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OPTIMAL CONTROL AND GLOBAL STABILITY OF THE SEIQRS EPIDEMIC MODEL

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Abstract. Medical treatment, vaccination, and quarantine are the most efficacious controls in preventing the spread of contagious epidemics such as COVID-19. In this paper, we demonstrate the global stability of the endemic and disease-free equilibrium by using the Lyapunov function. Moreover, we apply the three measures to minimize the density of infected people and also reduce the cost of controls. Furthermore, we use the Pontryagin Minimum Principle in order to characterize the optimal controls. Finally, we execute some numerical simulations to approve and verify our theoretical results using the fourth order Runge-Kutta approximation through Matlab.

Keywords: epidemiological models; optimal control; global stability; Covid-19; numerical simulation.

2020 AMS Subject Classification: 92D30.

1. Introduction

Epidemiology is the investigation of the occurrence of diseases in various groups of people and their reasons. Epidemiological findings are used to plan and measure disease prevention programs and also to guide the management of patients in whom disease has already appeared [1]. Like clinical results and pathology, the epidemiology of a disease is an important part

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of its basic description. It is important to note that this topic has its own data collection and interpretation techniques. Needless to say, one of the most dangerous and known contagious diseases of this decade is COVID-19. The latter is now a scientific concern in various fields [2].

A novel coronavirus called severe acute lung syndrome coronavirus 2 is the cause of Coronavirus 2019 (Covid-19), a lethal condition that worsens over time (SARS-CoV-2). We direct the reader to [3, 4] and the references therein for more information. Since the first case emerged in Wuhan, China, towards the end of 2019, the disease has spread quickly from one nation to another, causing significant financial losses and fatalities all over the world. A World Health Organization (WHO) statement from June 7, 2020 (see for instance [5]) states that SARS-CoV-2 has killed 397388 people and infected 6799713 people globally. Several mathematical techniques have been proposed and put into practice to understand the evolution of SARS-CoV-2 at this time. In particular, most of these mathematical models have described the spread of the virus in the human community.

As we all know, the emergence of contagious diseases has resulted in significant health care and disease control costs, and this has united the whole world to fight the spread of these risks [6]. In fact, researchers have employed diverse computer models to examine the rise and control inflationary diseases. On the other hand, classical infectious disease systems include three systems : the susceptible-infected (SI) system, the susceptible-infected-susceptible (SIS) system, and the susceptible-infected-recovered (SIR) system.

In the current situation, we are facing major health crises in both industrialized and less urbanized countries, and one of the main disciplinary domains covered by mathematical modeling is the health system. In this sense, Yves Cherruault [7] has described the process of mathematical modeling as a way of expressing a real-world issue in the context of abstract symbolism. In addition, the abstract expression that uses a mathematical formulation is referred to as the associated maternal model of the starting problem.

Let us emphasize that mathematical modeling of critical characteristics of outbreaks, such as the spread of infection, has been of great importance in understanding the dynamics of outbreaks. Furthermore, it should be noted that mathematical modeling of infectious diseases has three main objectives :

1) to better understand the mechanisms by which the disease spreads;

2) to predict the evolution of this epidemic;

3) to comprehend how to control the spread of this epidemic.

Our study design was based on a compartmental model divided into five classes S, E, I, Q and R (see for instance [8, 9]), including the consideration of the case of losing the immunity from a probable return of the recovered class to the susceptible class (see *e.g.* [10,11,12]).

A *SEIQRS* is a model in which the individual who has recovered has temporary immunity (the person may lose immunity and become susceptible again). For example, infections like Diphteria, Influenza, and COVID-19 have a latent period and temporary immunity.

Our work was motivated by the idea of complicating the chosen model. Indeed, this model represents a rotating system that considers the quarantine as a control parameter. The main objective of this work is to study the global stability of the proposed system using the Lyapunov function, as well as to identify the controls to minimize the cost of vaccination and treatment. Despite the fact that in this paper, vaccination was targeted not only on susceptible individuals but also on those who were most likely to become infected, in order to analyze the interaction of each class with its control It is important to mention that the resolution of such a problem requires different precautions from one compartmental class to another.

The paper is organized as follows:
In section 2, we present our mathematical model with a description depicting the transmission of the novel Coronavirus. The global stability and the basic reproduction number are proven and calculated respectively in section 3. Section 4 is devoted to explain our model with controls. In section 5, we demonstrate some basic properties of the *SEIQRS* system. In section 6, we study the existence and characterization of our optimal control using the Pontryagin Minimum Principle. Section 7 is assigned to some numerical simulation through Matlab. Finally, section 8 is reserved to conclude our work.

2. Description of Our SEIQRS Model

The *SEIR* model is very useful in modeling some diseases such as Covid-19. In fact, this system takes into consideration an important class of this type of epidemics which is the exposed population.

Our study is based on the *SEIR* and the *SIQRS* systems modeling the Corona-virus epidemic and other various epidemics respectively.

Inspired by the works of R. Engbert and L. Zhang [13, 14], we build a more complicated *SEIQRS* system.

As in the *SEIR* model, the idea of the first compartment of the system is that the number of healthy individuals who are also susceptible decreases over time in proportion to the number of contacts with the infected. After the infection, a healthy individual becomes a person carrying this disease, moreover, this compartment may have new proportions coming from other classes (the recovered ones) because of loss of immunity. The second compartment introduces a delay in the transition from the "contact" state to the infected state. This occurs after the incubation period of the disease. The third compartment presents the transition from the "contact" state to the "infected" state. The fourth compartment describes the transition from the "infected" state to the "sanitary isolation". Finally, the fifth compartment shows that the number of recovered people is relatively dependent on the recovery of infected people and those in quarantine. In an additional passage to this compartment, the latter faces a proportional decrease due to the loss of immunity. We note that, in each state, an individual may die, which accounts for the coefficient of the mortality rate d in each equation.

The diagram given in figure 1 presents a better description of our system.

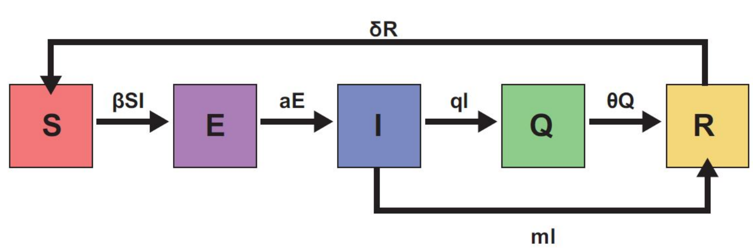

FIGURE 1. Flowchart of the SEIQRS model.

Throughout this paper, we use the following notations:

TABLE 1. System settings.

Parameters	Meanings
d	Coefficient of the natural mortality rate.
b	Coefficient of the birth rate.
β	Infection transmission rate.
a	The inverse of the average incubation period of the disease.
m	Coefficient of the recovery intensity of infected individuals.
q	Quarantined rate(Transmission rate from I to Q).
δ	Loss rate of immunity (Transmission rate from R to S).
θ	Cure rate by quarantine.

Now, we set the *SEIQRS* model as follows:

$$
(1) \qquad \begin{cases}
\dot{S}(t) = bN(t) - \beta S(t)\frac{I(t)}{N(t)} - dS(t) + \delta R(t), \\
\dot{E}(t) = \beta S(t)\frac{I(t)}{N(t)} - (a+d)E(t), \\
\dot{I}(t) = aE(t) - (q+m+d)I(t), \\
\dot{Q}(t) = qI(t) - \theta Q(t) - dQ(t), \\
\dot{R}(t) = mI(t) + \theta Q(t) - (\delta + d)R(t),
\end{cases}
$$

where all the parameters and their meanings are included in Table 1, and $N(t)$ is defined in function the states variables $S(t), E(t), I(t), Q(t)$ and $R(t)$ by

$$N(t) = S(t) + E(t) + I(t) + Q(t) + R(t), \qquad (\forall\, t \geqslant 0).$$

Moreover, $N(t)$ describes the size of the population at the moment t.

Firstly, we divide the model above by $N(t)$ and carry out some computations on the $SEIQRS$ model in order to have more relevant characteristics, see [15]. We take into account the fact that the population size is not constant and that it varies according to the following rule:

$$\dot{N}(t) = \dot{S}(t) + \dot{E}(t) + \dot{I}(t) + \dot{Q}(t) + \dot{R}(t) = (b-d)N(t).$$

Hence, our final system is

$$\begin{cases} \dot{S}(t) = b - \beta S(t)I(t) - bS(t) + \delta R(t), \\ \\ \dot{E}(t) = \beta S(t)I(t) - (a+b)E(t), \\ \\ \dot{I}(t) = aE(t) - (q+m+b)I(t), \\ \\ \dot{Q}(t) = qI(t) - \theta Q(t) - bQ(t), \\ \\ \dot{R}(t) = mI(t) + \theta Q(t) - (\delta+b)R(t). \end{cases} \tag{2}$$

Hereafter, we denote by $\mathcal{X}$ the set of the state variables:

$$\mathcal{X} := \{(S,E,I,Q,R) : \textit{the condition C1 holds}\},$$

where

$$C1 : \begin{cases} S(0), E(0), I(0), Q(0), R(0) \geq 0 \\ \\ S(t) + E(t) + I(t) + Q(t) + R(t) = 1 \end{cases}, \quad (\forall\, t \geqslant 0), \tag{3}$$

3. Stability and Basic Reproduction Number

In order to calculate the number of reproduction R_0, we use the next-generation matrix method illustrated by P. V. Driessche and J. Watmough [16,17].

Hereafter, we give the expression of this number as follows:

$$R_0 = \frac{a\beta}{(a+b)(q+m+b)}.$$

In our work, we are interested in the equilibrium points. In fact, we use the classical method in order to obtain the number of reproduction with the equilibrium point.

We can easily see that the point $P^0 = (1,0,0,0,0)$ is a disease-free equilibrium of the problem (2), and to find the endemic equilibrium $P^* = (S^*,E^*,I^*,Q^*,R^*)$, we set the right sides of the model (2) to zero gives. Hence, we get

$$E^* = \frac{\beta S^* I^*}{a+b}, I^* = \frac{aE^*}{q+m+b}, Q^* = \frac{q}{b+\theta} I^*, R^* = \frac{\theta + m(b+\theta)}{(b+\delta)(b+\theta)} I^*.$$

From the expressions of E^* and I^*, we notice that

$$S^* = \frac{(a+b)(q+m+b)}{a\beta}.$$

Now, we take the equations above into consideration in addition, we extend the calculations to get

$$I^* = \Gamma\left(\frac{a\beta}{(a+b)(q+m+b)} - 1\right) = \Gamma\left(R_0 - 1\right),$$

where

$$\Gamma = \frac{b(a+b)(q+m+b)(\theta+b)(\delta+b)}{\beta[(a+b)(b+m+q)(b+\theta)(\delta+b) - a\delta(\theta+m(b+\theta))]}.$$

Finally, the system (2) has an endemic equilibrium in the form of $P^* = (S^*,E^*,I^*,Q^*,R^*)$, where

$$S^* = \frac{(a+b)(q+m+b)}{a\beta}, E^* = \frac{q+m+b}{a} I^*, I^* = \Gamma\left(R_0 - 1\right), Q^* = \frac{q}{\theta+b} I^*, R^* = \frac{\theta + m(\theta+b)}{(b+\delta)(\theta+b)} I^*.$$

Global stability. When studying the stability of compartmental systems modeling epidemiology it is always assumed that immunity is durable. For the above reason, we take $\delta = 0$. Inspired by the work of A. Korobeinikov in [18].

Theorem 1. *The disease free equilibrium is globally asymptotically stable if $R_0 \leq 1$.*

Proof. By constructing the Lyapunov function as follows

$$V = aE + (a+b)I,$$

the derivative of this function with respect to time yields the following expression

$$V' = a\beta SI - (a+b)(q+m+b)I.$$

Moreover, a factorization by $a\beta I$ gives

$$V^{'} = a\beta I\left(S - \frac{(a+b)(q+m+b)}{a\beta}\right) = a\beta I\left(S - \frac{1}{R_0}\right).$$

Then, if $R_0 \leq 1$ from which we get the result. $\square$

Theorem 2. *Under the assumption* $\delta = 0$*. The endemic equilibrium is globally asymptotically stable if* $R_0 > 1$.

Proof. By using the assumption ($\delta = 0$), we reduce the main system (2) to an equivalent system containing only the first three equations

$$\begin{cases} \dot{S}(t) = b - \beta S(t)I(t) - bS(t), \\ \\ \dot{E}(t) = \beta S(t)I(t) - (a+b)E(t), \\ \\ \dot{I}(t) = aE(t) - (q+m+b)I(t). \end{cases} \tag{4}$$

We define a Lyapunov functional for this model in equilibrium as follows:

$$V = \left(S - S^{*}ln\left(\frac{S}{S^{*}}\right)\right) + \left(E - E^{*}ln\left(\frac{E}{E^{*}}\right)\right) + \frac{a+b}{a}\left(I - I^{*}ln\left(\frac{I}{I^{*}}\right)\right).$$

By computation of the time derivative of V, we get

$$V^{'} = b - bS - b\frac{S^{*}}{S} + bS^{*} - \beta\frac{SIE^{*}}{E} + (a+b)E^{*} - (a+b)\frac{EI^{*}}{I} + \frac{(a+b)(q+m+b)}{a}I^{*}.$$

In the equilibrium state, we have $b = \beta S^{*}I^{*} + bS^{*}$. By replacing b with its value and doing some computations, we find the following formula

$$V^{'} = bS^{*}\left(2 - \frac{S^{*}}{S} - \frac{S}{S^{*}}\right) + \beta S^{*}I^{*}\left(3 - \frac{S^{*}}{S} - \frac{SIE^{*}}{S^{*}I^{*}E} - \frac{I^{*}E}{IE^{*}}\right).$$

By means of the arithmetic–geometric inequality. Hence, we obtain $V' \leq 0$ which completes the proof. $\square$

4. Model With Control

In February 2020, the World Health Organization (WHO) stated that we cannot expect a COVID-19 vaccine for another 18 months; however, one year after the outbreak of the COVID-19 pandemic, China announced that one of the vaccines discovered against COVID-19 was 79.43% effective. Countries have begun to try to obtain other vaccines, such as Pfizer-BioNTech, which is 95% effective, and Moderna, which is 94.1% effective, and have started to vaccinate their citizens. As in Morocco, countries have begun vaccinating people at high risk of getting infected, such as doctors and security personnel. Subsequently, the circle of vaccine recipients has expanded. Therefore, in order to control the system, we propose to insert two vaccination controls ; one for the exposed population and the other for the susceptible population. In addition, we add a control that can be interpreted as a quarantine and health surveillance. For this reason, we take the deterministic variable q given in our main model describing ”the rate of being in quarantine” as a control, and the last one that we will insert is a treatment control.
Thus, our system with controls becomes:

$$
(5) \qquad \begin{cases}
\dot{S}(t) = b - \beta S(t)I(t) - (\alpha_1(t) + b)S(t) + \delta R(t), \\
\dot{E}(t) = \beta S(t)I(t) - (\alpha_2(t) + a + b)E(t), \\
\dot{I}(t) = aE(t) - (\mu(t) + q(t) + m + b)I(t), \\
\dot{Q}(t) = q(t)I(t) - (\theta + b)Q(t), \\
\dot{R}(t) = \alpha_1(t)S(t) + \alpha_2(t)E(t) + (\mu(t) + m)I(t) + \theta Q(t) - (\delta + b)R(t),
\end{cases}
$$

where all the parameters and their meanings are included in Table 2.
This problem with control is provided under the same conditions ($C1{:}3$) as in the system (2).

TABLE 2. Control settings.

Control	Meaning
α_1	Vaccinated rate(Transmission rate from S to R by vaccination).
α_2	Vaccinated rate(Transmission rate from E to R by vaccination).
μ	Cure rate only by treatment (Transmission rate from I to R).
q	Quarantined rate(Transmission rate from I to Q).

To end this section, we define the set of admissible controls as follows:

$$\Omega = \left\{ u(t) = (\alpha_1, \alpha_2, \mu, q) : 0 \leq \alpha_1(t), \alpha_2(t), \mu(t), q(t) \leq 1 \text{ for } t \in [0, t_f] \right\}. \tag{6}$$

5. Basic Properties of The SEIQRS Model With Controls

In this section, we study the existence of a global solution of system (5) by verifying some fundamentals properties. We start with a classical theorem which shows the positivity of solutions.

Theorem 3 (Positivity of solutions)**.** *Let $X = (S, E, I, Q, R)$ solution to the system (5) that veries the condition (C1:3). If $S(t)$ and $E(t)$ not equal to zero, then for all $t \geqslant 0$, the solutions $S(t)$, $E(t)$, $I(t)$, $Q(t)$ and $R(t)$ are non-negative.*

Proof. For some biological reason, we consider that the susceptible population $S(t)$ and the exposed population $E(t)$ are not equal to zero.
It follows from the first equation of the system (5) that

$$\begin{aligned} \dot{S}(t) &= b - \beta S(t)I(t) - (\alpha_1(t) + b)S(t) + \delta R(t) \\ &\geq -\beta S(t)I(t) - (\alpha_1(t) + b)S(t) + \delta R(t) \\ &\geq -S(t)\left(\beta I(t) - \delta \frac{R(t)}{S(t)} + \alpha_1(t) + b\right). \end{aligned}$$

We set $\psi(t) = \beta I(t) - \delta \frac{R(t)}{S(t)} + \alpha_1(t) + b$, then we obtain

$$\dot{S}(t) + \psi(t)S(t) \geq 0.$$

Using the Grönwall's lemma, we get

$$S(t) \geq S(0) exp \left(-\int_0^t \psi(s) ds \right).$$

Hence the desired result follows immediately. In the same way, we show that $E(t) \geq 0$.
It follows from the third equation of the system (5) that

$$\dot{I}(t) = aE(t) - (\mu(t) + q(t) + m + b)I(t).$$

Using the fact that $E(t) \geq 0$, we find the following inequality

$$\dot{I}(t) \geq -(\mu(t) + q(t) + m + b)I(t).$$

Now, we set $\psi(t) = \mu(t) + q(t) + m + b$. Therefore,

$$\dot{I}(t) + \psi(t)I(t) \geq 0.$$

Again by the Grönwall's lemma, we find

$$I(t) \geq I(0)exp\left(-\int_0^t \psi(s)ds\right).$$

Similarly, we show that $Q(t) \geq 0$ and $R(t) \geq 0$. □

Concerning the boundedness of the solutions, we can easily see that the solutions of our system are bounded, thanks to the theorem of positivity of solutions and the second equation of the condition ($C1$:3).

Existence of solution. In this section, we show the existence of the solution of the *SEIQRS* system [19].

Firstly, we write the system (5) in the form

$$f(X) = MX + h(X),$$

where X, $f(X)$ and $h(X)$ are column vectors given as follows

$$X = \begin{pmatrix} S \\ E \\ I \\ Q \\ R \end{pmatrix}, f(X) = \begin{pmatrix} \dot{S} \\ \dot{E} \\ \dot{I} \\ \dot{Q} \\ \dot{R} \end{pmatrix}, h(X) = \begin{pmatrix} b - \beta SI - \alpha_1 S \\ \beta SI - \alpha_2 E \\ -(\mu + q)I \\ qI \\ \alpha_1 S + \alpha_2 E + \mu I \end{pmatrix}$$

and

$$M = \begin{pmatrix} -b & 0 & 0 & 0 & \delta \\ 0 & -(a+b) & 0 & 0 & 0 \\ 0 & a & -(m+b) & 0 & 0 \\ 0 & 0 & -\theta & -b & 0 \\ 0 & 0 & m & \theta & -(\delta+b) \end{pmatrix}.$$

In order to show that the system has a solution, it is sufficient to verify that the function f is Lipschitz continuous. We have

$$||h(X_1)-h(X_2)||_1 = |-\beta(S_1I_1-S_2I_2)-\alpha_1(S_1-S_2)|+|\beta(S_1I_1-S_2I_2)-\alpha_2(E_1-E_2)| \\ +|(\mu+q)(I_1-I_2)|+|q(I_1-I_2)|+|\alpha_1(S_1-S_2)+\alpha_2(E_1-E_2)+\mu(I_1-I_2)|.$$

Furthermore, the triangular inequality yields

$$||h(X_1)-h(X_2)||_1 \leqslant 2\beta|S_1I_1-S_2I_2|+|\alpha_1||S_1-S_2|+|\alpha_2||E_1-E_2|+|\mu+q||I_1-I_2| \\ +|q||I_1-I_2|+|\alpha_1||S_1-S_2|+|\alpha_2||E_1-E_2|+|\mu||I_1-I_2|.$$

In addition, we know that

$$\begin{aligned} 2\beta|S_1I_1-S_2I_2| &= 2\beta|S_1I_1-S_1I_2+S_1I_2-S_2I_2| \\ &= 2\beta|S_1(I_1-I_2)+I_2(S_1-S_2)|, \end{aligned}$$

as well as, the state variables and the controls are increased by 1, then

$$\begin{aligned} ||h(X_1)-h(X_2)||_1 &\leqslant (2\beta+2)|S_1-S_2|+2|E_1-E_2|+(2\beta+4)|I_1-I_2| \\ &\leqslant (2\beta+4)(|S_1-S_2|+|E_1-E_2|+|I_1-I_2|). \end{aligned}$$

So, we get

$$||h(X_1)-h(X_2)||_1 \leqslant (2\beta+4)||X_1-X_2||_1.$$

Hence, $||f(X_1)-f(X_2)||_1 \leqslant k||X_1-X_2||_1$ with $k = max(2\beta+4, ||M||)$. Finally, the function f is Lipschitz and therefore the solution of system (5) exists.

6. THE OPTIMAL CONTROL PROBLEM

In the introduction of the *Optimal Control Systems* book [20], Naidu D. Subbaram gave a classification of the objective function that she called the performance index, which can take several forms depending on what we want to achieve with our control. The objective of this work is to minimize the density of people infected with the disease and to minimize the cost of treatment, quarantine, and vaccination campaigns. Thus, the objective function is expressed thematically as follows:

$$J(x(t),u(t)) = \int_0^{t_f} [L(x(t),u(t))]\,dt, \tag{7}$$

where $L(x(t),u(t)) = I(t) + \frac{1}{2}A\alpha_1^2(t) + \frac{1}{2}B\alpha_2^2(t) + \frac{1}{2}C\mu^2(t) + \frac{1}{2}Dq^2(t)$, and A,B,C and D are positives weighting factors.

We are interested in finding the optimal control $u^*(t)$ which will drive the *SEIQRS* system from the initial state to the final state, and at the same time will minimize the function J

$$J(\cdot,u^*(t)) = \min_{u(t)\in\Omega} J(\cdot,u(t)). \tag{8}$$

6.1. Existence of optimal control. We start this section with a theorem of the existence of the optimal solution.

Theorem 4. *There exists an optimal control $u^*(t) = (\alpha_1^*(t),\alpha_2^*(t),\mu^*(t),q^*(t))$ for the problem (5) which satisfies the equation (8).*

In order to prove this theorem, we recall a result given by Fleming and Rishel [21]. This result is about the existence of solutions for optimal control problems. see [Chapter 3, Theorem 4.1 and Corollary 4.1].

Theorem 5 (Existence of solution for the optimal control problem)**.** *Suppose that f and L are continuous such that :*

a) *$\mathcal{X}\times\Omega$ is not empty ;*

b) *Ω is closed ;*

c) *$\mathcal{X}$ is compact ;*

d) *Ω is convex, and $L(x,.)$ is convex on Ω ;*

e) *$L(x,.) \geqslant c_1||u||^n - c_2$, for $c_1 > 0$, $c_2 \geqslant 0$ and $n > 1$;*

f) *$||f(x,u)|| \leqslant c(1+||x||+||u||)$, for $c > 0$;*

g) *$||f(x_1,u) - f(x_2,u)|| \leqslant c||x_1 - x_2||(1+||u||)$, for $c > 0$.*

Then, there exists (x^*,u^*) minimizing J on $\mathcal{X}\times\Omega$.

Proof (Theorem 4). Firstly, we make sure that the functions f and L satisfy the conditions in theorem 5. We have L continuous function as the sum of the continuous functions. In section 5, we show that the function f is Lipschitz which implies that it is continuous.

By using a result given by Boyce and DiPrima, see [22, theorem 7.1.1], we obtain the condition (a). The two conditions (b) and (c) are verified thanks to the definitions of the domains $\mathcal{X}$ and Ω. Since $[0,1]$ is a convex domain, then Ω is a convex as the product of convex domains. Moreover, $L(x,.)$ is convex because it is quadratic, consequently (d).

In addition, we have

$$\begin{aligned} L(x,u) &= I(t) + \tfrac{1}{2}A\alpha_1^2(t) + \tfrac{1}{2}B\alpha_2^2(t) + \tfrac{1}{2}C\mu^2(t) + \tfrac{1}{2}Dq^2(t) \\ &\geqslant \frac{1}{2}A\alpha_1^2(t) + \frac{1}{2}B\alpha_2^2(t) + \frac{1}{2}C\mu^2(t) + \frac{1}{2}Dq^2(t) \\ &\geqslant c_1(\alpha_1^2(t) + \alpha_2^2(t) + \mu^2(t) + q^2(t)). \end{aligned}$$

So $L(x,u) \geqslant c_1||u||^2$ with $c_1 = \min(\frac{A}{2}, \frac{B}{2}, \frac{C}{2}, \frac{D}{2})$. Therefore, (e) is valid.

As $||f(x,u)|| \leqslant \sum_{i=1}^{5} |f_i(x,u)|$, for $i = 1$, we get

$$\begin{aligned} |f_1(x,u)| &= |b - \beta S(t)I(t) - (\alpha_1(t) + b)S(t) + \delta R(t)| \\ &\leqslant b + \beta|S(t)||I(t)| + |\alpha_1(t)||S(t)| + b|S(t)| + \delta|R(t)|. \end{aligned}$$

Besides, we know that $S(t)$ and $I(t)$ are bounded, then

$$|f_1(x,u)| \leqslant b + \beta|I(t)| + |\alpha_1(t)| + b|S(t)| + \delta|R(t)|.$$

Now, we set $\xi = \max(b, \beta, \delta)$, so we obtain

$$\begin{aligned} |f_1(x,u)| &\leqslant b + \xi(|S(t)| + |I(t)| + |R(t)|) + |\alpha_1(t)| \\ &\leqslant b + \xi||x|| + ||u|| \\ &\leqslant c_1(1 + ||x|| + ||u||), \end{aligned}$$

with $c_1 = \max(b, \xi, 1)$.

On the other hand, we apply the same procedure to the other f_i with $i = 2, ..., 5$. Thus, we get

$$||f(x,u)|| \leqslant c(1+||x||+||u||),$$

with $c = \max(c_1,c_2,c_3,c_4,c_5)$, therefore (f) is verified.

Furthermore, the last condition (g) is also valid because
$||f(x_1,u)-f(x_2,u)|| \leqslant \sum_{i=1}^{5}|f_i(x_1,u)-f_i(x_2,u)|$. For $i=1$, we have

$$\begin{aligned}|f_1(x_1,u)-f_1(x_2,u)| = \ & |\beta(S_2(t)I_2(t)-S_1(t)I_1(t))-\alpha_1(t)(S_1(t)-S_2(t)) \\ & -b(S_1(t)-S_2(t))+\delta(R_1(t)-R_2(t))|,\end{aligned}$$

then

$$\begin{aligned}|f_1(x_1,u)-f_1(x_2,u)| \leqslant \ & \beta|S_1(t)I_1(t)-S_2(t)I_2(t)|+b|S_1(t)-S_2(t)| \\ & +|\alpha_1(t)||S_1(t)-S_2(t)|+\delta|(R_1(t)-R_2(t))|.\end{aligned}$$

According to,

$$\begin{aligned}|S_1(t)I_1(t)-S_2(t)I_2(t)| & = |S_1(t)I_1(t)-S_1(t)I_2(t)+S_1(t)I_2(t)-S_2(t)I_2(t)| \\ & = |S_1(t)(I_1(t)-I_2(t))+I_2(t)(S_1(t)-S_2(t))|,\end{aligned}$$

knowing that $S(t)$ and $I(t)$ are bounded, then

$$|S_1(t)(I_1(t)-I_2(t))+I_2(t)(S_1(t)-S_2(t))| \leqslant |I_1(t)-I_2(t)|+|S_1(t)-S_2(t)|.$$

Thus,

$$\begin{aligned}|f_1(x_1,u)-f_1(x_2,u)| \leqslant \ & \beta(|I_1(t)-I_2(t)|+|S_1(t)-S_2(t)|)+b|S_1(t)-S_2(t)| \\ & +|\alpha_1(t)||S_1(t)-S_2(t)|+\delta|(R_1(t)-R_2(t))|.\end{aligned}$$

Now we put $\xi = \max(\beta+b,\delta)$, so we can see

$$\begin{aligned}|f_1(x_1,u)-f_1(x_2,u)| & \leqslant \xi||x_1(t)-x_2(t)||+||x_1(t)-x_2(t)||||u(t)|| \\ \\ & \leqslant c_1||x_1(t)-x_2(t)||(1+||u(t)||),\end{aligned}$$

with $c_1 = \max(\xi,1)$. Finally, we apply the same procedure to the other f_i with $i = 2,...,5$. Hence, we get

$$||f(x_1,u)-f(x_2,u)|| \leqslant c||x_1(t)-x_2(t)||(1+||u(t)||),$$

with $c = max(c_1,c_2,c_3,c_4,c_5)$, and therefore (g) is verified.

□

To get an optimal control for the previous system, we use the *Pontryagin Minimum Principle (PMP)* cited in [23].

Let us introduce the Hamiltonian of our model. We have

(9)

$$H(x(t),u(t),\lambda(t)) = L(x(t),u(t)) + \lambda_1(t)\dot{S}(t) + \lambda_2(t)\dot{E}(t) + \lambda_3(t)\dot{I}(t) + \lambda_4(t)\dot{Q}(t) + \lambda_5(t)\dot{R}(t),$$

where $\lambda(t) = (\lambda_1(t),\lambda_2(t),\lambda_3(t),\lambda_4(t))$ are the adjoint variables.

In the next stage, we resolve the optimal control, we begin this part by a fundamental theorem of *necessary optimality conditions*

Theorem 6. *Let $x^*(t)$ be the solution to the problem (5) in the optimal state, with the optimal control $u^*(t)$. Then, the adjoint problem is given by*

(10)

$$\begin{cases} \dot{\lambda_1}(t) = (b+\beta I(t)+\alpha_1(t))\lambda_1(t) - \beta I(t)\lambda_2(t) - \alpha_1(t)\lambda_5(t), \\ \\ \dot{\lambda_2}(t) = (a+b+\alpha_2(t))\lambda_2(t) - a\lambda_3(t) - \alpha_2(t)\lambda_5(t), \\ \\ \dot{\lambda_3}(t) = -1 + \beta S(t)(\lambda_1(t)-\lambda_2(t)) + (m+b+q(t)+\mu(t))\lambda_3(t) - q(t)\lambda_4(t) - (\mu(t)+m)\lambda_5(t), \\ \\ \dot{\lambda_4}(t) = (b+\theta)\lambda_4(t) - \theta\lambda_5(t), \\ \\ \dot{\lambda_5}(t) = -\delta\lambda_1(t) + (\delta+b)\lambda_5(t). \end{cases}$$

Furthermore, the optimal controls $\alpha_1^*(t),\alpha_2^*(t),\mu^*(t)$ and $q^*(t)$ are given by

(11)

$$\begin{cases} \alpha_1^*(t) = \min\left(\max\left(0, \frac{\lambda_1(t)-\lambda_5(t)}{A}S(t)\right),1\right), \\ \\ \alpha_2^*(t) = \min\left(\max\left(0, \frac{\lambda_2(t)-\lambda_5(t)}{B}E(t)\right),1\right), \\ \\ \mu^*(t) = \min\left(\max\left(0, \frac{\lambda_3(t)-\lambda_5(t)}{C}I(t)\right),1\right), \\ \\ q^*(t) = \min\left(\max\left(0, \frac{\lambda_3(t)-\lambda_4(t)}{D}I(t)\right),1\right). \end{cases}$$

Proof. Using the Pontryagin Minimum Principle [23] with the Hamiltonian function(9), we find the adjoint problem (10) by direct computations on the equations:

$$\dot{\lambda_1}(t) = -\frac{\partial H}{\partial S};\ \dot{\lambda_2}(t) = -\frac{\partial H}{\partial E};\ \dot{\lambda_3}(t) = -\frac{\partial H}{\partial I};\ \dot{\lambda_4}(t) = -\frac{\partial H}{\partial Q};\ \dot{\lambda_5}(t) = -\frac{\partial H}{\partial R}.$$

The controls are obtained by calculating the following computations

$$\begin{aligned}
\frac{\partial H}{\partial \alpha_1} &= A\alpha_1(t) - \lambda_1(t)S(t) + \lambda_5(t)S(t) = 0, & \frac{\partial H}{\partial \mu} &= C\mu(t) - \lambda_3(t)I(t) + \lambda_5(t)I(t) = 0,\\
&\Rightarrow \alpha_1(t) = \frac{(\lambda_1(t) - \lambda_5(t))}{A}S(t). & &\Rightarrow \mu(t) = \frac{(\lambda_3(t) - \lambda_5(t))}{C}I(t).\\
\frac{\partial H}{\partial \alpha_2} &= B\alpha_2(t) - \lambda_2(t)E(t) + \lambda_5(t)E(t) = 0, & \frac{\partial H}{\partial q} &= Dq(t) - \lambda_3(t)I(t) + \lambda_4(t)I(t) = 0,\\
&\Rightarrow \alpha_2(t) = \frac{(\lambda_2(t) - \lambda_5(t))}{B}E(t). & &\Rightarrow q(t) = \frac{(\lambda_3(t) - \lambda_4(t))}{D}I(t).
\end{aligned}$$

By taking in consideration the set of admissible controls, then we find the optimal controls given in the system (11). □

7. NUMERICAL SIMULATIONS

In this section, we use the fourth order Runge-Kutta algorithm in Matlab to illustrate the previous results numerically.

In order to do so, we use the following initial values : $S(0) = 0.7$, $E(0) = 0.1$, $I(0) = 0.1$, $Q(0) = 0$ and $R(0) = 0.1$. We need also to fix the values of weighting factors, thus we choose $A = 10$, $B = 10$, $C = 10$ and $D = 10$. And to deal with the system, we give values to the model parameters. Needless to say, these parameters change from one disease to another. Therefore, the parameters presented in the following table correspond to Covid-19 disease and are extracted from the articles [13, 24, 25, 26].

TABLE 3. Values of the parameters for problem.

Parameter	Meaning	value
b	Coefficient of the birth rate	0.1
β	Infection transmission rate	0.3
δ	Loss rate of immunity (Transmission rate from R to S)	0.1
a	The inverse of the average incubation period of the disease	1/5.2
m	Coefficient of the recovery intensity of infected individuals	0.05
θ	Cure rate by quarantine	0.1

To show the efficiency of the controls, we present a comparison between the state variables without controls and those with controls.

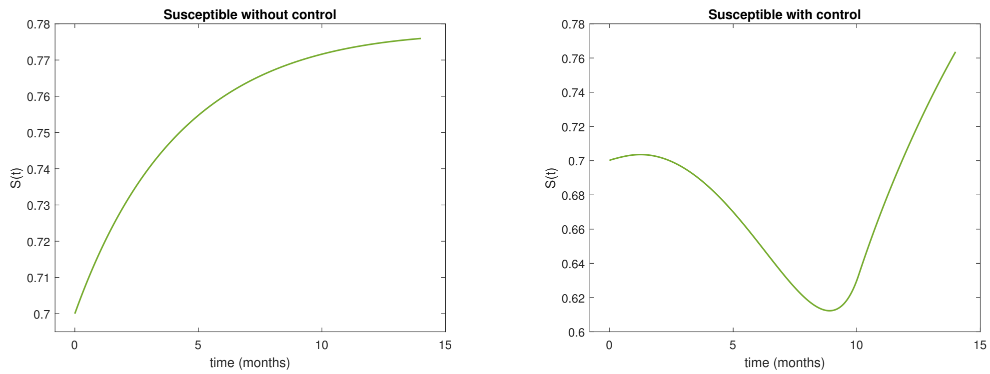

FIGURE 2. Susceptible population in time (without control vs with control).

In this illustration, we notice the effectiveness of the control , especially the control of the vaccination α_1 insert in the susceptible class, and we see it clearly in the first part of the second figure. In fact, instead of having an increase in the density of the susceptible people, we estimate a decrease in the level of this population, and this decrease is due to the fact that these people acquired immunity against the disease and they transferred directly to the class of the recovered. In addition, in the second part of the figure, we estimate an increase in the density of the susceptible people that we will explain later.

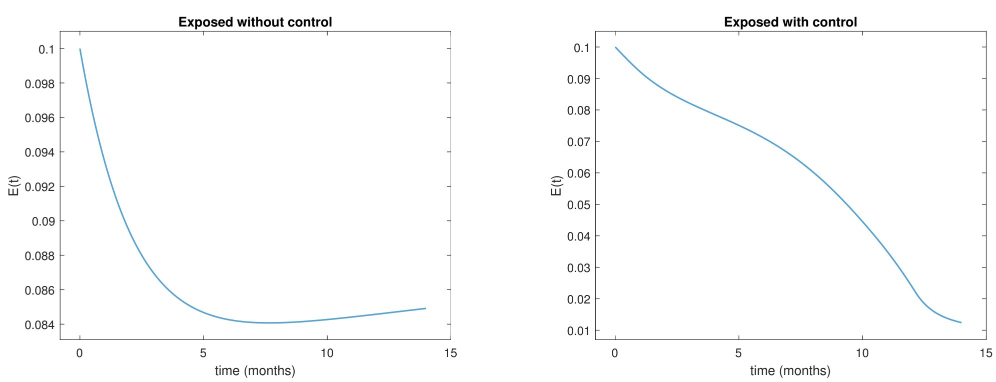

FIGURE 3. Exposed population in time (without control vs with control).

Once again, we see the effectiveness of the controls on the exposed compartment, which is the most susceptible to the disease. We notice that the control of vaccination α_2 has shown its efficiency in preventing the infection of the exposed people in a simple and fast way.

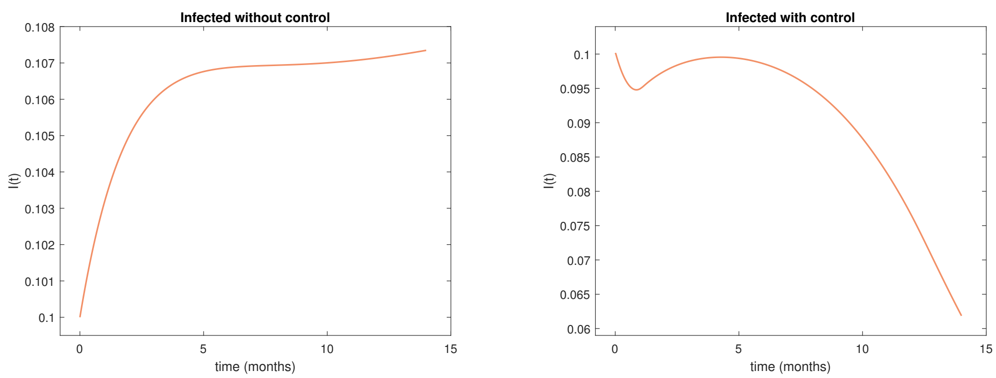

FIGURE 4. Infected population in time (without control vs with control).

Following a treatment applied to the infected, we also notice the effectiveness and success of the treatment over time, which can be seen by making a small comparison between the two figures. Let us see at the beginning of the virus propagation that the number of infected increases, while with control it remains almost stable at the same starting value, and in the last few days we can see more and more the efficiency of the control, since at time $t = 14$, the infected without control have a proportion of 0.107, while with control their proportion decreases to 0.062.

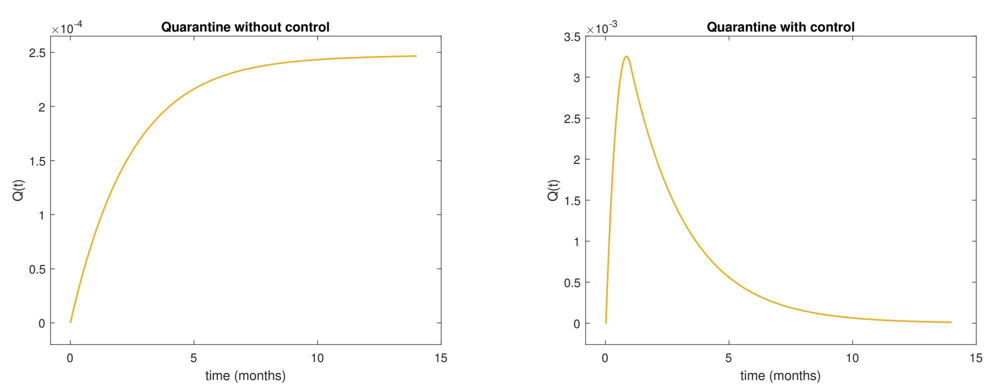

FIGURE 5. Quarantine population in time (without control vs with control).

Analyzing the category of quarantined people, we see that in the absence of the control, the curve that represents this class takes an ascending curve and it keeps its progression in time, while when we apply the control, we notice that a significant increase occurred in this category in a month and a half, after the curve of this class has a notable decrease, proving the effectiveness and efficiency of the control applied to people inside the quarantine.

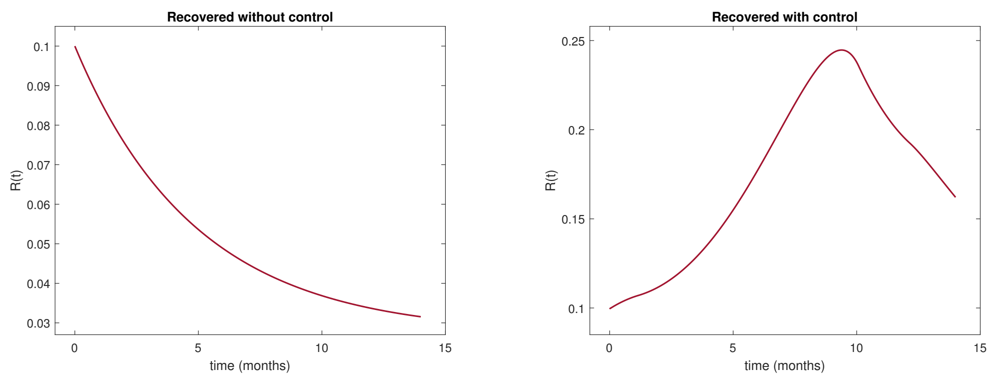

FIGURE 6. Recovered population in time (without control vs with control).

Moving on to the recovered compartment, we see that the density of recoveries increases with the control in a clear way until at a certain point we observe a decrease in recoveries due to loss of immunity. This decrease is related to the increase in the density of susceptible individuals, as shown in the figure below, and this shows the usefulness of the δ ratio in creating a rotating system.

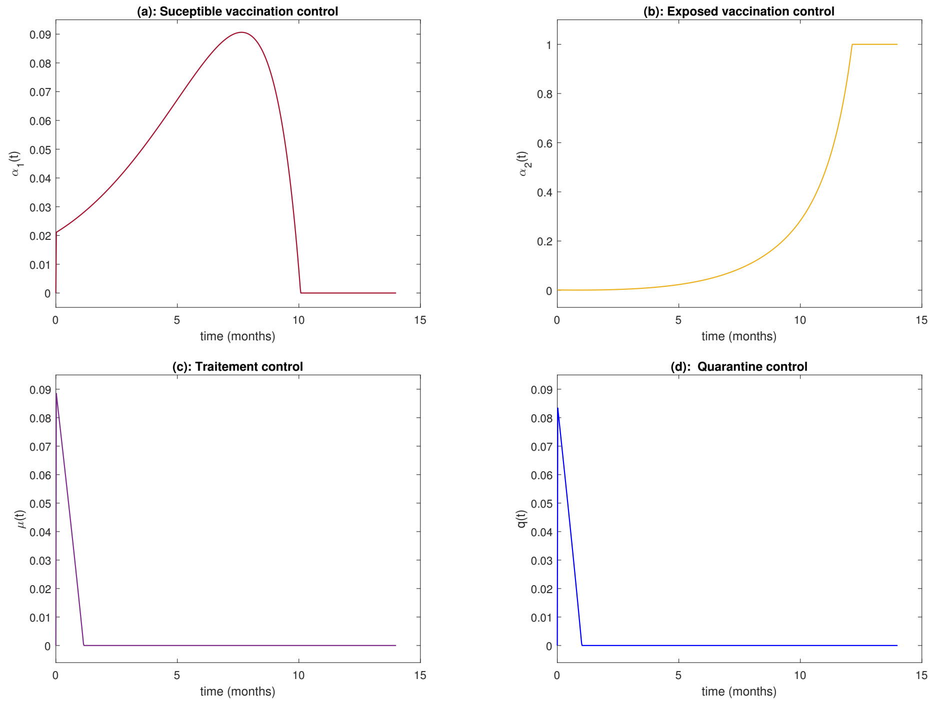

FIGURE 7. Vaccination control applied on the susceptible

Figure (a) shows the evolution of α_1 control during the vaccination period. We observe a remarkable increase in the density of people likely to be infected, but after several months and with an almost complete vaccination of the population, our control reaches zero as all the people will be vaccinated over time.

Figure (b) shows the evolution of the α_2 control exponentially during the period of vaccination and goes up to 1 and remains constant since the exposed people are the first to be infected due to the new mutations of COVID. Moreover, accompanied vaccination is important to secure them and stop the spread of the disease on a large scale.

Figure (c) shows the progress of the treatment applied to the infected people, an increase from the first days extracts from the exposed people who are infected and have to follow an obligatory treatment for their recovery, and after the first two controls that proved their effectiveness, the density of infected people decreases so that the treatment applied to these people tends towards zero with the days.

Figure (d) shows the evolution of the density of infected people who entered the quarantine, a curve similar to the one in figure (c), since after the treatment applied to the infected people, the next step is the quarantine as a precautionary measure recommended by the WHO and the national governments of each country.

8. Conclusion

In this paper, we introduced a *SEIQRS* mathematical model that describes the spread of the COVID-19 virus. We divided the population denoted by N into five compartments ; the susceptible population S, the exposed E, the infected I, the quarantined Q, and the recovered R. In fact, we analyzed a mathematical model and the overall stability of the free COVID-19 equilibrium and the endemic equilibrium were achieved. In addition, in order to reduce the density of infected, we introduced four controls that, respectively, represent two vaccination controls enforced on the susceptible and exposed population and one control interpreted as the quarantine plus the last one given as the treatment control. We also studied the optimal control, which represents the scale of the intervention with treatment means or education campaigns, and found that if preventive and proactive measures are implemented, such as education and quarantine campaigns, worldwide, the spread of the COVID-19 epidemic will be decreased. Thus, the density of people infected with the virus and the number of deaths will be reduced. Furthermore, we applied the findings of the control model theory and succeeded in getting the characterizations of the optimal controls. Finally, the numerical simulations of the obtained results showed the efficiency of the suggested control strategies.

Conflict of Interests

The authors declare that there is no conflict of interests.

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