1, b \in [11.5, 12.5]$.

Example 5.2. *We select the parameter values and initial values as follows:*

$$a = 1.5, c = 1.8, m = 2, \alpha = 0.8, x(0) = 0.5, y(0) = 0.85.$$

For these values, the equilibrium point of (3) *is* $E = (0.444444, 0.8)$. *The eigenvalues of* $J(E)$ *for these values are* $\theta_1 = -0.35 - 0.93675i, \theta_2 = -0.35 + 0.93675i$ *satisfying* $|\theta_{1,2}| = 1$, *which confirms that the system* (3) *undergoes Neimark-Sacker bifurcation at* $(0.444444, 0.8)$ *as* *bifurcation parameter b passes through* $b_3 = 1.875$. *We plot bifurcation diagrams for both prey and predator populations for* $b \in [0.94, 2.14]$ *(see Figure 2a and Figure 2b).*

(A) (B)

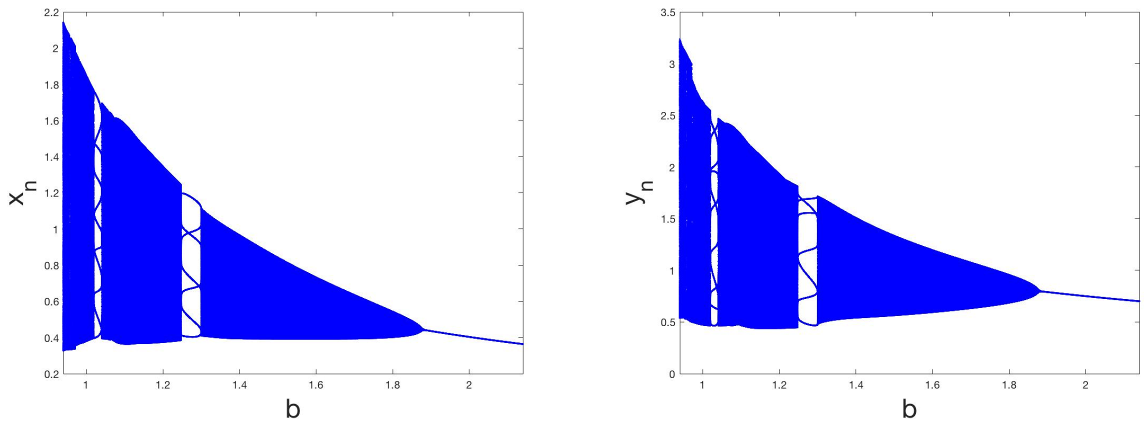

FIGURE 2. Bifurcation diagrams of (3) for $a = 1.5, c = 1.8, m = 2, \alpha = 0.8, x(0) = 0.5, y(0) = 0.85, b \in [0.94, 2.14]$.

The equilibrium point E is a sink for these parameter values iff $b > 1.875$. *Figures 3a-3e depict phase portraits of system* (3) *for different values of b. The figures show that the equilibrium point E is a sink for* $b > 1.875$ *but becomes unstable at* $b \approx 1.875$, *where the system* (3) *experiences Neimark-Sacker bifurcation. A smooth invariant curve appears for* $b \leq 3.1746$, *increasing its radius as b decreases. By decreasing the value of b, the invariant curve disappears suddenly, and some periodic orbit appears, and then again, we have an invariant curve in place of a periodic orbit.*

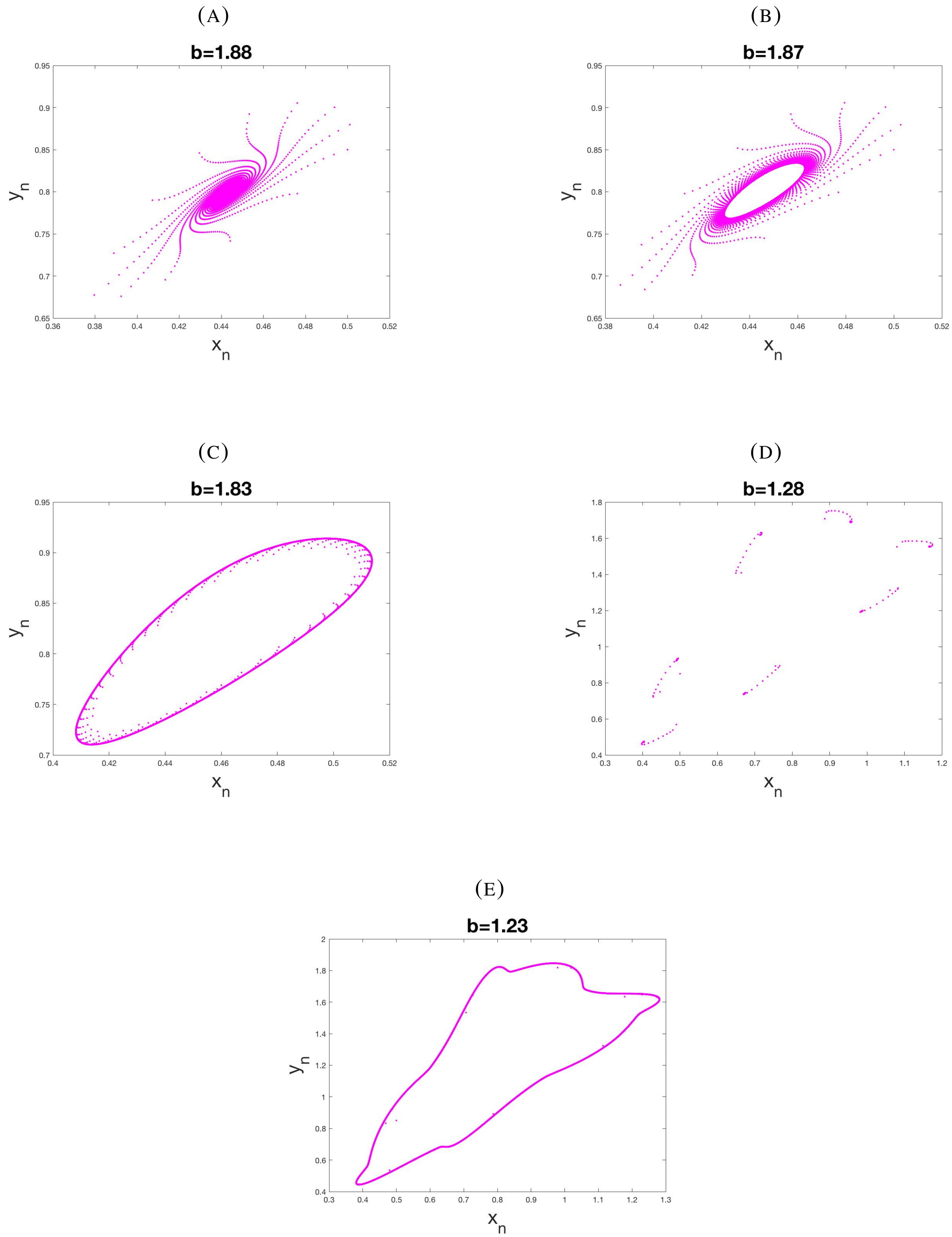

FIGURE 3. Phase portraits of (3) for $a = 1.5, c = 1.8, m = 2, \alpha = 0.8, x(0) = 0.5, y(0) = 0.85, a \in \{1.23, 1.28, 1.83, 1.87, 1.88\}$.

For controlled system (20)*, we consider the same parameter values with* $\beta = 0.95$*. The bifurcation diagram for the controlled system depicts that Neimark-Sacker bifurcation has been delayed. See figures 4a,4b. The controlled system is experiencing Neimark-Sacker bifurcation when b passes through $b_3 = 1.38587$ (See Figure 4a and Figure 4b). In the original system* (3)*, the equilibrium point E is stable for $b > 1.875$, whereas in the controlled system* (20)*, the equilibrium point E is stable for $b > 1.38587$.*

(A) (B)

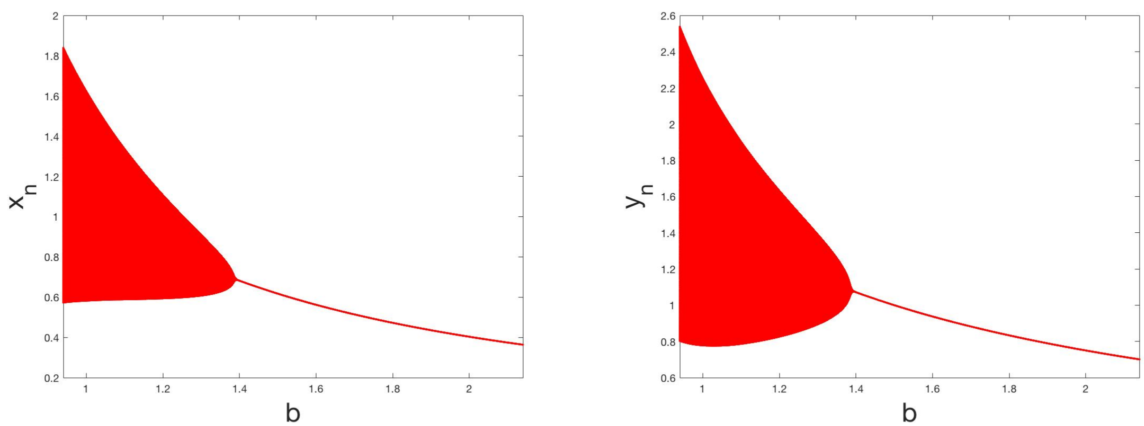

FIGURE 4. Bifurcation diagrams of (20) for $\beta = 0.95, a = 1.5, c = 1.8, m = 2, \alpha = 0.8, x(0) = 0.5, y(0) = 0.85, b \in [0.94, 2.14]$.

6. CONCLUSION

We suggested and investigated the discrete-time predator-prey system (3) with the additive Allee effect in this study utilizing the constant piecewise argument approach. The authors of [23] investigated the bifurcation and chaos control of the discrete-time version of system (2) by using the forward Euler approach with step size δ as the bifurcation parameter. The numerical results in [23] reveal that when a large step size is taken in Euler's approach, Neimark-Sacker bifurcation occurs; this fact contradicts the precision of the numerical method for discretization. We employ the constant piecewise argument approach to compensate for this shortcoming to obtain system (3). The presence of equilibrium points and their topological classification is examined. Furthermore, it is demonstrated that the population may maintain both period-doubling and Neimark-Sacker bifurcation at the equilibrium point E. Using bifurcation theory and the center manifold theorem, the parametric conditions for the direction and presence of both types of bifurcations are derived. A hybrid control method is applied to regulate the chaotic and bifurcating behaviors of a discrete-time system. Finally, a few numerical examples are given to back up the analytical and theoretical findings.

APPENDIX A

$$
\begin{aligned}
a_1 &= \frac{a^2(4+(-2+a)c)^2}{2(4+(-4+a)c)^2\alpha^2},\ a_2=\frac{a^3(4+(-2+a)c)^2 m}{12c^2(4+(-4+a)c)\alpha^2},\ a_3=\frac{a(4+(-2+a)c)}{(4+(-4+a)c)\alpha},\\
a_4 &= \frac{a^2(4+(-2+a)c)m}{4c^2\alpha},\ a_5=-1,\ a_6=-\frac{a(4+(-4+a)c)m}{4c^2},\\
b_1 &= -\frac{4c^7(4+(-2+a)c)^2\left(6-6c+c^2\right)}{3(4+(-4+a)c)^5m^3\alpha^2},\\
b_2 &= -\frac{c^4(4+(-2+a)c)^2\left(8+2(-6+a)c+(2-4a)c^2+ac^3\right)}{(4+(-4+a)c)^4m^2\alpha^2},\\
b_3 &= -\frac{c(4+(-2+a)c)^2\left(32+16(-5+a)c+2\left(20-18a+a^2\right)c^2-4\left(2-3a+a^2\right)c^3+a^2c^4\right)}{4(4+(-4+a)c)^3m\alpha^2},\\
b_4 &= -\frac{1}{48c^2(4+(-4+a)c)^2\alpha^2}(4+(-2+a)c)^2\Bigg(384+96(-10+3a)c+8\left(80-72a+9a^2\right)c^2\\
&\quad +6\left(-24+44a-18a^2+a^3\right)c^3-6a\left(4-5a+a^2\right)c^4+a^3c^5\Bigg),\\
b_5 &= -\frac{2(-2+c)c^5(4+(-2+a)c)}{(4+(-4+a)c)^3m^2\alpha},\\
b_6 &= -\frac{c^2(4+(-2+a)c)\left(-4-(-2+a)c+ac^2\right)}{(4+(-4+a)c)^2m\alpha},\\
b_7 &= -\frac{(4+(-2+a)c)\left(-32-16(-2+a)c-2\left(4-6a+a^2\right)c^2+a^2c^3\right)}{8c(4+(-4+a)c)\alpha},\\
b_8 &= \frac{c^4\left(8+2(-12+a)c+(14-4a)c^2+(-2+a)c^3\right)}{a(4+(-4+a)c)^2m^2},\\
b_9 &= \frac{c\left(32+16(-6+a)c+2\left(32-20a+a^2\right)c^2-4\left(3-5a+a^2\right)c^3+(-2+a)ac^4\right)}{2a(4+(-4+a)c)m},\\
b_{10} &= \frac{1}{16ac^2}\Bigg(384+32(-32+9a)c+8\left(100-76a+9a^2\right)c^2+2\left(-112+168a-56a^2+3a^3\right)c^3\\
&\quad +\left(16-48a+38a^2-6a^3\right)c^4+(-2+a)a^2c^5\Bigg),
\end{aligned}
$$

$$b_{11}=\frac{(-1+c)c^2\,\alpha}{am},\ b_{12}=\frac{(4+(-4+a)c)\left(-8-2(-3+a)c+ac^2\right)\,\alpha}{4ac},\ b_{13}=\frac{(4+(-4+a)c)^2\alpha^2}{2a(4+(-2+a)c)},$$

$$b_{14}=-\frac{c(4+(-4+a)c)\left(2-4c+c^2\right)\alpha^2}{4a^2m},$$

$$b_{15}=-\frac{(4+(-4+a)c)^2\left(24+6(-6+a)c+(10-6a)c^2+ac^3\right)\alpha^2}{16a^2c^2},$$

$$b_{16}=-\frac{(4+(-4+a)c)^3\left(-8-2(-2+a)c+(-2+a)c^2\right)\alpha^3}{8a^2c(4+(-2+a)c)^2},$$

$$b_{17}=\frac{(4+(-4+a)c)^4\left(96+48(-3+a)c+2\left(56-30a+3a^2\right)c^2+\left(-28+26a-6a^2\right)c^3+(-2+a)^2c^4\right)\alpha^4}{48a^3c^2(4+(-2+a)c)^3}.$$

APPENDIX B

$$c_1=\frac{(4+(-2+a)c)^2(192+96(-5+a)c+8(40-21a+3a^2)c^2+(-72+72a-18a^2+a^3)c^3)}{6c^2(4+(-4+a)c)^2(-4+ac)\alpha^2},$$

$$c_2=-\Bigg((4+(-2+a)c)^2(-128-16(-20+7a)c-4(60-46a+5a^2)c^2-2(-28+38a-10a^2+a^3)c^3$$
$$+(-2+a)^2ac^4)\Bigg)\Bigg/\Bigg(2c^2(4+(-4+a)c)^2(-4+ac)\alpha^2\Bigg),$$

$$c_3=-\Bigg((4+(-2+a)c)^2(-128-64(-4+a)c-4(44-34a+9a^2)c^2-(-40+72a-36a^2+a^3)c^3$$
$$+a(8-9a+a^2)c^4)\Bigg)\Bigg/\Bigg(2c^2(4+(-4+a)c)^2(-4+ac)\alpha^2\Bigg),$$

$$c_4=\Bigg((4+(-2+a)c)^2(-48(-2+c)^2+a^3c^2(2-6c+c^2)+6a^2c(12-10c+3c^2)$$
$$-24a(-8+14c-7c^2+c^3))\Bigg)\Bigg/\Bigg(12c(4+(-4+a)c)^2(-4+ac)\alpha^2\Bigg),$$

$$c_5=-\frac{(4+(-2+a)c)(16+4(-4+a)c+(-2+a)^2c^2)}{c(4+(-4+a)c)(-4+ac)\alpha},\ c_6=-\frac{2(4+(-2+a)c)(2+(-1+a)c)}{c(-4+ac)\alpha},$$

$$c_7=\frac{(4+(-2+a)c)(-32+32c-4(2-a+a^2)c^2+a^2c^3)}{2c(4+(-4+a)c)(-4+ac)\alpha},$$

$$c_8=-\frac{(192+16(-32+7a)c+8(50-26a+3a^2)c^2+2(-56+52a-14a^2+a^3)c^3-(-2+a)^3c^4)}{2ac^2(-4+ac)},$$

$$c_9=\frac{1}{ac^2(-4+ac)}\Bigg(-128-16(-20+7a)c-4(68-55a+7a^2)c^2+(88-124a+37a^2-2a^3)c^3$$
$$+(-8+18a-9a^2+a^3)c^4\Bigg),$$

$$c_{10}=-\frac{1}{4ac^2(-4+ac)}\Big(256+64(-10+3a)c+8(64-42a+7a^2)c^2+2(-80+96a-42a^2+3a^3)c^3$$
$$+(16-32a+30a^2-6a^3)c^4+(-2+a)a^2c^5\Big),$$
$$c_{11}=\frac{(8+(-6+a)c)(4+(-4+a)c)\alpha}{ac(-4+ac)},\ c_{12}=-\frac{(-2+c)(4+(-4+a)c)(2+ac)\alpha}{ac(-4+ac)},$$
$$c_{13}=-\frac{2(4+(-4+a)c)^2\alpha^2}{a(4+(-2+a)c)(-4+ac)},\ c_{14}=-\frac{(4+(-4+a)c)^2(-12-2(-9+a)c+(-5+a)c^2)\alpha^2}{2a^2c^2(-4+ac)},$$
$$c_{15}=\frac{(4+(-4+a)c)^2(16+(-20+6a)c-6(-1+a)c^2+ac^3)\alpha^2}{4a^2c^2(-4+ac)},$$
$$c_{16}=\frac{(4+(-4+a)c)^3(-8-2(-2+a)c+(-2+a)c^2)\alpha^3}{2a^2c(4+(-2+a)c)^2(-4+ac)},$$
$$c_{17}=-\Big(((4+(-4+a)c)^4(96+48(-3+a)c+2(56-30a+3a^2)c^2+(-28+26a-6a^2)c^3$$
$$+(-2+a)^2c^4)\alpha^4\varepsilon^3\Big)\Big/\Big(12a^3c^2(4+(-2+a)c)^3(-4+ac))\Big),$$
$$d_1=\frac{a(4+(-2+a)c)^2(-96-48(-5+a)c-2(80-42a+7a^2)c^2+9(-2+a)^2c^3)}{12c(4+(-4+a)c)^2(-4+ac)\alpha^2},$$
$$d_2=\Big(a(4+(-2+a)c)^2(-128-64(-5+2a)c-8(30-23a+2a^2)c^2-2(-28+38a-10a^2+a^3)c^3$$
$$+(-2+a)^2ac^4)\Big)\Big/\Big(8c(4+(-4+a)c)^2(-4+ac)\alpha^2\Big),$$
$$d_3=(a(4+(-2+a)c)^2(-128-32(-8+3a)c-8(22-17a+3a^2)c^2-2(-20+36a-18a^2+a^3)c^3$$
$$+a(8-9a+a^2)c^4))/(8c(4+(-4+a)c)^2(-4+ac)\alpha^2),$$
$$d_4=-\Big(a(4+(-2+a)c)^2(-48(-2+c)^2+a^3c^2(6-6c+c^2)+2a^2c(16-30c+9c^2)$$
$$-24a(-12+14c-7c^2+c^3))\Big)\Big/\Big(48(4+(-4+a)c)^2(-4+ac)\alpha^2\Big),$$
$$d_5=\frac{a(4+(-2+a)c)(16+4(-4+a)c+(-2+a)^2c^2)}{4(4+(-4+a)c)(-4+ac)\alpha},$$
$$d_6=\frac{ac(4+(-2+a)c)(8(-3+c)-10a(-2+c)+a^2c)}{4(4+(-4+a)c)(-4+ac)\alpha},$$
$$d_7=-\frac{ac(4+(-2+a)c)(-8(-4+c)+4a(-4+c)+a^2(-2+c)c)}{8(4+(-4+a)c)(-4+ac)\alpha},$$
$$d_8=\frac{(192+16(-32+7a)c+8(50-26a+3a^2)c^2+2(-56+52a-14a^2+a^3)c^3-(-2+a)^3c^4)}{8c(-4+ac)},$$

$$d_9 = \frac{1}{4c(-4+ac)}\Big(128+16(-19+7a)c+4(68-57a+7a^2)c^2+2(-44+62a-18a^2+a^3)c^3$$
$$-(-8+18a-9a^2+a^3)c^4\Big),$$
$$d_{10} = \frac{1}{16c(-4+ac)}\Big(256+192(-3+a)c+8(64-46a+7a^2)c^2+2(-80+96a-40a^2+3a^3)c^3$$
$$+(16-32a+30a^2-6a^3)c^4+(-2+a)a^2c^5\Big),$$
$$d_{11} = -\frac{(8+(-6+a)c)(4+(-4+a)c)\alpha}{4(-4+ac)},\ d_{12} = \frac{(-2+c)(4+(-4+a)c)(2+ac)\alpha}{4(-4+ac)},$$
$$d_{13} = \frac{c(4+(-4+a)c)^2\alpha^2}{2(4+(-2+a)c)(-4+ac)},\ d_{14} = \frac{(4+(-4+a)c)^2(-12-2(-9+a)c+(-5+a)c^2)\alpha^2}{8ac(-4+ac)},$$
$$d_{15} = -\frac{(4+(-4+a)c)^2(16+(-20+6a)c-6(-1+a)c^2+ac^3)\alpha^2}{16ac(-4+ac)},$$
$$d_{16} = -\frac{(4+(-4+a)c)^3(-8-2(-2+a)c+(-2+a)c^2)\alpha^3}{8a(4+(-2+a)c)^2(-4+ac)},$$
$$d_{17} = \frac{(4+(-4+a)c)^4(96+48(-3+a)c+2(56-30a+3a^2)c^2+(-28+26a-6a^2)c^3+(-2+a)^2c^4)\alpha^4}{48a^2c(4+(-2+a)c)^3(-4+ac)},$$

APPENDIX C

$$b_1 = -\frac{c^3e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}(a^2+2\alpha\gamma-a(1+\alpha\gamma))(-4\alpha\gamma+a(-2+c+2\alpha\gamma))}{2(-2+a)^3m^2\alpha(-2\alpha\gamma+a(-1+\alpha\gamma))^2},$$
$$b_2 = \Big(c^2e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}(a^2+2\alpha\gamma-a(1+\alpha\gamma))(-12\alpha^2\gamma^2+a^3(-1+c+\alpha\gamma)+4a\alpha\gamma(-2+c+3\alpha\gamma)$$
$$-a^2(1+2(-1+c)\alpha\gamma+3\alpha^2\gamma^2))\Big)\Big/\Big((-2+a)^2m\alpha(-2\alpha\gamma+a(-1+\alpha\gamma))^3\Big),$$
$$b_3 = -\Big(ce^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}(a^2+2\alpha\gamma-a(1+\alpha\gamma))(-48\alpha^3\gamma^3+a^5(-2+c+2\alpha\gamma)+8a\alpha^2\gamma^2(-3+2c+9\alpha\gamma)$$
$$+a^4(2-4c\alpha\gamma-6\alpha^2\gamma^2)-4a^2\alpha\gamma(1+4c\alpha\gamma+9\alpha^2\gamma^2)+2a^3(-1+(-3+4c)\alpha\gamma+(9+2c)\alpha^2\gamma^2+3\alpha^3\gamma^3))\Big)$$
$$\Big/\Big(2(-2+a)\alpha(a+2\alpha\gamma-a\alpha\gamma)^4\Big),$$
$$b_4 = \Big(c^4e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}(a^2+2\alpha\gamma-a(1+\alpha\gamma))^2(24\alpha^2\gamma^2-12a\alpha\gamma(-2+c+2\alpha\gamma)+a^2(c^2+6c(-1+\alpha\gamma)$$
$$+6(-1+\alpha\gamma)^2))\Big)\Big/\Big(6(-2+a)^5am^3\alpha^2(-2\alpha\gamma+a(-1+\alpha\gamma))^3\Big),$$
$$b_5 = -\Big(c^3e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}(a^2+2\alpha\gamma-a(1+\alpha\gamma))^2(48\alpha^3\gamma^3-4a\alpha^2\gamma^2(-14+9c+18\alpha\gamma)$$

$$-a^3(-2+c+2\alpha\gamma-2c\alpha\gamma+2c^2\alpha\gamma-6\alpha^2\gamma^2+9c\alpha^2\gamma^2+6\alpha^3\gamma^3)+a^4(c^2+4c(-1+\alpha\gamma)+2(-1+\alpha\gamma)^2)$$
$$+4a^2\alpha\gamma(5+c^2-12\alpha\gamma+9\alpha^2\gamma^2+c(-5+9\alpha\gamma)))\Big)\Big/\Big(2(-2+a)^4am^2\alpha^2(a+2\alpha\gamma-a\alpha\gamma)^4\Big),$$

$$b_6=\Big(c^2e^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}(a^2+2\alpha\gamma-a(1+\alpha\gamma))^2(96\alpha^4\gamma^4-16a\alpha^3\gamma^3(-6+7c+12\alpha\gamma)$$
$$+8a^2\alpha^2\gamma^2(4-7c+2c^2-12\alpha\gamma+21c\alpha\gamma+18\alpha^2\gamma^2)+2a^5(-1+c+3\alpha\gamma+2c\alpha\gamma-2c^2\alpha\gamma+\alpha^2\gamma^2$$
$$-7c\alpha^2\gamma^2-3\alpha^3\gamma^3)-4a^3\alpha\gamma(-2+c-2\alpha\gamma+4c^2\alpha\gamma+21c\alpha^2\gamma^2+12\alpha^3\gamma^3)+a^6(c^2+4c(-1+\alpha\gamma)$$
$$+2(-1+\alpha\gamma)^2)+2a^4(1-12\alpha^2\gamma^2+12\alpha^3\gamma^3+3\alpha^4\gamma^4+2c^2\alpha\gamma(2+\alpha\gamma)+c(-1-11\alpha\gamma+21\alpha^2\gamma^2$$
$$+7\alpha^3\gamma^3)))\Big)\Big/\Big(2(-2+a)^3am\alpha^2(-2\alpha\gamma+a(-1+\alpha\gamma))^5\Big),$$

$$b_7=-\Big(ce^{\frac{(-2+a)c\alpha\gamma}{-2\alpha\gamma+a(-1+\alpha\gamma)}}(a^2+2\alpha\gamma-a(1+\alpha\gamma))^2(192\alpha^5\gamma^5-96a\alpha^4\gamma^4(1+4c+5\alpha\gamma)+32a^2\alpha^3\gamma^3(-3-3c$$
$$+2c^2+15\alpha\gamma+24c\alpha\gamma+15\alpha^2\gamma^2)-48a^3\alpha^2\gamma^2(-1+(-3+4c+2c^2)\alpha\gamma+3(5+4c)\alpha^2\gamma^2+5\alpha^3\gamma^3)$$
$$-2a^5(3+3(1-8c)\alpha\gamma+6(-19+3c+4c^2)\alpha^2\gamma^2+2(51+60c+2c^2)\alpha^3\gamma^3+3(25+4c)\alpha^4\gamma^4+3\alpha^5\gamma^5)$$
$$+a^8(c^2+6c(-1+\alpha\gamma)+6(-1+\alpha\gamma)^2)-3a^7(2c^2\alpha\gamma+c(-3-2\alpha\gamma+9\alpha^2\gamma^2)+2(3-7\alpha\gamma+\alpha^2\gamma^2$$
$$+3\alpha^3\gamma^3))+6a^6(3-6\alpha\gamma-14\alpha^2\gamma^2+18\alpha^3\gamma^3+3\alpha^4\gamma^4+2c^2\alpha\gamma(1+\alpha\gamma)+c(-1-9\alpha\gamma+15\alpha^2\gamma^2$$
$$+7\alpha^3\gamma^3))+12a^4\alpha\gamma(1-16\alpha\gamma+6\alpha^2\gamma^2+40\alpha^3\gamma^3+5\alpha^4\gamma^4+4c^2\alpha\gamma(1+\alpha\gamma)$$
$$+2c(-1-3\alpha\gamma+18\alpha^2\gamma^2+8\alpha^3\gamma^3)))\Big)\Big/\Big(6(-2+a)^2a\alpha^2(a+2\alpha\gamma-a\alpha\gamma)^6\Big).$$

APPENDIX D

$$C_1=-\frac{(-1+a)^2a^4c^2(-6+ac)}{48(-2+a)^2\alpha^2},\ C_2=\frac{(-1+a)^2a^2(-ac(-4+ac))^{3/2}}{16(-2+a)^2\alpha^2},$$
$$C_3=\frac{(-1+a)^2a^3c(-4+ac)(-2+ac)}{16(-2+a)^2\alpha^2},\ C_4=-\frac{(-1+a)^2a^2(-ac(-4+ac))^{3/2}}{48(-2+a)^2\alpha^2},$$
$$C_5=\frac{(-1+a)a^2c(-4+ac)}{8(-2+a)\alpha},\ C_6=\frac{(-1+a)a\sqrt{-ac(-4+ac)}(-2+ac)}{4(-2+a)\alpha},$$
$$C_7=-\frac{(-1+a)a^2c(-4+ac)}{8(-2+a)\alpha},$$
$$D_1=-\Bigg(\Big((-1+a)^2a^3c(-96+48(3+a)c-2(32+36a+9a^2)c^2+(12+36a+12a^2+11a^3)c^3$$
$$-6a(2-a+2a^2)c^4+2a^3c^5)\Big)\Big/\Big(48(-2+a)^2\sqrt{-ac(-4+ac)}\alpha^2\Big)\Bigg),$$

$$D_2 = -\frac{1}{16(-2+a)^2\alpha^2}(-1+a)^2a^2c\Big(4(4-6c+c^2)+a^3c^2(11-12c+2c^2)$$
$$+2a^2c(-6-2c+5c^2)-4a(-4+4c-7c^2+3c^3)\Big),$$
$$D_3 = -\frac{1}{16(-2+a)^2\alpha^2}(-1+a)^2ac\sqrt{-ac(-4+ac)}\Big(-4(2+c)+a(8+28c-12c^2)$$
$$+a^3c(11-12c+2c^2)+2a^2(-3-10c+7c^2)\Big),$$
$$D_4 = \frac{(-1+a)^2ac^2(-4+ac)\left(-12-12a(-3+c)+18a^2(-2+c)+a^3\left(11-12c+2c^2\right)\right)}{48(-2+a)^2\alpha^2},$$
$$D_5 = \frac{(-1+a)a^2c\left(-16+4(4+3a)c-\left(4+4a+5a^2\right)c^2+2a^2c^3\right)}{8(-2+a)\sqrt{-ac(-4+ac)}\alpha},$$
$$D_6 = \frac{(-1+a)ac\left(4+6a-4c+a^2c(-5+2c)\right)}{4(-2+a)\alpha},\ D_7 = \frac{(-1+a)c\sqrt{-ac(-4+ac)}\left(-4+4a+a^2(-5+2c)\right)}{8(-2+a)\alpha}.$$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A.J. Lotka, Elements of physical biology, Williams & Wilkins, (1925).

[2] V. Volterra, Variazioni e fluttuazioni del numero d'individui in specie animali conviventi, Memorie della Reale Accademia Nazionale dei Lincei, 6 (1926), 31–113.

[3] S. Pal, N. Pal, S. Samanta, et al. Effect of hunting cooperation and fear in a predator-prey model, Ecol. Complex. 39 (2019), 100770. https://doi.org/10.1016/j.ecocom.2019.100770.

[4] S. Kumar, H. Kharbanda, Chaotic behavior of predator-prey model with group defense and non-linear harvesting in prey, Chaos Solitons Fractals. 119 (2019), 19–28. https://doi.org/10.1016/j.chaos.2018.12.011.

[5] Y. Zhou, W. Sun, Y. Song, et al. Hopf bifurcation analysis of a predator–prey model with Holling-II type functional response and a prey refuge, Nonlinear Dyn. 97 (2019), 1439–1450. https://doi.org/10.1007/s11071-019-05063-w.

[6] S. Akhtar, R. Ahmed, M. Batool, et al. Stability, bifurcation and chaos control of a discretized Leslie prey-predator model, Chaos Solitons Fractals. 152 (2021), 111345. https://doi.org/10.1016/j.chaos.2021.111345.

[7] H. Deng, F. Chen, Z. Zhu, et al. Dynamic behaviors of Lotka–Volterra predator–prey model incorporating predator cannibalism, Adv. Differ. Equ. 2019 (2019), 359. https://doi.org/10.1186/s13662-019-2289-8.

[8] P.H. Leslie, Some further notes on the use of matrices in population mathematics, Biometrika. 35 (1948), 213-245. https://doi.org/10.2307/2332342.

[9] P.H. Leslie, A stochastic model for studying the properties of certain biological systems by numerical methods, Biometrika, 45 (1958), 16-31.

[10] W.C. Allee, Animal aggregations: A study in general sociology, University of Chicago Press, Chicago, (1931).

[11] S. Vinoth, R. Sivasamy, K. Sathiyanathan, et al. The dynamics of a Leslie type predator–prey model with fear and Allee effect, Adv. Differ. Equ. 2021 (2021), 338. https://doi.org/10.1186/s13662-021-03490-x.

[12] Y. Du, B. Niu, J. Wei, Dynamics in a predator–prey model with cooperative hunting and Allee effect, Mathematics. 9 (2021), 3193. https://doi.org/10.3390/math9243193.

[13] H. Molla, S. Sarwardi, S.R. Smith, et al. Dynamics of adding variable prey refuge and an Allee effect to a predator–prey model, Alexandria Eng. J. 61 (2022), 4175–4188. https://doi.org/10.1016/j.aej.2021.09.039.

[14] Z. Shang, Y. Qiao, Bifurcation analysis of a Leslie-type predator–prey system with simplified Holling type IV functional response and strong Allee effect on prey, Nonlinear Anal.: Real World Appl. 64 (2022), 103453. https://doi.org/10.1016/j.nonrwa.2021.103453.

[15] K. Fang, Z. Zhu, F. Chen, et al. Qualitative and bifurcation analysis in a Leslie-Gower model with Allee effect, Qual. Theory Dyn. Syst. 21 (2022), 86. https://doi.org/10.1007/s12346-022-00591-0.

[16] L. Lai, Z. Zhu, F. Chen, Stability and bifurcation in a predator–prey model with the additive Allee effect and the fear effect, Mathematics. 8 (2020), 1280. https://doi.org/10.3390/math8081280.

[17] R. Ahmed, Complex dynamics of a fractional-order predator-prey interaction with harvesting, Open J. Discrete Appl. Math. 3 (2020), 24–32. https://doi.org/10.30538/psrp-odam2020.0040.

[18] R. Ahmed, M.S. Yazdani, Complex dynamics of a discrete-time model with prey refuge and Holling type-II functional response, J. Math. Comput. Sci. 12 (2022), 113. https://doi.org/10.28919/jmcs/7205.

[19] R. Ahmed, A. Ahmad, N. Ali, Stability analysis and Neimark-Sacker bifurcation of a nonstandard finite difference scheme for Lotka-Volterra prey-predator model, Commun. Math. Biol. Neurosci. 2022 (2022), 61. https://doi.org/10.28919/cmbn/7534.

[20] Y. Chou, Y. Chow, X. Hu, et al. A Ricker–type predator–prey system with hunting cooperation in discrete time, Mathematics and Computers in Simulation. 190 (2021), 570–586. https://doi.org/10.1016/j.matcom.2021.06.003.

[21] Q. Shu, J. Xie, Stability and bifurcation analysis of discrete predator–prey model with nonlinear prey harvesting and prey refuge, Math. Methods Appl. Sci. 45 (2021), 3589–3604. https://doi.org/10.1002/mma.8005.

[22] L. Fei, X. Chen, B. Han, Bifurcation analysis and hybrid control of a discrete-time predator–prey model, J. Differ. Equ. Appl. 27 (2021), 102–117. https://doi.org/10.1080/10236198.2021.1876038.

[23] S. Isik, F. Kangalgil, On the analysis of stability, bifurcation, and chaos control of discrete-time predator-prey model with Allee effect on predator, Hacettepe J. Math. Stat. 51 (2022), 404-420. https://doi.org/10.15672/hujms.728889.

[24] A. Tassaddiq, M.S. Shabbir, Q. Din, et al. Discretization, bifurcation, and control for a class of predator-prey interactions, Fractal Fract. 6 (2022), 31. https://doi.org/10.3390/fractalfract6010031.

[25] Q. Zhou, F. Chen, S. Lin, Complex dynamics analysis of a discrete amensalism system with a cover for the first species, Axioms. 11 (2022), 365. https://doi.org/10.3390/axioms11080365.

[26] D. Mukherjee, Global stability and bifurcation analysis in a discrete-time two prey-one predator model with help, Int. J. Model. Simul. (2022). https://doi.org/10.1080/02286203.2022.2121676.

[27] S. Lin, F. Chen, Z. Li, L. Chen, Complex dynamic behaviors of a modified discrete Leslie–Gower predator–prey system with fear effect on prey species, Axioms. 11 (2022), 520. https://doi.org/10.3390/axioms11100520.

[28] A.C.J. Luo, Regularity and complexity in dynamical systems, Springer, New York, 2012. https://doi.org/10.1007/978-1-4614-1524-4.