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DYNAMIC MODEL OF SMOKERS AND ITS SENSITIVITY ANALYSIS

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Abstract. We study a smoking dynamical model with two types of smokers: beginners and heavy smokers. The qualitative behavior of the model, such as the stability of the equilibrium points and the basic reproduction number, is investigated. We show some simulations to validate the analytical findings, such as solution dynamics at different time scales and phase portraits of solutions with varying initial conditions. We also present a normalized sensitivity analysis of the basic reproduction number to discover which parameter has the most impact on smoking transmission, and perform a time-dependent sensitivity analysis of parameters to examine their impact on population dynamics.

Keywords: smoking model; two smoker subclasses; elasticity sensitivity index; time-dependent sensitivity.

2020 AMS Subject Classification: 37N25, 37C75, 93B35.

1. INTRODUCTION

Smoking tobacco increases the chance of several long-term health conditions, and it is the most common preventable disease, accounting for about 19% of adult deaths in the UK [1, 2]. More than 16 million persons in the USA suffer illnesses as a result of tobacco use, and 527,736 of those fatalities are preventable, or 17.9% of all deaths each year [3]. The cost of treating illnesses brought on by smoking is believed to be $467 billion globally, with Europe and North

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America bearing the heaviest financial burden [4]. More than seven million people die each year as a direct result of smoking; six million more people die as a result of smoking; and approximately 900,000 nonsmokers are harmed by smokers, also known as passive smokers [5]. Researchers from a variety of fields are studying the dynamics of smoking, particularly with the help of mathematical models.

In order to characterize the dynamics of drug use among teenagers, specifically tobacco use, authors in [6] presented a general epidemiological model, then they constructed specific models by taking other factors into account that have been identified to have an impact on the rising trend of tobacco use. Authors in [7] offered a thorough mathematical analysis for evaluating the dynamics of smoking and its effects on community public health. Authors in [8] introduced a novel model for quitting smoking in which the interaction term is the square root of current and potential smokers. To investigate how media campaigns affect smoking cessation, authors in [9] examined a nonlinear mathematical model, with the focus of the analysis being on backward bifurcation. Authors in [10] suggested a mathematical model to investigate the dynamics of smoking habit under the influence of educational initiatives as well as human willpower to give up smoking. Authors in [11] examined the qualitative behavior of a smoking model in which the population is split into five classes: non-smokers, smokers, smokers who have temporarily given up smoking, smokers who have permanently given up smoking, and smokers who have a smoking-related ailment. In a delayed quitting smoking model with harmonic mean type incidence rate and relapse, authors in [12] examined the stability and Hopf bifurcation. Authors in [13] We investigated the existence of Hopf bifurcation and global stability in a delayed smoking model that included potential smokers, infrequent smokers, smokers, temporary quitters, permanent quitters, and smokers with some disease.

Because fractional order displays the past history and hereditary qualities in models, notably in the models of infectious diseases, fractional order mathematical models have been shown to be beneficial in mathematically displaying a wide range of phenomena than integer-order models [14], a fractional dynamic model of tobacco smoking is used by some scientists. Authors in [14] studied a Caputo fractional-order tobacco smoking model with snuffing class. Authors in

[15] examined of the fractional order smoking model through computation. In [16], the fractional order smoking model is investigated and solved using the generalized Mittag-Leffler function method and the Sumudu transform method. Authors in [17] employed a numeric-analytic approach to approximate a fractional derivative-based model of quitting smoking. Authors in [18] considered the Atangana-Baleanu derivative to analyze the dynamics of the smoking model and its impact on public health.

Numerous scholars also investigated the smoking mathematical model in conjunction with the implementation of various prevention strategies. Authors in [19] investigated potential light-smoker-quit smokers with two possible control variables in the form of education and therapy campaigns aimed at decreasing smoking attitudes. Authors in [20] studied the optimal control method for a discrete time smoking model with a fixed saturation incidence rate. Authors in [21] examined the optimal control scheme for a new model of quitting smoking that incorporates the class of chain smokers' continuous age-structure. Authors in [22] used four control variables–educational campaigns, anti-smoking gum, medications, and government bans on smoking in public spaces–along with a mathematical study of the harmonic mean type incidence rate of giving up smoking in order to reduce the use of smoking in the community. In a harmonic mean type dynamics of a delayed giving up smoking model, authors in [23] investigated the best legislative control strategy to reduce the number of smokers. Authors in [24] developed a control problem taking into account three control measures, namely; education campaign, anti-nicotine gum, and anti-nicotine medications, in order to manage the smoking behavior in the population of a giving up smoking model with relapse and harmonic mean type incidence rate. Recently, in a model of interactions between smokers in mixed populations of beginners and heavy smokers, authors in [25] examined a smoking cessation control, namely educational campaign and nicotine therapy counselling.

The model in [25] considers the untreated and treated populations of smokers, since it incorporates a control strategy. In this research, we investigate the model without the controls, or, in other words, we investigate the reduced model by combining the untreated and treated beginners into a single population. The model's qualitative behavior, sensitivity of the basic reproduction number, and sensitivity of the model's parameters are then investigated. By using this strategy, we can identify the most sensitive parameters that have the greatest impact on smoking transmission and population dynamics.

2. MODEL

Herdiana *et al.* [25] proposed a mathematical model of the dynamics of active smokers in mix population incorporating with cessation controls. The model is as follows

$$(1)\qquad \begin{cases} \dfrac{dP}{dt} = \Lambda - (\alpha B_U + \beta S_U)P - \mu P + \sigma B_U + \varphi \nu_1(t) B_T + \theta \nu_2(t) S_T \\ \dfrac{dB_U}{dt} = (\alpha B_U + \beta S_U)P - \delta B_U S_U - (\sigma + r_1 + \mu) B_U \\ \dfrac{dB_T}{dt} = r_1 B_U - \nu_1(t) B_T - \mu B_T \\ \dfrac{dS_U}{dt} = \delta B_U S_U - (r_2 + \mu) S_U \\ \dfrac{dS_T}{dt} = r_2 S_U - \nu_2(t) S_T - \mu S_T \\ \dfrac{dQ}{dt} = (1-\varphi)\nu_1(t) B_T + (1-\theta)\nu_2(t) S_T - \mu Q \end{cases}$$

where P is non-smokers or potential smokers, B_U is untreated beginner smokers, B_T is treated beginner smokers, S_U is untreated smokers, S_T is treated smokers, and Q is smokers who quit smoking permanently. The cessation controls are ν_1 and ν_2. The control ν_1 is educational campaign, while ν_2 is counseling with nicotine replacement.

When there is no control, that is $\nu_1 = \nu_2 = 0$, the model (1) becomes

$$(2)\qquad \begin{cases} \dfrac{dP}{dt} = \Lambda - (\alpha B_U + \beta S_U)P - \mu P + \sigma B_U \\ \dfrac{dB_U}{dt} = (\alpha B_U + \beta S_U)P - \delta B_U S_U - (\sigma + r_1 + \mu) B_U \\ \dfrac{dB_T}{dt} = r_1 B_U - \mu B_T \\ \dfrac{dS_U}{dt} = \delta B_U S_U - (r_2 + \mu) S_U \\ \dfrac{dS_T}{dt} = r_2 S_U - \mu S_T \\ \dfrac{dQ}{dt} = -\mu Q \end{cases}$$

In this paper, we reduce the model (2) by grouping the untreated and treated beginners as one population, and also for the untreated and treated smokers, by defining $B = B_U + B_T$ and

$S = S_U + S_T$. We also neglect the smoking quit population, since in (2) it is a standalone equation. Thus, now we have a simpler model as follows

$$\begin{cases} \dfrac{dP}{dt} = \Lambda - (\alpha_1 B + \beta_1 S)P - \mu P + \sigma_1 B \\ \dfrac{dB}{dt} = (\alpha_1 B + \beta_1 S)P - \delta_1 BS - (\sigma_1 + \mu)B \\ \dfrac{dS}{dt} = \delta_1 BS - \mu S \end{cases} \tag{3}$$

where the description of the parameters and their value are given in Table 1. The population of potential smokers grows with constant rate Λ. The potential smokers interact with beginner or smokers with effective interaction rates α_1 and β_1, respectively. This interaction makes the potential smokers becoming smokers. The beginner smokers may control theirselves to quit smoking with rate σ_1. The beginner smokers can continue their behavior on smoking if they interact with smokers with effective rate δ_1. All populations can die naturally with the rate μ.

TABLE 1. Description of parameters.

Parameter	Description	Value	Source
Λ	Constant growth rate of non-smokers population	0.25	[26]
α_1	Effective interaction rate of non-smokers with beginner smokers	0.00014	[19]
β_1	Effective interaction rate of non-smokers with smokers	0.0024	[19]
σ_1	Self-control quit rate of beginner smokers	0.0001	Assumed
δ_1	Effective contact rate of beginner smokers with smokers	0.0004	Assumed
μ	Natural death rate	0.0031	[19]

3. STABILITY ANALYSIS

The equilibrium of system (3) is obtained by taking $\frac{dP}{dt} = \frac{dB}{dt} = \frac{dS}{dt} = 0$. We get three equilibriums, namely

$$\text{Smoking-free equilibrium:} \quad E_0 = \left(\frac{\Lambda}{\mu}, 0, 0\right),$$

$$\text{Beginners equilibrium:} \quad E_B = \left(\frac{\sigma_1 + \mu}{\alpha_1}, \frac{\Lambda\alpha_1 - \mu(\sigma_1 + \mu)}{\alpha_1 \mu}, 0\right),$$

$$\text{Smokers equilibrium:} \quad E_S = \left(P^*, \frac{\mu}{\delta_1}, S^*\right),$$

where P^* is the root of $P(Z)$,

$$P(Z) = \beta_1\delta_1\mu Z^2 - (\Lambda\beta_1\delta_1 + \mu^2[\alpha_1 + \delta_1 - \beta_1])Z + (\Lambda\delta_1\mu + \sigma_1\mu^2),$$

and

$$S^* = \frac{\Lambda\delta_1 + \sigma_1\mu - (\alpha_1 + \delta_1)\mu P^*}{\beta_1\delta_1 P^*}.$$

The Jacobian matrix of system (3) evaluated at any point $E = (P, B, S)$ is as follows

$$J(E) = \begin{bmatrix} -(\alpha_1 B + \beta_1 S) - \mu & -\alpha_1 P + \sigma_1 & -\beta_1 P \\ \alpha_1 B + \beta_1 S & \alpha_1 P - \delta_1 S - (\sigma_1 + \mu) & \beta_1 P - \delta_1 B \\ 0 & \delta_1 S & \delta_1 B - \mu \end{bmatrix}. \tag{4}$$

The local stability of smoking-free and beginners equilibriums are given in Theorem 1 and Theorem 2, respectively. Before that, the basic reproduction number is calculated. We use the next generation matrix method [27, 28, 29]. In model (3), there are two ”infected” populations, that is beginner smokers B and smokers S. We split the vector $(\frac{dB}{dt}, \frac{dS}{dt})'$ as substraction of vector of new infection namely $\mathscr{G}'$ and vector of other transitions namely $\mathscr{M}'$, where

$$\mathscr{G} = \begin{bmatrix} (\alpha_1 B + \beta_1 S)P \\ 0 \end{bmatrix} \quad \text{and} \quad \mathscr{M} = \begin{bmatrix} \delta_1 BS + (\sigma_1 + \mu)B \\ -\delta_1 BS + \mu S \end{bmatrix}.$$

Evaluating the Jacobian matrix of $\mathscr{G}$ and $\mathscr{M}$ at the smoking-free equilibrium E_0 yields

$$G = \begin{bmatrix} \frac{\Lambda\alpha_1}{\mu} & \frac{\beta_1\Lambda}{\mu} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} \sigma_1 + \mu & 0 \\ 0 & \mu \end{bmatrix}.$$

Then, we have

$$GM^{-1} = \begin{bmatrix} \frac{\Lambda\alpha_1}{\mu(\sigma_1+\mu)} & \frac{\beta_1\Lambda}{\mu^2} \\ 0 & 0 \end{bmatrix}.$$

The basic reproduction number R_0 is the spectral radius of matrix GM^{-1} which is

$$R_0 = \frac{\Lambda\alpha_1}{\mu(\sigma_1 + \mu)}. \tag{5}$$

Theorem 1. *The smoking-free equilibrium E_0 is locally stable if $R_0 < 1$.*

Proof. Evaluating the Jacobian matrix (4) at $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0\right)$ gives

$$J(E_0) = \begin{bmatrix} -\mu & -\frac{\Lambda\alpha_1}{\mu} + \sigma_1 & -\frac{\Lambda\beta_1}{\mu} \\ 0 & \frac{\Lambda\alpha_1 - \mu(\sigma_1+\mu)}{\mu} & \frac{\Lambda\beta_1}{\mu} \\ 0 & 0 & -\mu \end{bmatrix}.$$

We have the eigen values $\lambda_1 = \lambda_2 = -\mu < 0$ and $\lambda_3 = \frac{\Lambda\alpha_1 - \mu(\sigma_1+\mu)}{\mu}$. The equilibrium E_0 will be locally stable if $\lambda_3 = \frac{\Lambda\alpha_1 - \mu(\sigma_1+\mu)}{\mu} < 0$, or $R_0 = \frac{\Lambda\alpha_1}{\mu(\sigma_1+\mu)} < 1$. □

Theorem 2. *The beginners equilibrium E_B is locally stable if* $\frac{\delta_1(\Lambda\alpha_1 - \mu(\sigma_1+\mu))}{\alpha_1\mu^2} < 1$.

Proof. We can observe that the beginners equilibrium $E_B = \left(\frac{\sigma_1+\mu}{\alpha_1}, \frac{\Lambda\alpha_1 - \mu(\sigma_1+\mu)}{\alpha_1\mu}, 0\right)$ has bilogically meaning if $R_0 > 1$. Assessing the Jacobian matrix at E_B produces

$$J(E_B) = \begin{bmatrix} -\frac{\Lambda\alpha_1 - \mu(\sigma_1+\mu)}{\mu} - \mu & -\mu & -\frac{\beta_1(\sigma_1+\mu)}{\alpha_1} \\ \frac{\Lambda\alpha_1 - \mu(\sigma_1+\mu)}{\mu} & 0 & \frac{\beta_1(\sigma_1+\mu)}{\alpha_1} - \frac{\delta_1(\Lambda\alpha_1 - \mu(\sigma_1+\mu))}{\alpha_1\mu} \\ 0 & 0 & \frac{\delta_1(\Lambda\alpha_1 - \mu(\sigma_1+\mu))}{\alpha_1\mu} - \mu \end{bmatrix}.$$

The eigen values of $J(E_B)$ are $\lambda_1 = -\frac{\Lambda\alpha_1 - \mu(\sigma_1+\mu)}{\mu}$, $\lambda_2 = -\mu$, and $\lambda_3 = \frac{\delta_1(\Lambda\alpha_1 - \mu(\sigma_1+\mu))}{\alpha_1\mu} - \mu$. It is clear that $\lambda_2 < 0$, and by the condition $R_0 > 1$, we have also $\lambda_1 < 0$. The equilibrium E_B is locally stable if $\lambda_3 < 0$, or $\frac{\delta_1(\Lambda\alpha_1 - \mu(\sigma_1+\mu))}{\alpha_1\mu^2} < 1$. □

The smokers equilibrium E_S is too complicated to be studied analitically. Thus, we study it numerically by substituting the parameters value from Table 1 into E_S. We have two equilibriums, namely $E_{S1} = (1.43, 7.75, 71.47)$ and $E_{S2} = (73.21, 7.75, -0.32)$. The equilibrium E_{S2} is neglected here since it contains negative value. By evaluating the Jacobian matrix at $E_S = E_{S1}$, we have

$$J(E_S) = \begin{bmatrix} -0.1757 & 0.0001 & -0.0034 \\ 0.1726 & -0.0316 & 0.00003 \\ 0 & 0.0286 & 0 \end{bmatrix},$$

and its eigen values $\lambda_1 = -0.0031$, $\lambda_2 = -0.0279$, and $\lambda_3 = -0.1763$. Thus, E_S is locally stable.

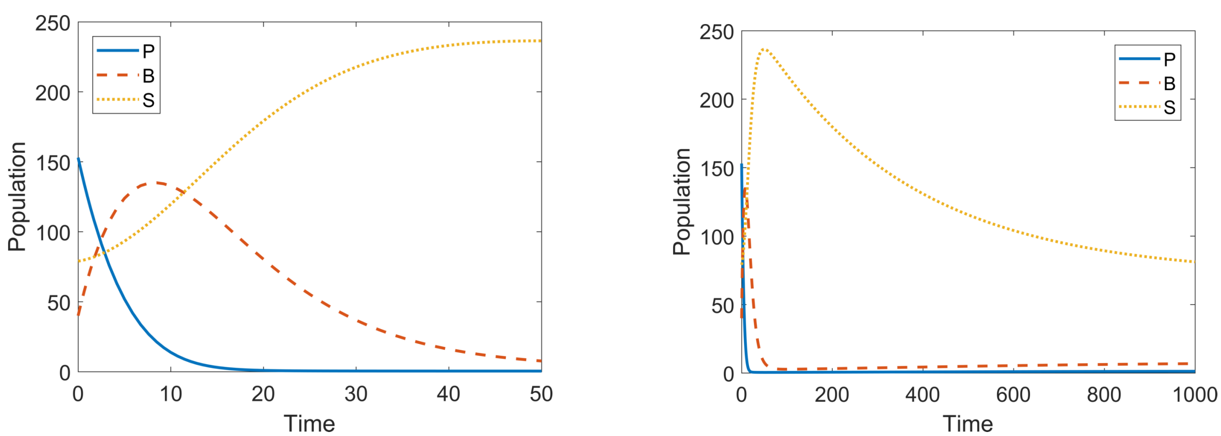
Another view of determining the stability of E_S is from previous theorems. We can deduce that the equilibrium E_S will be locally stable if $R_0 > 1$ and $\frac{\delta_1(\Lambda\alpha_1 - \mu(\sigma_1+\mu))}{\alpha_1\mu^2} > 1$. We can see that

$$R_0 = \frac{\Lambda\alpha_1}{\mu(\sigma_1+\mu)} = 3.53 > 1 \quad \text{and} \quad \frac{\delta_1(\Lambda\alpha_1 - \mu(\sigma_1+\mu))}{\alpha_1\mu^2} = 7.47 > 1.$$

Thus, E_S is locally stable.

4. Simulation of the Solution

We simulate system (3) using parameters value in Table 1 and initial conditions [19, 25]: $P(0) = 153$, $B(0) = 40$, and $S(0) = 79$. The solution of system (3) with respect to time t is given in Figure 1. The non-smokers population declines as time goes by, while the beginner smokers population grows rapidly in fisrt short period but delines after that. While the other populations decline, the smokers population grows until reach highest number but then declines slowly and reaching equilibrium in a long time.



(A) (B)

Figure 1. Solution of system (3) in different time scales.

The phase portraits of system (3) with various initial conditions is presented in Figure 2. Starting from any initial point, the trajectory $(P(t), B(t), S(t))$ declines in P-axis but at the same time it grows until reaching the highest level of B, and then it declines in $B-axis$ but it grows until reaching the highest level of S, and then converging to the equilibrium point E_S.

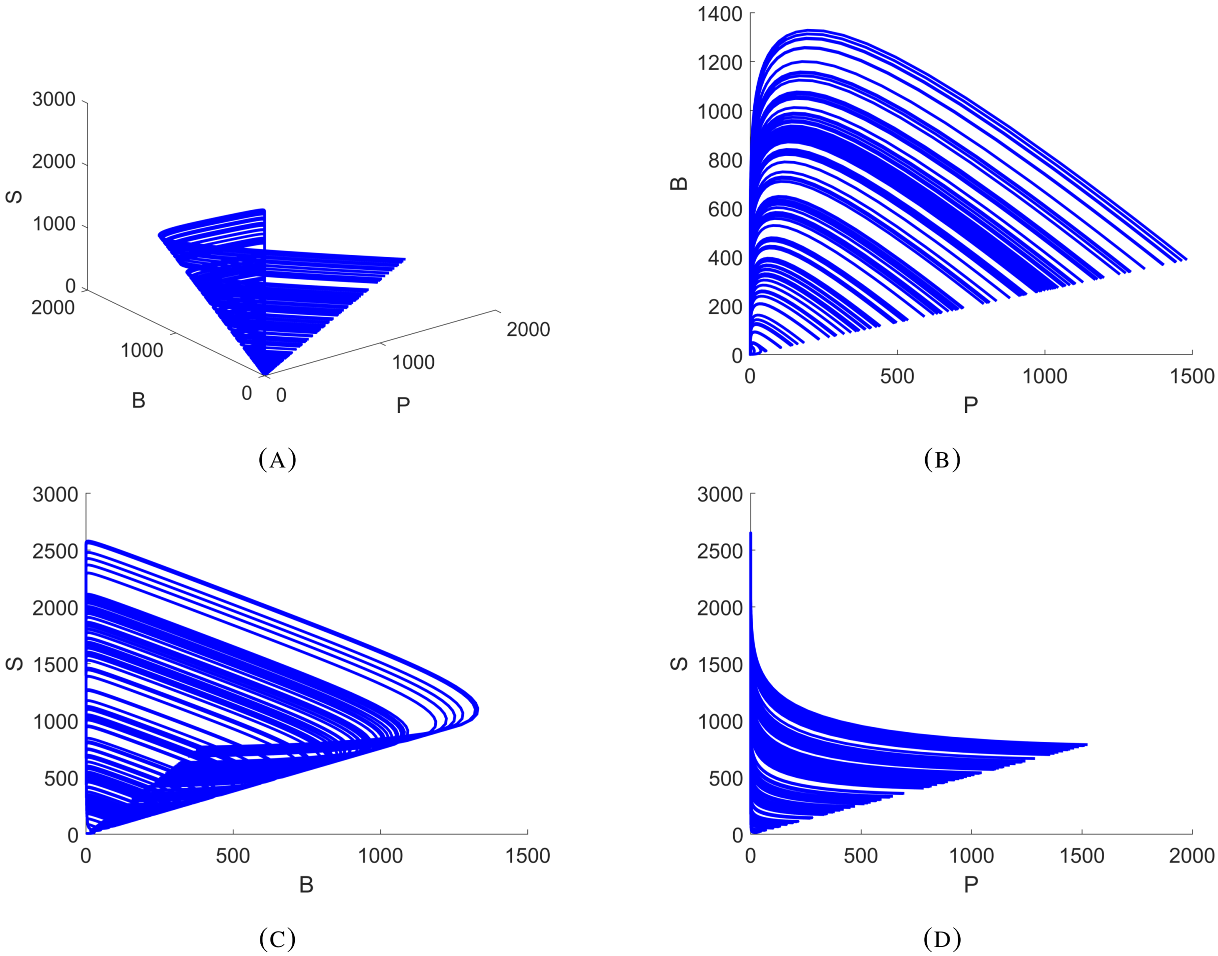


FIGURE 2. Phase portraits of system (3) with various initial conditions.

5. SENSITIVITY ANALYSIS

From previous analysis, we know that the basic reproduction number R_0 acts as initially smoking behavior transmission. The next step is to determine the normalized sensitivity index to see the the relative change of parameter (appeared in R_0) on the value of R_0. This can be used to measure which parameter having most impact on R_0. The normalized sensitivity index is defined as follows [29],

$$I_q^{R_0} = \frac{\partial R_0}{\partial q} \times \frac{q}{R_0}, \tag{6}$$

where $q \in \{\Lambda, \alpha_1, \sigma_1, \mu\}$.

By the definition (6), we have $I_\Lambda^{R_0} = 1$, $I_{\alpha_1}^{R_0} = 1$, $I_{\sigma_1}^{R_0} = -\frac{\sigma_1}{\sigma_1+\mu}$, and $I_\mu^{R_0} = -\frac{\sigma_1+2\mu}{\sigma_1+\mu}$. Hence, $-1 < I_{\sigma_1}^{R_0} < 0$ and $I_\mu^{R_0} < -1$. This means that parameters σ_1 and μ have negative impact on

R_0, while parameters Λ and α_1 have positive impact on R_0. But, the natural death parameter μ is the most sensitive parameter, and thus reducing it will have highest proportional impact on the smoking behavior transmission, followed by Λ and α_1. The comparison between these parameters by substituting their value from Table 1 is presented in Figure 3.

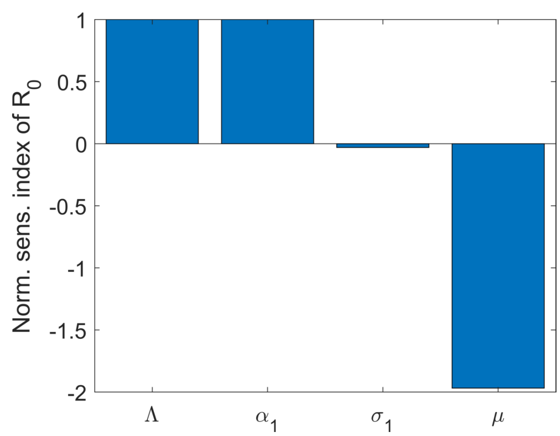


FIGURE 3. Normalized sensitivity index of R_0 with respect to Λ, α_1, σ_1, and μ.

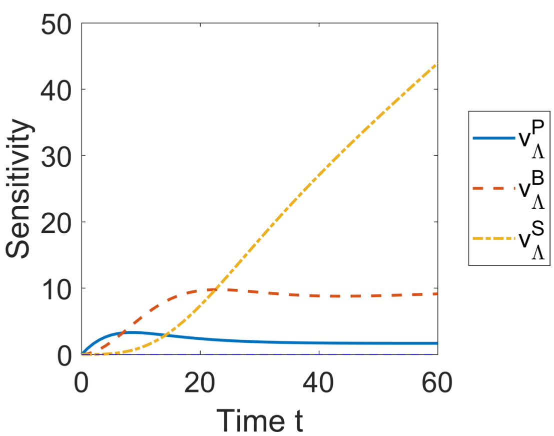
Another interesting examination is to study the impact of changes of all parameters on the dynamics of populations. By this purpose, we perform a time-dependent sensitivity analysis. This sensitivity analysis have been used in many papers, for example [30, 31, 32, 33]. Let $X = (P, B, S)$ be vector of populations, $Q = (\Lambda, \alpha_1, \beta_1, \sigma_1, \delta_1, \mu)$ be vector of parameters, and $F = \frac{dX}{dt}$ be the vector equations of the righ-side of (3). To see the effect of changes of parameters on populations, let us define a sensitivity function $V = \frac{\partial X}{\partial Q}$. Now, by seeing V as a function of time t, we can make total derivation of V as follows

$$\frac{dV}{dt} = \frac{d}{dt}\frac{\partial X}{\partial Q} = \frac{\partial}{\partial Q}\frac{dX}{dt} = \frac{\partial F}{\partial X}\frac{\partial X}{\partial Q} + \frac{\partial F}{\partial Q} = \frac{\partial F}{\partial X}V + \frac{\partial F}{\partial Q}. \tag{7}$$

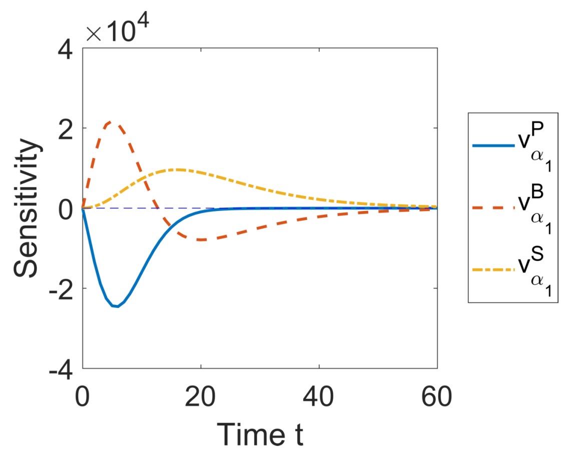
The term $\frac{\partial F}{\partial X}$ is 3×3 Jacobian matrix $J(X)$ as appeared in (4), V is a 3×6 matrix, and $\frac{\partial F}{\partial Q}$ is a 3×6 matrix which given as follows

$$\frac{\partial F}{\partial Q} = \begin{bmatrix} 1 & -BP & -SP & B & 0 & -P \\ 0 & BP & SP & -B & -BS & -B \\ 0 & 0 & 0 & 0 & BS & -S \end{bmatrix}.$$

We solve the system of 18 differential equations (7) numerically with initial conditions all zeros, and then plot the solution. We write the sensitivity function as $v_q^x = \frac{\partial x}{\partial q}$, where $x \in \{P, B, S\}$ and $q \in \{\Lambda, \alpha_1, \beta_1, \sigma_1, \delta_1, \mu\}$. The plot of v_q^x for each parameter is presented in Figure 4. Parameter Λ has positive impact on the populations. Parameters α_1 and β_1 have similar impact on the populations, where they affect positively on population S, negatively on population P, and first positively but then negatively on population B. Parameter σ_1 has positive impact on population P, negative impact on population S, and first negative but then positive impact on population B. Parameter δ_1 affect positively on population S, but negatively on populations P and B. Parameter μ produce negative impact on population S, first negative but then positive impact on both populations P and B.



(A)



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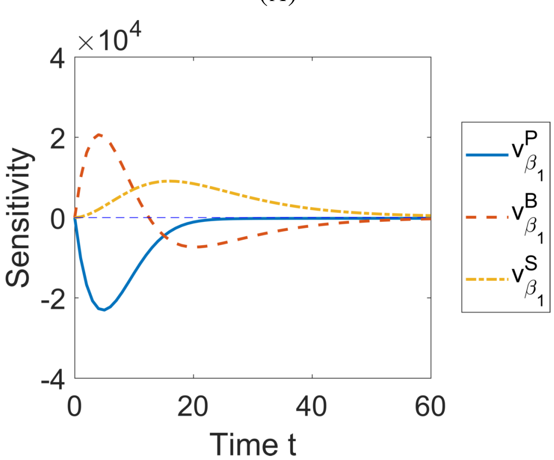

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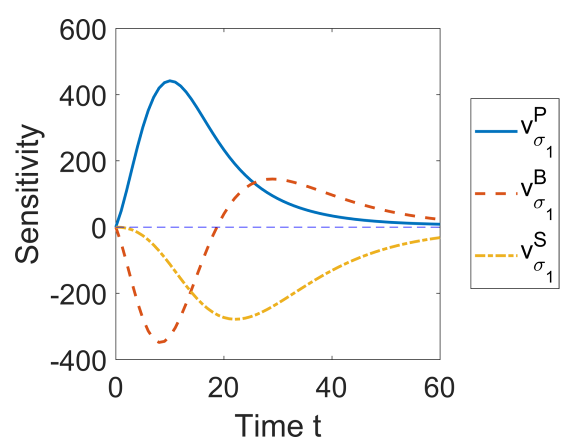

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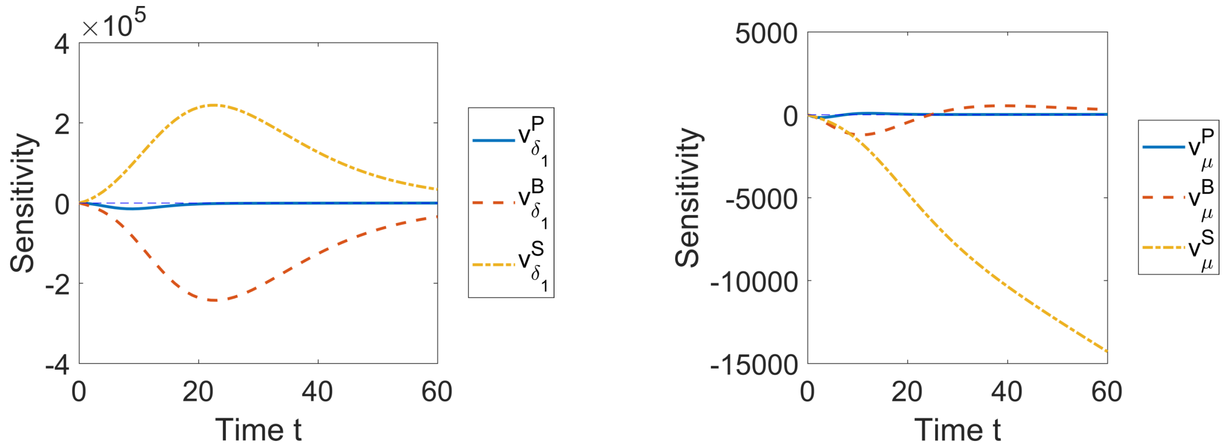

(E) (F)

FIGURE 4. Time-dependent sensitivity of parameters.

We can observe that some parameters give positive and negative impact on populations as time goes by. To see which one of parameters that produces highest impact on all populations, we plot the sensitivity index of the sensitivity function after arriving at the equilibrium. We plot their comparison in Figure 5. We can see that parameter μ is the most sensitive parameter on population S, and it si followed by parameter δ_1. On the other hand, parameter δ_1 is the most sensitive parameter on population B, and it is followed by μ. In the case of population P, parameter β_1 is the most sensitive parameter.

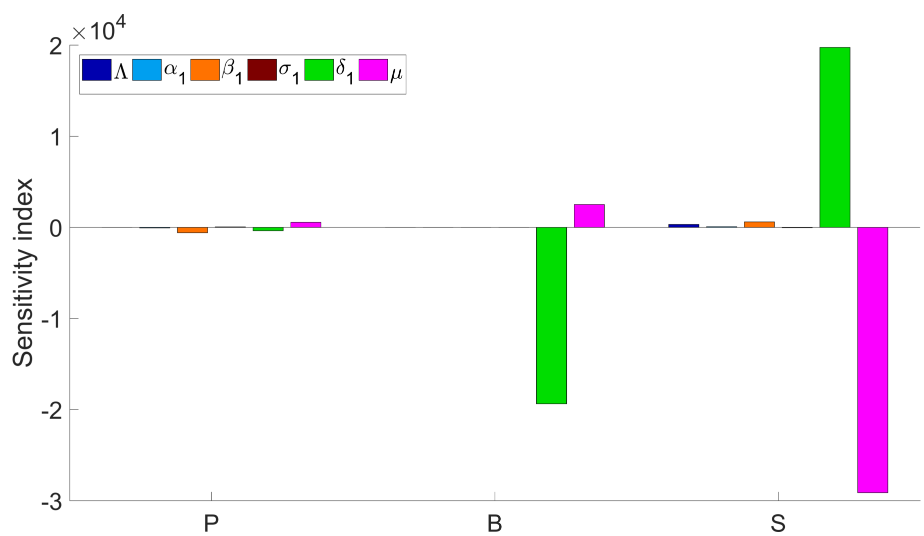

FIGURE 5. Sensitivity index of all parameters after reaching equilibrium.

6. CONCLUSIONS

The model studied in this paper considers a population related to smoking behaviour of a system that consists of three compartments, namely potential or non-smokers, beginner smokers, and smokers. The stability of the system depends on the basic reproduction number. If the basic reproduction number is less than one, then the system converges to smoking-free equilibrium, if it is bigger than one, the system converges to smokers equilibrium. The normalized sensitivity analysis of the basic reproduction number reveals that the natural death rate parameter gives highest impact on the smoking behaviour transmission. But, in the time-dependent sensitivity analysis, this parameter gives highest impact only on the smokers population. Meanwhile the parameter of effecive contact rate between beginners and smokers gives highest impact on the beginners population, and the parameter of effective interaction rate between non-smokers and smokers has highest impact on the non-smokers population. Thus, to reduce the smoking behaviour, non-smokers population should avoid contact with smokers, and beginners should also avoid contact with smokers.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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